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SOME RESULTS AND APPLICATIONS ABOUT THE VECTOR ϵ-ALGORITHM

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The ϵ-algorithm is a device found by Wynn [7] to accelerate the convergence of sequences. It is closely related to the Padé table in the following way (Wynn [8]): if we apply the ϵ-algorithm to the partial sums of the power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$ then $\epsilon_{2k}^{(n)} = f_{k,n+k}(x)$ where $f_{k,n+k}(x)$ is the Padé approximant to $f(x)$ the denominator of which is of degree k and the numerator of degree $n + k$. Wynn [9] has also proposed a non-scalar ϵ-algorithm working with vectors or matrices or with elements of an associative division algebra over the complex numbers.

There is still no connection between this non-scalar ϵ-algorithm and a Padé table because of non-existence of a theory for the non-scalar Padé table. Yet I think that two recent papers by Wynn [10], [11] are the beginning of such a theory. It is the reason why, in this paper, I should like to speak about convergence theorems for the non-scalar ϵ-algorithm and give an application of the vector ϵ-algorithm to the solution of systems of nonlinear equations.

The non-scalar ϵ-algorithm satisfies the relationship

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + (\epsilon_k{}^{(n+1)} - \epsilon_k{}^{(n)})^{-1}$$

with the initial conditions $\epsilon_{-1}^{(n)} = 0$ and $\epsilon_0{}^{(n)} = S_n$ where the S_n are non-scalar quantities. The inverse of a vector is defined by $z^{-1} = \bar{z}/(z, z)$ where $\bar{z}$ denotes the complex conjugate of z and (z, z) is the scalar product. Let us first give two results concerning the application of the non-scalar ϵ-algorithm to sequences of matrices.

THEOREM 1. *If $S_n = \sum_{k=0}^{n} A^k$ where A is a nonsingular matrix and if $I - A$ is nonsingular, then*

$$\epsilon_2{}^{(n)} = (I - A)^{-1} \quad \forall n.$$

This theorem is a generalization of a result of Householder [4] for the scalar ϵ-algorithm.

THEOREM 2. *If $\{S_n\}$ is a sequence of square matrices so that*

$$S_{n+1} - S = (A + E_n) \cdot (S_n - S) \text{ or } S_{n+1} - S = (S_n - S) \cdot (A + E_n)$$

where A and E_n are square matrices, with $\|A\| < 1$ and $\lim_{n\to\infty} E_n = 0$ *then*:

$$\lim_{n\to\infty} \epsilon_2^{(n)} = S \textit{ and } \lim_{n\to\infty} (\epsilon_2^{(n)} - S)(S_n - S)^{-1} = 0.$$

This theorem is a generalization of a theorem by Henrici [3]. The demonstration is quite the same. If $E_n = 0$ $\forall n$ then $\epsilon_2^{(n)} = S\,\forall n$.

We now give a theorem concerning the application of the non-scalar ϵ-algorithm to vectors.

THEOREM 3. *Let $\{S_n\}$ be a sequence of k dimensional vectors so that* $S_{n+1} - S = A(S_n - S)$ *where A is a matrix so that $I - A$ is non-singular, then* $\epsilon_{2k}^{(n)} = S$ $\forall n$.

A very powerful method for solving systems of nonlinear equations has been obtained by Brezinski [1], [2] using the vector ϵ-algorithm. This method is quadratic without calculating any derivative. If we want to find the solution s of $x = F(x)$ where F is a transformation of R^p into itself, we propose the following algorithm: x_0 given

$$\text{iteration } n \begin{cases} u_0 = x_n, & \\ u_k = F(u_{k-1}), & k = 1, \cdots, 2p, \\ x_{n+1} = \epsilon_{2p}^{(0)}. & \end{cases}$$

THEOREM 4. *If F is Fréchet-differentiable in the neighbourhood of s and if $I - F'(s)$ is non singular, then there exists a neighbourhood of s so that for each x_0 belonging to this neighbourhood the preceding algorithm converges at least quadratically to s.*

If we know some supplementary properties of F, we prove:.

THEOREM 5. *If, for* $\lambda > 1$, $F(x) - s - F'(s)\cdot(x - s) = O(\|x - s\|^\lambda)$ *then* $x_{n+1} - s = O(\|x_n - s\|^\lambda)$.

We have the same result with o instead of O.

This algorithm is the generalization of Steffensen's method for equations and theorem 5 is similar to that given by Ostrowski [5] for equations. We notice that the convergence result of theorem 4 does not need the convergence of the basic iterations $u_k = F(u_{k-1})$.

The application of the previous algorithm to the solution of two point nonlinear boundary value problems has been studied by Rieu

[6]. The solution of such a problem can be reduced to the solution of a system of nonlinear equations after integrating the differential system.

EXAMPLE.

$$\begin{aligned} x'(t) &= y(t) - z(t) \qquad & x(0) &= 1, \\ y'(t) &= x^2(t) + y(t) \qquad & y(1) &= -4 - e(2-e), \\ z'(t) &= x^2(t) + z(t) \qquad & z(1) &= -4 - e(1-e). \end{aligned}$$

The missing initial conditions are $y(0) = -1$ and $z(0) = 0$. Starting from the initial guess $\bar{y}(0) = -25$ and $\bar{z}(0) = 35$ we obtain:

iteration	$\bar{y}(0)$	$\bar{z}(0)$
1	-0.99268	$0.877(10^{-2})$
2	-1.00000022	$-0.184(10^{-6})$
3	-1.00000000065	$-0.572(10^{-9})$

The following iterations give the same result, which is in fact the solution of the discretized problem. The error has the same magnitude as the discretization error. It is possible to use the same method for solving some partial differential equations of evolution with boundary conditions.

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