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MATRIX INTERPRETATIONS AND APPLICATIONS OF THE CONTINUED FRACTION ALGORITHM

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1. **Introduction.** This paper is concerned with certain aspects of the one to one correspondence between real sequences $\{c_n\}_0^\infty$, formal Laurent series $f(z) = \sum_0^\infty c_n/z^{n+1}$ and infinite Hankel matrices $C = (c_{i+j})_{i,j=0}^\infty$. The finite 'connected' submatrices of C will be denoted by $C_n^{(m)} \equiv (c_{m+i+j})_{i,j=0}^{n-1}$, with $C_n \equiv C_n^{(0)}$, and their determinants by $c_n^{(m)} = \det C_n^{(m)}$.

Also associated with $\{c_n\}$ is the linear functional c^* which acts on the vector space of real polynomials and is determined by $c^*(z^n) = c_n$, $n \geqq 0$. With the ordinary (Cauchy) product of two polynomials $c^*(pq)$ becomes a (Cauchy) bilinear functional on the algebra of real polynomials. If $p(z) = \sum a_i z^i$, $q(z) = \sum b_j z^j$ and $a = (a_0, a_1, a_2, \cdots)^T$, $b = (b_0, b_1, b_2, \cdots)^T$ are the column vectors of coefficients then $c^*(pq) = \sum a_i c_{i+j} b_j = a^T C b$.

The functional $c^*(pq)$ is an *inner product* if and only if $\{c_n\}$, f and C are *positive definite*, that is $c^*(p^2) > 0$ if $p \neq 0$, or equivalently $c_n^{(0)} > 0$ for $n \geqq 0$. An alternative characterization is that $p \neq 0$ and $p(x) \geqq 0$ for $-\infty < x < +\infty$ imply $c^*(p) > 0$. This involves the (unique) decomposition of such a (positive) polynomial p as the sum of two squares of real polynomials whose zeros interlace (strictly) and gives rise to the geometric theory of moment spaces [**20**, **18**, **19**]. If the coefficients $c_n = \int_{-\infty}^{+\infty} t^n d\mu(t)$ are moments of a bounded nondecreasing function μ with infinitely many points of increase then all $c_n^{(2m)} > 0$, since $c^*(t^{2m}p^2) = \int_{-\infty}^{+\infty} t^{2m}\ [p(t)]^2 d\mu(t)$. Conversely if all $c_n^{(0)} > 0$ then the existence of such a μ follows by compactness arguments from the algebraic results to be given below [**12**, **34**, **1**].

2. **Lanczos polynomials.** The algebraic aspects of the theory of orthogonal polynomials carry over to the case in which all $c_n^{(0)} \neq 0$. *Hence this will be assumed.* The material of this section is readily adapted from [**29**, **32**, **11**], for example.

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THEOREM 1. *There exists a unique sequence* $\{q_n(z)\}_0^\infty = \{\sum_{j=0}^n \ell_{n,j} z^j\}_{n=0}^\infty$ *of monic polynomials for which* $c^*(q_m q_n) = 0$ *when* $m \neq n$. *The determinant representations*

$$q_n(z) = \frac{1}{c_n^{(0)}} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & z & \cdots & z^n \end{pmatrix}$$

are valid. The sequence $\{q_n\}_0^\infty$ *satisfies the three term recurrence relation*

$$\begin{aligned} &q_{-1}(z) = 0, \quad q_0(z) = 1, \\ &q_{n+1}(z) = (z - \alpha_{n+1}) q_n(z) - \beta_n^2 q_{n-1}(z), \qquad (1) \\ &n = 0, 1, 2, \cdots, \end{aligned}$$

in which

$$\begin{aligned} \alpha_{n+1} &= \pi_{2n+1}/\pi_{2n}, \quad \beta_n^2 = \pi_{2n}/\pi_{2n-2} \quad (\beta_0^2 \equiv c_0), \\ \pi_{2n} &\equiv c^*(q_n^2) = c^*(z^n q_n) = c_{n+1}^{(0)}/c_n^{(0)}, \\ \pi_{2n+1} &\equiv c^*(z q_n^2). \end{aligned}$$

In accordance with their use in numerical linear algebra [16] the generalized orthogonal polynomials will be called the *Lanczos polynomials of the first kind* for $\{c_n\}$. They are related to the denominators of the $(n-1, n)$ Padé fractions for the formal power series $\sum_0^\infty c_n z^n$ by $q_n(z) = z^n q_{n-1,n}(1/z)$. In this context (1) is a special Frobenius identity. Observe that $\{c_n\}$ is positive definite if and only if all $\pi_{2n} > 0$, or equivalently all $\beta_n^2 > 0$. In general β_n^2 may be negative.

According to Wall [32], Chapter 11, the following algorithmic consequence of (1) dates back to Chebyshev [3]. It is more general than the quotient-difference algorithm [25, 13, 14, 17] which serves a similar purpose but also requires all $c_n^{(m)} \neq 0$ (When $c_n = \int_0^\infty t^n d\mu(t)$ one has all $c_n^{(m)} > 0$ and this is implied by $c_n^{(0)} > 0$ and $c_n^{(1)} > 0$ for $n \geqq 0$ [28].)

THEOREM 2. *The coefficients* β_{n-1}^2, α_n, $\{\ell_{n,j}\}_{j=0}^n$, $n = 1, 2, \cdots, N$, *may be computed recursively from the sequence* $\{c_n\}_0^{2N-1}$ *by the rational* $0(N^2)$ *process*:

$$\sigma_{-1} = 1, \tau_0 = 0, \quad \ell_{0,0} = 1,$$

$$\text{for } n = 0, 1, \cdots, N-1$$

$$\sigma_n = \sum_{j=0}^{n} \ell_{n,j} c_{n+j},$$

$$\tau_{n+1} = \left(\sum_{j=1}^{n} \ell_{n,j} c_{n+j+1} \right) / \sigma_n,$$

$$\beta_n^2 = \sigma_n / \sigma_{n-1}, \alpha_{n+1} = \tau_{n+1} - \tau_n,$$

$$\ell_{n-1,n} = \ell_{n,-1} = 0, \ell_{n+1,\, n+1} = 1,$$

$$\text{for } j = 0, 1, \cdots, n$$

$$\ell_{n+1,j} = \ell_{n,j-1} - \alpha_{n+1} \ell_{n,j} - \beta_n^2 \ell_{n-1,j}.$$

PROOF. Let

$$\sigma_n \equiv \pi_{2n} = c^*(q_n^2) = \beta_0^2 \beta_1^2 \cdots \beta_n^2$$

$$= c^*(z^n q_n) = \sum_{j=0}^{n} \ell_{n,j} c_{n+j}$$

and

$$\sigma_n \tau_{n+1} \equiv c^*(z^{n+1} q_n) = \sum_{j=0}^{n} \ell_{n,j} c_{n+j+1}.$$

Then

$$\sigma_n \tau_{n+1} = c^*(z^n(q_{n+1} + \alpha_{n+1} q_n + \beta_n^2 q_{n-1}))$$

$$= \alpha_{n+1} \sigma_n + \beta_n^2 \sigma_{n-1} \tau_n$$

$$= \sigma_n(\tau_n + \alpha_{n+1}),$$

so

$$\tau_n = \alpha_1 + \alpha_2 + \cdots + \alpha_n \ (\tau_0 \equiv 0).$$

The rest follows by equating coefficients in (1).

It is known from practical experience with the positive definite case that $\beta_n^2 = \beta_n^2(c_0, c_1, \cdots, c_{2n})$ and $\alpha_{n+1} = \alpha_{n+1}(c_0, c_1, \cdots, c_{2n+1})$ are ill conditioned functions of the moments $\{c_n\}$ although this has not been quantified precisely. Hence the algorithm can only be recommended if the $\{c_n\}$ are rational and rational arithmetic is used. However see [26] for a treatment of 'modified moments', [6, 7] for related analysis and [33] for an application.

The nth *reproducing kernel function,*

$$K_n(z, w) \equiv \sum_{j=0}^{n} \frac{q_j(z)q_j(w)}{\pi_{2j}} = K_n(w, z)$$

satisfies $p(z) = c_t^*(K_n(z, t)p(t))$ when $\deg p \leqq n$.

THEOREM 3. 1. (CHRISTOFFEL-DARBOUX). *The following relations hold:*

$$K_n(z, w) = \frac{q_{n+1}(z)q_n(w) - q_n(z)q_{n+1}(w)}{\pi_{2n}(z - w)}, \quad z \neq w,$$

$$= \frac{q'_{n+1}(z)q_n(z) - q_n{}'(z)q_{n+1}(z)}{\pi_{2n}}, \quad z = w.$$

2. *The alternative representations*

$$K_n(z, w) = \frac{-1}{c_{n+1}^{(0)}} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n & 1 \\ c_1 & c_2 & \cdots & c_{n+1} & z \\ \vdots & \vdots & & \vdots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} & z^n \\ 1 & w & \cdots & w^n & 0 \end{pmatrix}$$

$$= (-1)^n \frac{\det(c_t^*(t^{i+j}(t - z)(w - t))_{i,j=0}^{n-1}}{c_{n+1}^{(0)}}$$

are valid.

3. (CHRISTOFFEL). *If* $q_n(a) \neq 0$, $n \geqq 0$, *then the Lanczos polynomials of the first kind associated with the linear functional* $c_a^*(p) \equiv c^*((z - a)p)$ *also exist and are given by*

$$q_n{}^a(z) = \pi_{2n}K_n(z, a)/q_n(a), \quad n \geqq 0.$$

The *Lanczos polynomial of the second kind* for $\{c_n\}$ are denoted by $\{p_n(z)\}_0^\infty$. They form a second linearly independent solution of (1):

$$p_{-1}(z) \equiv -1, p_0(z) \equiv 0, \tag{2}$$
$$p_{n+1}(z) = (z - \alpha_{n+1})p_n(z) - \beta_n{}^2 p_{n-1}(z), \quad n = 0, 1, 2, \cdots.$$

Note that $p_n(z) = c_0 z^{n-1} + \cdots$ for $n \geqq 0$. The twin recurrence relations (1) and (2) lead to the following, via the elementary theory of continued fractions.

THEOREM 4. *There is a one to one correspondence between formal Laurent series* $f(z) = \sum_0^\infty c_n/z^{n+1}$ *with all* $c_n^{(0)} \neq 0$ *and formal (associated) continued fractions*

$$F(z) = \frac{\beta_0^2}{z-\alpha_1} - \frac{\beta_1^2}{z-\alpha_2} - \frac{\beta_2^2}{z-\alpha_3} - \cdots$$

with all $\beta_n^2 \neq 0$. *If*

$$w_n(z) \equiv p_n(z)/q_n(z) \quad (n \geqq 0)$$

is the nth approximant of $F(z)$, *then*

$$p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z) \equiv \pi_{2n-2} \neq 0 \tag{3}$$

and the Taylor expansion of $w_n(z)$ *about* $z = \infty$ *coincides with* $f(z)$ *precisely through the term in* $1/z^{2n}$:

$$f(z) = w_n(z) + \pi_{2n}/z^{2n+1} + 0(1/z^{2n+2}). \tag{4}$$

The following representations hold:

$$p_n(z) = \frac{(-1)^n}{c_n^{(0)}} \det\begin{pmatrix} 0 & c_0 & c_0z + c_1 & \cdots & c_0z^{n-1} + \cdots + c_{n-1} \\ c_0 & c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}$$
$$= c_t^* \left(\frac{q_n(z) - q_n(t)}{z-t}\right).$$

The polynomials $\{p_n\}$ are related to the numerators of the $(n-1, n)$ Padé fractions for $\sum_0^\infty c_n z^n$ by $p_n(z) = z^{n-1}p_{n-1,n}(1/z)$. In this context the use of bigradient determinants [17] provides a short proof of a generalization of (4).

It will be assumed that the zeros $\{z_{n,k}\}_{k=1}^n$ *of* q_n *are distinct for each* $n \geqq 1$. One may then obtain a 'generalized' Gaussian quadrature formula. For the construction of such formulas in the positive definite case see [5, 8].

THEOREM 5. *The partial fraction expansion of* $w_n(z)$ *is*

$$w_n(z) = \sum_{k=1}^n \frac{w_{n,k}}{z - z_{n,k}} \quad \text{with } w_{n,k} = \frac{1}{K_n(z_{n,k}, z_{n,k})}.$$

Consequently from (4),

$$c^*(p) = \sum_{k=1}^{n} w_{n,k}p(z_{n,k}) \text{ if } \deg p < 2n$$

and

$$c^*(z^{2n}) - \sum_{k=1}^{n} w_{n,k}z_{n,k}^{2n} = \pi_{2n} \neq 0.$$

In particular $\{c_n\}$ is positive definite if and only if all $w_{n,k} > 0$ and then the zeros $\{z_{n,k}\}_{k=1}^{n}$ are real, distinct and (strictly) separated by those of p_n and q_{n-1}.

3. **Matrix interpretations.** The results of the previous section become more transparent when viewed in terms of matrices.

Let the unit left triangular matrices

$$L \equiv (\ell_{i,j})_{i,j=0}^{\infty} \quad \text{and} \quad L_n \equiv (\ell_{i,j})_{i,j=0}^{n-1}$$

The 'orthogonality' of the polynomials $\{q_n\}$ is then equivalent with

$$LCL^T = P \equiv \operatorname{diag}(\pi_0, \pi_2, \pi_4, \cdots).$$

and with

$$(5) \qquad L_nC_nL_n{}^T = P_n \equiv \operatorname{diag}(\pi_0, \pi_2, \cdots, \pi_{2n-2})$$

for $n \geqq 0$. In other words

$$C = L^{-1}P(L^{-1})^T$$

is the Gauss-Banachiewicz LDR factorization of the symmetric Hankel matrix C. This factorization exists if and only if all $c_n{}^{(0)} \neq 0$. The algorithm of theorem 2 effects the factorization (5) in $0(n^2)$ operations as compared with the usual $0(n^3)$ for an arbitrary (symmetric) $n \times n$ matrix.

The recurrence relations (1) and (2) are related to the tridiagonal matrix

$$J \equiv \operatorname{tridiag}\begin{pmatrix} 1, & 1, & 1, & & \cdots \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \cdots \\ \beta_1{}^2, & \beta_2{}^2, & \beta_3{}^2, & & \cdots \end{pmatrix},$$

its nth leading principal submatrices J_n and the submatrices $J_n{}'$ of J_n in which the first row and column are deleted. One has

$$p_n(z) = c_0 \det(zI_{n-1} - J_n'),$$
$$q_n(z) = \det(zI_n - J_n),$$

with I_n the $n \times n$ identity matrix.

Let the translation matrix

$$T \equiv (\delta_{i+1,j})_{i,j=0}^{\infty}$$

and the Frobenius matrices

$$F_n \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\ell_{n,0} & -\ell_{n,1} & -\ell_{n,2} & \cdots & -\ell_{n,n-2} & -\ell_{n,n-1} \end{pmatrix}.$$

The F_n are the companion matrices of their characteristic polynomials q_n. From theorem 2 one then finds

$$JL = LT \quad \text{and} \quad J_n L_n = L_n F_n.$$

The first of these relations was given by Stieltjes [27]; the second is its finite analogue. Finally, denote by

$$V_n \equiv \left(z_{n,j}^{\,i-1} \right)_{i,j=1}^{n}$$

the Vandermonde matrix of the zeros of q_n. Then

$$F_n V_n = V_n D_n, \; D_n \equiv \operatorname{diag}(z_{n,1}, z_{n,2}, \cdots, z_{n,n})$$
$$Q_n \equiv L_n V_n = (q_{i-1}(z_{n,j}))_{i,j=1}^{n},$$

and

$$J_n Q_n = Q_n D_n,$$

showing explicitly the similarities among the matrices J_n, F_n and D_n.

Further consequences arise from theorem 3. First of all the Christoffel-Darboux formula shows that the matrices

$$(K_n(z_{n,i}, z_{n,j})) = Q_n{}^T P_n{}^{-1} Q_n$$

are diagonal. There follows

(6) $$Q_n W_n Q_n{}^T = P_n, \quad W_n \equiv \operatorname{diag}(w_{n,1}, w_{n,2}, \cdots, w_{n,n}),$$

and with (5),

$$C_n = V_n W_n V_n^T, \tag{7}$$

exhibiting the explicit congruences among C_n, P_n and W_n. Equation (6) shows the 'orthogonality' of the polynomials $\{q_k\}_0^{n-1}$ with respect to 'weighted' summation over the zeros of q_n, and (7) essentially contains the 'quadrature formula'. From theorem 3.2 $K_{n-1}(z, w) = \sum b_{i,j} w^{i-1} z^{j-1}$ is the generating function for the elements of $B_n \equiv C_n^{-1}$. Theorem 2 and the Christoffel-Darboux formula thus provide an $0(n^2)$ algorithm for the inversion of $n \times n$ Hankel matrices:

for $i = 0, 1, \cdots, n-1$

  $b_{i,n+1} = 0,$

  for $j = i+1, i+2, \cdots, n$

    if $i = 0$ then $b_{0,j} = 0,$

    $b_{i+1,j} = b_{j,i+1} = b_{i,j+1} + (\ell_{n-1,i}\ell_{n,j} - \ell_{n,i}\ell_{n-1,j})/\sigma_{n-1}.$

See also [**30**, **31**].

Let the nth *resolvent*

$$R_n(z) \equiv \left(r_{i,j}^{(n)}(z) \right)_{i,j=0}^{n-1} \equiv (zI_n - J_n)^{-1}$$

$$= \sum_{m=0}^{\infty} J_n{}^m / z^{m+1} \equiv \sum_{k=1}^{n} \frac{R_{n,k}}{z - z_{n,k}} .$$

Then the residues $R_{n,k}$ satisfy

$$I_n = \sum_{k=1}^{n} R_{n,k} \text{ and } J_n = \sum_{k=1}^{n} z_{n,k} R_{n,k}.$$

Moreover from the above

$$R_n(z) = Q_n(zI_n - P_n)^{-1} W_n Q_n{}^T P_n{}^{-1}$$

giving

$$\pi_{2j} r_{i,j}^{(n)}(z) = \sum_{k=1}^{n} \frac{w_{n,k} q_i(z_{n,k}) q_j(z_{n,k})}{z - z_{n,k}}$$

and

$$R_{n,k} = w_{n,k}(q_i(z_{n,k}) q_j(z_{n,k}) / \pi_{2j})_{i,j=0}^{n-1}.$$

In particular the nth approximant of the continued fraction $F(z)$ is

$$w_n(z) = p_n(z)/q_n(z) = c_0 r_{0,0}^{(n)}(z)$$

and likewise

$$r_{n-1,\, n-1}^{(n)}(z) = q_{n-1}(z)/q_n(z).$$

The Christoffel-Darboux formula shows that the residues are projectors: $R_{n,k}^2 = R_{n,k}$. When $\{c_n\}$ is positive definite the polynomials $\{\tilde{q}_n\} \equiv \{q_n/\pi_{2n}^{1/2}\}$ are orthonormal, $\tilde{Q}_n \equiv P_n^{-1/2} Q_n W_n^{1/2}$ is an orthogonal matrix, $\tilde{J}_n \equiv P_n^{-1/2} J_n P_n^{1/2}$ is symmetric and $\tilde{J}_n \tilde{Q}_n = \tilde{Q}_n D_n$. The residues $\tilde{R}_{n,k} = P_n^{-1/2} R_{n,k} P_n^{1/2}$ of the resolvent of $\tilde{J}_n$ are then also orthogonal projectors: $\tilde{R}_{n,k} = \tilde{R}_{n,k}^T$.

Let A be a real $N \times N$ matrix and consider the *Krylov sequences*

$$\boldsymbol{x}_n \equiv A\boldsymbol{x}_{n-1} = A^n \boldsymbol{x}_0,\ \boldsymbol{y}_n \equiv A^T \boldsymbol{y}_{n-1} = (A^T)^n \boldsymbol{y}_0$$

of A and A^T with respect to initial vectors $\boldsymbol{x}_0$, $\boldsymbol{y}_0$ for which $\boldsymbol{y}_0{}^T \boldsymbol{x}_0 \neq 0$. For $n \geqq 1$ put

$$X_n \equiv (\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_{n-1}),\ Y_n \equiv (\boldsymbol{y}_0, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_{n-1})$$

and

$$C^n \equiv Y_n{}^T X_n = (\boldsymbol{y}_i{}^T \boldsymbol{x}_j) = (\boldsymbol{y}_0{}^T A^{i+j} \boldsymbol{x}_0) = (c_{i+j}).$$

The matrices C_n are ultimately singular, at least for $n > N$. Let ν be such that $c_1{}^{(0)} c_2{}^{(0)} \cdots c_\nu{}^{(0)} \neq 0$ but $c_{\nu+1}^{(0)} = 0$. Then the sequence $\{q_n\}$ 'terminates' with q_ν but all previous relations, as well as the following, hold for $n \leqq \nu$. If

$$\hat{\boldsymbol{x}}_n \equiv q_n(A)\boldsymbol{x}_0 = \sum_{k=0}^{n} \ell_{n,k} \boldsymbol{x}_k, \tag{8}$$

$$\hat{\boldsymbol{y}}_n \equiv q_n(A^T)\boldsymbol{y}_0 = \sum_{k=0}^{n} \ell_{n,k} \boldsymbol{y}_k$$

then

$$\hat{X}_n \equiv (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \cdots, \hat{\boldsymbol{x}}_{n-1}) = X_n L_n{}^T,$$

$$\hat{Y}_n \equiv (\hat{\boldsymbol{y}}_0, \hat{\boldsymbol{y}}_1, \cdots, \hat{\boldsymbol{y}}_{m-1}) = Y_n L_n{}^T$$

and (5) shows that

$$\hat{Y}_n{}^T \hat{X}_n = P_n. \tag{9}$$

That is $\{\hat{\boldsymbol{x}}_n\}_0^\nu$ and $\{\hat{\boldsymbol{y}}_n\}_0^\nu$ are biorthogonal. From (8)

$$AX_n = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n) = X_n F_n{}^T + (\boldsymbol{0}, \cdots, \boldsymbol{0}, \hat{\boldsymbol{x}}_n)$$

and the relation $J_n L_n = L_n F_n$ gives

$$A\hat{X}_n = \hat{X}_n J_n{}^T + (0, \cdots, 0, \hat{x}_n) L_n{}^T.$$

Consequently from (9)

$$\hat{Y}_n{}^T A \hat{X}_n = P_n J_n{}^T = J_n P_n. \tag{10}$$

The three term recurrence relation now provides the algorithm

$$\hat{x}_{-1} = \hat{y}_{-1} = 0,\ \hat{x}_0 = x_0,\ \hat{y}_0 = y_0,$$
$$\hat{x}_{n+1} = (A - \alpha_{n+1} I)\hat{x}_n - \beta_n{}^2 \hat{x}_{n-1},$$
$$\hat{y}_{n+1} = (A^T - \alpha_{n+1} I)\hat{y}_n - \beta_n{}^2 \hat{y}_{n-1},$$
$$n = 0, 1, \cdots, \nu - 1,$$

in which, on comparing coefficients in (9) and (10),

$$\beta_n{}^2 = \frac{\hat{y}_n{}^T \hat{x}_n}{\hat{y}_{n-1}^T \hat{x}_{n-1}}, \quad \alpha_{n+1} = \frac{\hat{y}_n{}^T A \hat{x}_n}{\hat{y}_n{}^T \hat{x}_n}.$$

Moreover

$$c_n{}^{(0)} = \prod_{k=0}^{n-1} \hat{y}_k{}^T \hat{x}_k.$$

This is the essence of the *Lanczos algorithm* [21] for tridiagonalization. If the algorithm can be completed ($\nu = N$), which is possible if A is symmetric with distinct eigenvalues and x_0, y_0 are chosen appropriately, then A is similar to $J_N{}^T$:

$$\hat{X}_N{}^{-1} A \hat{X}_N = J_N{}^T.$$

In particular if A is symmetric and $y_0 = x_0$ then $\tilde{X}_N \equiv \hat{X}_N P_N{}^{-1/2}$ is orthogonal and $\tilde{X}_N{}^T A \tilde{X}_N = \tilde{J}_N$ is also symmetric. It is always theoretically possible to complete a modified version of the algorithm [**16**].

3. **Concluding remarks.** The material of this section will be developed in detail elsewhere, but is mentioned here for the sake of completeness.

The Lanczos polynomials $\{p_n\}_0^\infty$ and $\{q_n\}_0^\infty$ may be generalized to maintain their connection with the Padé numerators and denominators, which are defined even though nontrivial blocks may occur in the Padé table; see [**11**]. The $\{q_n\}$ remain monic of degree n but the 'orthogonality' is lost. The matrix $P = LCL^T$ is now block diagonal and the diagonal blocks are *lower* triangular (Δ) Hankel matrices which are nonsingular, except possibly for a last one which is the infinite null matrix ($f(z)$ rational). This 'left triangular congruence' arose in connection with the determination of the signature of a general Hankel matrix and certain theorems from [4] were used in the proof. The matrix J becomes block tridiagonal and the diagonal blocks are companion matrices. The off diagonal blocks are null apart from their lower left element which is unity for the superdiagonal and nonnull for the subdiagonal. These results are intimately related to P-fractions [22, 23, 24] which, however, are developed in the ascending power notation. A generalization of the algorithm of theorem 2 follows from the theory of continued fractions.

Inclusion *disks* for the approximants of positive definite 'J-fractions' $F(z)$ are classical [12, 32, 36, 1]. When μ has a restricted set S of points of increase these classical regions are not best possible. In [1] the best regions are described when S is the complement of a finite interval, in [15] the Stieltjes case $S = [0, +\infty)$ is treated and in [9, 10] the extended Hausdorff case $S = [a, b]$. The best regions are now intersections of two circular disks, or *lunes*. It will now be indicated how to construct the best inclusion lunes $L_n(z)$ for $f(z) = \int_a^b d\mu(t)/(z-t)$ $(-\infty < a < b < +\infty)$ when the moments $\{c_k\}_0^n$ are known.

Thus let $K_n(z)$ be the classical disk for $f(z)$ when $\{c_k\}_0^{2n-2}$ are known. The functions

$$f_a(z) = \int_a^b \frac{(t-a)d\mu(t)}{z-t} = (z-a)f(z) - c_0,$$

$$f_b(z) = \int_a^b \frac{(b-t)d\mu(t)}{z-t} = (b-z)f(z) + c_0,$$

$$f_{ab}(z) = \int_a^b \frac{(t-a)(b-t)d\mu(t)}{z-t}$$

$$= (z-a)(b-z)f(z) + (z-a-b)c_0 + c_1$$

each have (convergent) J-fraction expansions. Let $\hat{K}_n{}^a(z)$, $\hat{K}_n{}^b(z)$ and $\hat{K}_n{}^{ab}(z)$ be the corresponding classical disks. Transforming back to $f(z)$ gives

$$K_n{}^a(z) \equiv [\hat{K}_n{}^a(z) + c_0]/(z-a),$$

$$K_n{}^b(z) \equiv [\hat{K}_n{}^b(z) - c_0]/(b-z),$$

$$K_n{}^{ab}(z) \equiv [\hat{K}_n{}^{ab}(z) - (z-a-b)c_0 - c_1]/(z-a)(b-z).$$

Observe that $K_n{}^a(z)$, $K_n{}^b(z)$ require the moments $\{c_k\}^{2n-1}$ and $K_n{}^{ab}(z)$ requires $\{c_k\}_0^{2n}$. The lunes are thus given by

$$L_{2n-1}(z) = K_n{}^a(z) \cap K_n{}^b(z),$$
$$L_{2n}(z) = K_{n+1}(z) \cap K_n{}^{ab}(z).$$

An application of Christoffel's formula, theorem 3.3, then shows that the vertices of the lunes $L_n(z)$ may be expressed in terms of $\{p_k(z)\}$, $\{q_k(z)\}$, $\{q_k(a)\}$ and $\{q_k(b)\}$. In particular only one application of the continued fraction algorithm of theorem 2 is necessary.

The advantage of this technique over those using continued fractions of special form is its extension to the case when S is the union of a number of disjoint intervals. For example if $S = [a_1, b_1] \cup [a_2, b_2]$ then each inclusion region is the intersection of four circular disks. The polynomials corresponding to 1, $t - a$, $b - t$, $(t - a)(b - t)$ above are now 1, $t - a_1$, $b_2 - t$, $(t - a_1)(b_2 - t)$, $(t - b_1)(t - a_2)$, $(t - a_1)(t - b_1)(t - a_2)$, $(t - b_1)(t - a_2)(b_2 - t)$ and $(t - a_1)(t - b_1)(t - a_2)(b_2 - t)$.

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