ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5 Number 1, Winter 1975

SOME REMARKS ON KERNELS, RECOVERY FORMULAS, AND EVOLUTION EQUATIONS

ROBERT CARROLL

1. This article is a revised version of a talk given at the Conference on Evolution Equations and Functional Analysis, University of Kansas, June 28–July 18, 1970. The contents form a partial survey of some work done a few years ago (see [**14**; **15**; **16**; **17**; **18**]) plus some reorganization and modification to simplify and improve the material. We were motivated originally in part by a desire to "understand" the recovery formula

$$u(t) = G(t, \tau)u(\tau) + \int_\tau^t G(t, \xi)f(\xi)\, d\xi \tag{1.1}$$

which, under suitable hypotheses, expresses the solution of the evolution equation $u' + A(\,\cdot\,)u = f$ where $u(\tau)$ and f are prescribed and $A(t)$ is a family of linear operators. One can ask for solutions to this problem in various contexts and in various senses of the word solution and as a guide to the literature we mention [**1**; **6**; **7**; **8**; **11**; **12**; **15**; **16**; **17**; **19**; **20**; **28**; **29**; **30**; **31**; **33**; **34**; **35**; **36**; **37**; **38**; **40**; **41**; **42**; **42a**; **43**; **44**; **45**; **45a**; **46**; **47**; **48**; **49**; **50**; **52**; **56**; **57**; **58**; **59**; **60**; **65**; **66**; **67**; **69**; **70**; **72**; **73**; **74**]. This list is certainly not complete and we only mean it to be representative. For simplicity we will work in a separable Hilbert space F with $A(t)$ a closed densely defined linear operator having domain $D(A(t)) \subset F$. Let us recall that $A(t)$ is said to be closed if the graph $G(A(t)) \subset F \times F$ is closed. In practice $A(t)$ is often determined by an elliptic differential polynomial $\sum a_\alpha(t, x)D^\alpha$ where $|\alpha| = \sum \alpha_i \leqq 2m$ and $D_1^{\alpha_1} \cdots D_n^{\alpha_n} = D^\alpha$ with $D_k = (1/i)\partial/\partial x_k$ (cf. [**19**; **47**]) while in the simplest cases $A(t) = -A$ with $G(t, s) = \exp[-(t - s)A]$ the semigroup generated by $-A$ (cf. [**19**; **30**; **37**]).

Now we want to assume as little as possible about the $A(t)$ and even about $G(t, s)$ in order to see what consequences follow from (1.1) itself. First we pick for convenience the space $H = L^2(\tau, T; F)$ in which to study (1.1). Thus H consists of F valued functions w defined a.e. on $[\tau, T]$, $T < \infty$, with $\int_\tau^T |w(t)|^2\, dt < \infty$ where $|\,|$(resp. $(\,,\,)$) denotes the norm (resp. scalar product) in F (we will not distinguish between functions and equivalence classes). We can regard any $u \in H$ as an F valued distribution on (τ, T) and hence u' makes sense

Received by the Editors March 6, 1972 and in revised form January 12, 1973.

in $D'(F)$ (cf. [19]). Thus for any $h \in F$ $(u(\cdot), h)' = (u'(\cdot), h)$ in D', and when $u' \in H$ this means $\int_\tau^T (u'(t), h)\varphi(t)\, dt = -\int_\tau^T (u(t), h)\varphi'(t)\, dt$ for all $\varphi \in C_0^\infty(\tau, T)$. ($C_0^\infty(\tau, T)$ is the space of C^∞ functions on (τ, T) with compact support where support φ = supp φ is the smallest closed set outside of which φ vanishes identically). We define operators $L_0 = d/dt$ and $L_1 = d/dt$ in H by the prescription of dense domains $D(L_0) = \{u \in H;\ u' \in H;\ u(\tau) = 0\}$ and $D(L_1) = \{u \in H; u' \in H\}$; it is easy to see that $u \in H$ and $u' \in H$ together imply that u can be identified with the continuous function $\int_\tau^t u'(\xi)\, d\xi + c$ so that it makes sense to speak of point values $u(\tau)$ (see [19]). An elementary distribution argument shows that L_0 and L_1 are closed. Indeed, if, for example, $u_n \to u$ with $u_n' \to v$ in H, then in $D'(F)$ $u_n' \to v$ and $u_n' \to u'$ by the continuity of differentiation which implies that $v = u'$ (cf. [19]). The preservation of initial value zero for $u_n \in D(L_0)$ follows from the representation of u_n as a continuous function given above. We will assume now that the $A(t)$ determine a closed densely defined operator A in H with $D(A) = \{u \in H; u(t) \in D(A(t))$ a.e.; $A(\cdot)u(\cdot) \in H\}$. That A be closed is no problem (cf. [15; 19; 47]) and $D(A)$ dense can be assured for example by imposing measurability requirements on the way in which the $A(t)$ vary. Thus if $A(t)$ is 1-1 onto F with $A^{-1}(t) \in L(F)$ weakly measurable and uniformly bounded, then $D(A)$ is dense ($L(F)$ denotes the space of continuous linear maps $F \to F$). To see this note that $(A^{-1}(\cdot)h, k)$ measurable imples $A^{-1}(\cdot)^*$ is weakly measurable and hence, for $u \in H$, $(A^{-1}(\cdot)u, h) = (u, A^{-1}(\cdot)^*h)$ is measurable with $A^{-1}(\cdot)u \in H$ (cf. [19], and recall that weak and strong measurability coincide for F separable). Therefore if $((A^{-1}(\cdot)u, v)) = 0$ for all $u \in H$, $((\ ,\ ))$ denoting the scalar product in H, it follows that $A^{-1}(t)^*\, v(t) = 0$ a.e. or that $v = 0$. We write now $S_0 = L_0 + A$ and $S_1 = L_1 + A$ with $D(S_0) = D(L_0) \cap D(A)$ and $D(S_1) = D(L_1) \cap D(A)$. By imposing smoothness requirements on the variation of the $A(t)$ one can assure that $D(S_0) \subset D(S_1)$ is dense; for example (cf. [19; 40]) this holds if $A^{-1}(\cdot) \in C^1(L_s(F))(L_s(F)$ is $L(F)$ with the strong operator topology where $A_\alpha \to A$ means $A_\alpha h \to Ah$ for each fixed $h \in F$). In particular there will be many realistic instances with A and S_0 densely defined and we need not go into this further in view of our intention to concentrate on (1.1). We remark that in general S_0 is not closed but it frequently is closeable with range $R(\bar{S}_0) = H$ (cf. [15; 19; 29; 48; 39a]).

Thus we can formulate the evolution problem in H, $u' + A(\cdot)u = f$, $u(\tau) = u_0$, in the form $S_1 u = f$ with $u(\tau) = u_0$, and we will give a diagrammatic sketch of this operator equation in § 2 in connection with a study of (1.1) (cf. also [**57**; **72**]). Further in § 3 we will give a diagrammatic presentation of certain more general abstract boundary value problems and recovery formulas (cf. [**5**; **13**; **14**; **15**; **18**; **27**; **39**; **58**; **71**]). Then the nature of some of these recovery formulas will be indicated in terms of the reproducing kernels of Aronszajn (cf. [**2**; **3**; **4**; **9**; **10**; **55**]) and (in § 4) in terms of Schwartz kernels (cf. [**61**; **62**; **63**; **64**]). These general formulas apply in particular to a version of (1.1), and an example appears in [**15**]. § 4 represents an improvement and clarification of the formulation of [**18**].

2. Let $H_0 = D(S_0)$ (resp. $H_1 = D(S_1)$) with the graph Hilbert structure (i.e. for $u, v \in H_0$ $((u, v)) = (u, v) + (S_0u, S_0v))$ and consider the following diagram of linear maps

(2.1)
$$\begin{tikzcd}
0 \arrow[r] & H_0 \arrow[r, "i"] \arrow[d, shift left, "S_0"] & H_1 \arrow[r, "j"] \arrow[dl, "S_1"] \arrow[dr, "\delta"] & \Gamma \arrow[r] & 0 \\
 & R \subset H \arrow[u, shift left, "G"] & & I \subset F \arrow[u, "\rho"'] &
\end{tikzcd}$$

Here we write $R = R(S_0) = R(S_1)$, $G = S_0{}^{-1}$ (defined on R), $\Gamma = \ker S_1$ where $H_1 = H_0 + \Gamma$ (topological direct sum) with j the associated projection, $H_0 = \ker \delta$, and ρ is obtained from the induced algebraic isomorphisms between I, Γ, and H_1/H_0. Thus i, j, S_0, and S_1 are continuous maps but nothing is assumed for the moment about G, δ, or ρ. We think here of I as the space of initial values, and in general one does not expect R or I to be closed. On the other hand one usually deals with situations where R and I are dense (cf. in particular [**7**; **51**] concerning I). Let us now index all spaces and maps in (2.1) with a parameter τ so that (2.1) implies

(2.2) $$u = \rho_\tau\delta_\tau u + G_\tau S_1{}^\tau u; \; f = S_1{}^\tau G_\tau f = S_0{}^\tau G_\tau f$$

for $u \in H_1$ and $f \in R_\tau$. The first equation is an abstract recovery formula corresponding to (1.1) with $\delta_\tau u = u(\tau)$, $S_1{}^\tau u = f$, G_τ an integral operator with kernel $G(t, s)$, and $\rho_\tau = G(t, \tau)$. We remark that a priori there is only the connection (2.2) between ρ_τ and G_τ, but the formula (1.1) will in fact follow from (2.2) under certain weak hypotheses (cf. theorem 2.3).

Definition 2.1. An operator valued function $G(t, s) \in L(F)$, $\tau \leqq s \leqq t \leqq T$, will be called a propagator if (1) $G(s, s) = I$ and (2) $G(t, s)G(s, r) = G(t, r)$ for $r \leqq s \leqq t$. If in addition (3) $(t, s) \to G(t, s) \in C^0(L_s(F))$ for $s \leqq t$, then G will be called a strongly continuous propagator.

Now, as indicated in § 1, $D(S_1) \subset D(L_1) \subset C^0(F)$, so that for $h \in I_\tau$, $\rho_\tau(h) \in \Gamma \subset H_1 = D(S_1)$ has point values $\rho_\tau(h)(t) = P(t, \tau)h$ where $P(t, \tau)$ is obviously then a linear map from R_τ to F. For $r \leqq s$ let $\lambda(s, r)$: $H_1{}^r \to H_1{}^s$ and $\beta(s, r) : R_r \to R_s$ be restriction maps so that $S_1{}^s\lambda(s, r) = \beta(s, r)\ S_1{}^r$ (more general maps λ and β enjoying this property might be envisioned, but it would be somewhat artificial to work at that level). Then from (2.2) a routine calculation (see [16]) yields

$$\rho_s\delta_s\lambda(s, r)\rho_r\delta_r = \lambda(s, r)\rho_r\delta_r \tag{2.3}$$

Setting now $\delta_s u = u(s)$, which makes sense, we obtain on I_r

$$P(t, s)P(s, r) = P(t, r);\ P(r, r) = I \tag{2.4}$$

PROPOSITION 2.2. *Given $\delta_s u = u(s)$, (2.2) implies that P is a propagator on its domain of definition.*

We assume now that the linear operator G_τ has a kernel $G(t, s) \in L(F)$ with $s \to G(t, s) \in C^0(L_s(F))$ for $t \geqq s$ so that for fixed t

$$(G_\tau f)(t) = \int_\tau^t G(t, s)f(s)\, ds. \tag{2.5}$$

To check that everything makes sense we note first that since $[\tau, t]$ is compact the set $Q = \{G(t, s) : s \in [\tau, t]\}$ is compact in $L_s(F)$ and hence bounded there. By the Banach-Steinhaus theorem (see [19]), Q is then norm bounded. Since $f \in L^2 \subset L^1$ is measurable on $[\tau, t]$ (with values in F), we have $(G(t, s)f(s), h) = (f(s), G^*(t, s)h)$ measurable for $h \in F$ (since $G(t, \cdot)k \in C^0(F)$ for $k \in F$ implies $G^*(t, \cdot)h$ is weakly continuous and hence strongly measurable); this implies that $s \to G(t, s)f(s)$ is weakly and hence strongly measurable. Hence the integral in (2.5) makes sense. Finally we note as above that $R(G_\tau) = D(S_0{}^\tau) \subset C^0(F)$, so point values make sense. Now one expects that $P(t, s) = G(t, s)$, but normally one has various stronger continuity or differentiability properties of P and G available to prove this. In fact, however, it is true under much weaker hypotheses, and $G(t, s)$ will be a propagator essentially by virtue of the recovery formula (2.2) which now takes the form

$$u(t) = P(t, \tau)u(\tau) + \int_\tau^t G(t, s)f(s)\, ds. \tag{2.6}$$

Now we say $u \in C^1(F) \cap C^0(D(A))$ if $u \in C^1(F)$ with $A(\cdot)u(\cdot) \in C^0(F)$. The following theorem was proved in [17] where the explicitly used assumption of uniformity (see proposition 3) was somehow omitted from the statement of theorem 1.

THEOREM 2.3. *Let (2.6) represent the unique solution of* $S_1{}^{\tau}u = f$ *with* $u(\tau) \in I_\tau$ *and* $f \in R_\tau$ *given. Assume* $s \to G(t, s) \in C^0(L_s(F))$ *for* $t \geqq s$, $C^1(F) \cap C^0(D(A))$ *has a dense set of initial values* $\hat{I}_s \subset I_s$ *on any interval* $[s, T]$ *for* $s \geqq \tau$, *and* $P(t, s)$ *is uniformly extendable by continuity from* I_s *to* F *for* $s \in [\tau, T]$ *fixed. Then* $P(t, s) = G(t, s)$ *on* I_s, *and* $G(t, s)$ *is a strongly continuous propagator.*

REMARK 2.4. The condition on $C^1(F) \cap C^0(D(A))$ is perfectly natural and we refer to [47] for further information on this. The results of [16] yield a similar conclusion under a hypothesis that $D_0 = \cap D(A(t))$ belong to I_s for all s and be dense in F (instead of the hypothesis about $\hat{I}_s$). A stronger continuity assumption on $G(\cdot, \cdot)$ is made in [16] but this could be weakened to that of [17].

REMARK 2.5. By $P(t, s)$ being uniformly extendable we mean that $P(t, s)$ extends by continuity from I_s to a (unique) $\bar{P}(t, s) \in L(F)$ for each $t \in [s, T]$ while $|\bar{P}(t, s)h| \leqq c(h)$ for $h \in F$. Consequently the set $Q = \{\bar{P}(t, s)\}$ for $t \in [s, T]$ is weakly bounded, and by the Banach-Steinhaus theorem Q is norm bounded and equicontinuous; this is used in proving theorem 2.3 (see below). As a natural example where this occurs we consider $u' + Au = 0$, take scalar products with u and real parts, and integrate to obtain

$$(2.7) \qquad \tfrac{1}{2}(|u(t)|^2 - |u(\tau)|^2) + \mathrm{Re} \int_\tau^t (A(\xi)u(\xi), u(\xi))\, d\xi = 0.$$

The standard evolution problems involve situations where $\mathrm{Re}(A(\xi)u(\xi), u(\xi)) \geqq -\lambda|u(\xi)|^2$, and using the Gronwall inequality (2.7) then yields (cf. [19])

$$(2.8) \qquad |u(t)|^2 = |P(t, \tau)u(\tau)|^2 \leqq e^{2\lambda(t-\tau)}|u(\tau)|^2.$$

Consequently $|P(t, \tau)h| \leqq c|h|$ for $h \in I_\tau$ (since $T < \infty$) and $\|P(t, \tau)\| \leqq c$; the uniform extendability is immediate.

We will indicate briefly the method of proving theorem 2.3 used in [17] by stating some of the ingredients. First one knows $t \to P(t, s)h$ is continuous for $h \in I_s$ since $H_1{}^s \subset C^0(F)$. However on the set $Q = \{\bar{P}(t, s)\}$, $t \in [s, T]$, of remark 2.5 the topology of simple convergence (i.e., the strong operator topology) is equivalent to the topology of simple convergence on a dense set such as $I_s \subset F$ (cf. [19]). Hence $t \to \bar{P}(t, s)$ is strongly continuous. If one can show that $s \to \bar{P}(t, s)$ is also strongly continuous then using proposition 2.2 (which extends by continuity to $\bar{P}(t, s)$) it follows that $(t, s) \to \bar{P}(t, s) \in C^0(L_s(F))$. This is achieved by showing that $P(t, s) = G(t, s)$ on $\hat{I}_s$ so that $G(t, s) = \bar{P}(t, s)$ is then a strongly continuous propagator. The argument of

[17] proving that $P(t, s) = G(t, s)$ on $\hat{I}_s$ is somewhat lengthy and we omit the details here (essentially the same argument appears in [16] under slightly different hypotheses).

REMARK 2.6. The "meaning" of the formula $\partial G(t, s)/\partial s = G(t, s)A(s)$ (suitably interpreted) in the integrated form $\int_\tau^t G_s(t, s)\, ds = I - G(t, \tau) = \int_\tau^t G(t, s)A(s)\, ds$ can be abstracted from (2.1) as follows. Assume for simplicity that $u_0 \in I_\tau$ belongs to $D_0 = \bigcap D(A(t))$. Then $u_0 \in H_1$, with $L_1 u_0 = 0$, and we let $u \in H_1$ be any element satisfying $\delta_\tau u = u(\tau) = u_0$. Then $u - u_0 \in H_0$, and hence $u - u_0 = G_\tau S_1{}^\tau(u - u_0) = G_\tau S_1{}^\tau u - G_\tau S_1{}^\tau u_0$. But $u = \rho_\tau u_0 + G_\tau S_1{}^\tau u$ by (2.2) so we obtain $\rho_\tau u_0 = u - G_\tau S_1{}^\tau u = u_0 - G_\tau S_1{}^\tau u_0$. Since $S_1{}^\tau u_0 = Au_0$ and $P(t, s) = G(t, s)$ on I_s we have $G(t, \tau)u_0 = u_0 - \int_\tau^t G(t, s)A(s)u_0\, ds$.

3. We take H to be a separable Hilbert space with S_0 and $S_0{}'$ closed densely defined operators in H. The pair $(S_0, S_0{}')$ is formally adjoint if $(S_0 x, y) = (x, S_0{}'y)$ for $x \in D(S_0)$ and $y \in D(S_0{}')$. Define $S_1 = (S_0{}')^* \supset S_0$ and set $H_0 = D(S_0)$ with $H_1 = D(S_1)$ (in graph Hilbert structure). We recall that $x \in D((S_0{}')^*)$ if the map $y \to (x, S_0{}'y) : D(S_0) \to C$ is continuous in the topology of H so that $(x, S_0{}'y) = ((S_0{}')^* x, y)$. One writes $H_1 = H_0 + \Gamma$ where Γ denotes "abstract" boundary conditions (cf. [5; 13; 27; 39; 58; 71]) and $+$ means topological direct sum. We assume S_0 is 1-1 with closed range and that $R(S_1) = H$. Operators $\hat{S}$ with $S_0 \subset \hat{S} \subset S_1$ are characterized by linear subspaces $\hat{\Gamma}$ of Γ such that $\hat{H} = D(\hat{S}) = \{u_1 \in H_1;\ ju_1 \in \hat{\Gamma}\}$ where $j : H_1 \to \Gamma$ is the projection determined by H_1 and Γ (thus $\hat{H} = H_0 + \hat{\Gamma}$). The following diagram gives a more refined breakdown of the situation where $\Gamma = \ker S_1 + \tilde{\Gamma}$

$$
\begin{array}{c}
H_0 + \ker S_1 + \tilde{\Gamma} \xrightarrow{j} \{0\} + \Gamma_1 + \tilde{\Gamma} \\
\Big\downarrow S_1 \\
R(S_0) + \{0\} + \tilde{H}
\end{array}
\tag{3.1}
$$

Here $\tilde{\Gamma}$ (identified with $j\tilde{\Gamma}$) is any topological supplement in H_1 of $H_0 + \ker S_1$ and, with $\ker S_1$, determines j as above, $\Gamma_1 = j(\ker S_1) = \ker S_1$, and $\tilde{H} = S_1\tilde{\Gamma}$ with $H = R(S_0) + \tilde{H}$. One checks that everything makes sense and that the Green's operator $G : (ju_1, S_1 u_1) \to u_1 : \Gamma \times H \to H_1$ is well defined; it is easy to verify that G is also continuous.

Now decompose G as follows. If $ju_1 = 0$ (i.e., $u_1 \in H_0$), then $G(0, S_1 u_1)$ defines a continuous map $G_2 : R(S_0) \to H_1$. By [13] there exists a solvable realization operator $\tilde{S}$ for (S_0, S_0') (i.e., $\tilde{S}$ is 1-1 onto H with $S_0 \subset \tilde{S} \subset S_1$), and on $R(S_0)$ one has $G_2 = S_0^{-1} = \tilde{S}^{-1}$ which means that G_2 extends (as $\tilde{S}^{-1}$) to a continuous map $G_2 : H \to H_0 + \tilde{\Gamma}$. Similarly if $u_1 \in \ker S_1$ then $G(ju_1, 0)$ determines a continuous map $G_1 : \Gamma_1 \to H_1$ (the identity, which extends, as the identity) to a continuous map $G_1 : \Gamma_1 + \tilde{\Gamma} \to H_1$. Then for $u_1 \in H_0 + \ker S_1$ one has $u_1 = G_1(ju_1) + G_2(S_1u_1)$ whereas for $u_1 \in \tilde{\Gamma}$, $u_1 = G_1(ju_1) = G_2(S_1u_1)$. There are now two recovery formulas for $u_1 \in H_1$. Thus in the first place

$$u_1 = G_2(\rho S_1 u_1) + G_1(ju_1) \tag{3.2}$$

where $\rho : H \to R(S_0)$ is the projection determined by $R(S_0)$ and $\tilde{H}$. On the other hand, if $\hat{\rho} : \Gamma \to \Gamma_1$ is the projection determined by Γ_1 and $\tilde{\Gamma}$ in Γ, then

$$u_1 = G_2(S_1u_1) + G_1(\hat{\rho} j u_1). \tag{3.3}$$

Evidently (3.3) is a possible interpretation of (1.1) and one can obtain information from both (3.2) and (3.3) (cf. [**14**; **15**]); we refer to [**15**] for concrete examples from evolution equations.

We recall the notion of kernel following Aronszajn for a linear map $T : E \to F$ (here E and F are separable Hilbert spaces of equivalence classes of measurable functions on a suitable space X). Thus T has a kernel $T(y, \cdot)$ if (1) for all $y \in X$, $T(y, \cdot) \in E$ (2) the map $y \to T(y, \cdot) :$ $X \to E$ is measurable, and (3) for all $e \in D(T)$, $(Te)(y) = (e, T(y, \cdot))_E$ almost everywhere. Suppose that G_1 and G_2 have kernels g_1 and $g_2 (G_1 : \Gamma \to H_1$ and $G_2 : H \to H_1)$. Then, writing ${}^tT$ (resp. T^*) for the adjoints of continuous (resp. unbounded) maps, (3.2) and (3.3) become (suppressing the y variable)

$$u_1 = (u_1, {}^tS_1{}^t\rho g_2 + {}^tjg_1)_{H_1}, \tag{3.4}$$

$$u_1 = (u_1, {}^tS_1 g_2 + {}^tj{}^t\hat{\rho} g_1)_{H_1}. \tag{3.5}$$

THEOREM 3.1. *If G_1 and G_2 have kernels as above then H_1 has a reproducing kernel (i.e., the identity operator on H_1 has a kernel) given by either of the formulas*

$$h_1 = {}^tS_1{}^t\rho g_2 + {}^tjg_1, \tag{3.6}$$

$$h_1 = {}^tS_1 g_2 + {}^tj{}^t\hat{\rho} g_1. \tag{3.7}$$

One notes here that if h_1 is given then $g_1 = Jh_1$ is the component of h_1 in Γ in the decomposition $H_1 = \Gamma \oplus (H_1 \ominus \Gamma)$ where $\oplus$ denotes the orthogonal direct sum. Thus

THEOREM 3.2. *If H_1 has a reproducing kernel h_1, then G_2 has a kernel determined by* (3.7).

For purposes of calculations we observe that if ${}^t S_1 v = w$ then $w = S_0'(v - S_1 w)$ so that $w \in R(S_0')$ and therefore (3.7) can be written

$$g_2 = [(S_0')^{-1} + S_1](h_1 - {}^t j\, {}^t \hat{\rho} g_1). \tag{3.8}$$

Such formulas can be used in concrete examples as in [**15**] (for (3.6)). If one is given h_1 and uses (3.6) then the element φ determined by ${}^t S_1 \varphi = h_1 - {}^t j g_1$ lies in $H \ominus \tilde{H} = R({}^t\rho)$, but since $\ker {}^t\rho = H \ominus R(S_0)$ this only determines g_2 up to an arbitrary element of $H \ominus R(S_0)$. Writing $\varphi = {}^t\rho g_2$ in this case for some such nonunique g_2, one gets $(\rho S_1 u_1, g_2) = (S_1 u_1, \varphi) = (u_1, h_1 - {}^t j g_1) = G_2(\rho S_1 u_1)$ and we could of course use here the uniquely specified component $\tilde{g}_2 = {}^t\rho^{-1}\varphi$ of g_2 in $R(S_0)$ to produce a kernel for $G_2 : R(S_0) \to H_1$. There is no need to insist on $H = R(S_0) \oplus \tilde{H}$ as in [**14**].

THEOREM 3.3. *If H_1 has a reproducing kernel h_1, then one can specify a kernel $\tilde{g}_2 = {}^t\rho^{-1}\, {}^t S_1^{-1}(h_1 - {}^t j g_1)$ for $G_2 : R(S_0) \to H_1$.*

We remark that $(\tilde{S}, S_0')$, $(\tilde{S}, \tilde{S}^*)$, and $(S_0, \tilde{S}^*)$ are formally adjoint pairs and various formulas become more manageable for calculations when these pairs are used (provided $\tilde{S}$ and/or $\tilde{S}^*$ is known). Thus, if one looks at $(\tilde{S}, S_0')$ then $\tilde{H} = \{0\} = \tilde{\Gamma}$, so ρ = identity = $\hat{\rho}$, and (3.6) = (3.7). If we consider $(S_0, \tilde{S}^*)$, then $S_1 = \tilde{S}$, so $\ker S_1 = \Gamma_1 = \{0\}$ and we have $\rho = 0 = {}^t\hat{\rho}$; in this case $H_1 = H_0 \oplus \tilde{\Gamma}$ can be envisioned (note Γ_1 is not orthogonal to H_0). Finally using $(\tilde{S}, \tilde{S}^*)$ with $H_0 = H_1$, we obtain from (3.8) (cf. also theorem 4.1)

$$g_2 = [(\tilde{S}^*)^{-1} + \tilde{S}]\, h_1. \tag{3.9}$$

This formula was used in [**15**] to compute h_1 and g_2 in an evolution problem (the discussion in [**15**] indicates situations and contexts where the procedure is justified, and we refer also to [**19**]). Various diagrams are drawn in [**14**; **15**] to show how the maps j, J, etc. behave and some criteria are established for an element of H_1 to belong to various subspaces (note for example that $g_1 = Jh_1$ means $h_1 - Jh_1 \in H_1 \ominus \Gamma$ while ${}^t jJh_1 = {}^t jg_1 = {}^t jh_1 \in H_1 \ominus H_0$ with $h_1 - {}^t jh_1 \in H_1 \ominus \Gamma$). Thus, taking $H = R(S_0) \oplus \tilde{H}$ for convenience, one proves

THEOREM 3.4. *Assume $H = R(S_0) \oplus \tilde{H}$ (with $\tilde{\Gamma} = \tilde{S}^{-1}\tilde{H}$). Then $u \in H_1 \ominus \Gamma$ if and only if $u \in R(S_0')$ with $[(S_0')^{-1} + S_1]\, u \in R(S_0)$. Similarly $u \in H_1 \ominus H_0$ if and only if $(1 + S_0^* S_1)u = 0$.*

4. We recall first a few ideas and facts about Schwartz kernels, and in order to simplify the notation we will work with the equivalent notion of antikernel (see [**19**; **61**; **62**; **63**; **64**]). The idea is to characterize Hilbert subspaces of very general topological vector spaces in terms of certain "intrinsic" operators (either kernels or anitkernels). By way of application this idea was used by Schwartz to describe elementary particles in quantum mechanics (cf. [**62**; **63**]).

We restrict ourselves here to Hilbert subspaces of Hilbert spaces, and thus some of the constructions will be seen to be related to techniques and ideas in interpolation theory, for example, but we will not attempt to be biographical in this respect. Thus let $V \subset H$ be separable Hilbert spaces and let V' denote the dual of V (i.e., V' is the space of continuous linear functionals on V). Note that one has a canonical antiisomorphism $\theta : V' \to V$ determined by $\langle w', w\rangle = ((w, \theta w'))$ (similarly $\hat{\theta} : H' \to H$ is defined). The Schwartz operator (or antikernel) L of V relative to H is the composition $L = i\,\theta i^* : H' \to H$ where $i : V \to H$ is the injection. The operator L is characterized by the property $\langle h', w\rangle = ((w, Lh'))$ for $h' \in H'$ and $w \in V$ where $\langle\ ,\ \rangle$ denotes the $H - H'$ pairing (we write $((\ ,\ ))$ for the scalar product in V and $(\ ,\ )$ for that in H); note that L is antilinear (i.e., conjugate linear). Define now $T = \hat{\theta}L^{-1} : R(L) \subset H \to H$; then T is an unbounded self adjoint positive definite operator mapping onto H, and V is characterized as $D(T^{1/2})$ (cf. [**18**; 19; **64**]). We write $T^{1/2} = S$ and call S the standard operator for V in H((note that $(Sx, Sy) = ((x, y)))$. Let us mention that the use of such standard operators in describing the variation of domains $V(t) \subset H$ has been systematically exploited in the study of variable domain abstract evolution equations and has led to general existence-uniqueness results in coercive and noncoercive situations (see [17; **19**; **20**; **22**; 23; 24; 25; **26**; 53; 54; 54a; **68**]). In [18] there are also some preliminary applications of standard operators to homotopy properties of operators.

We will now indicate another way to look at some of the results of § 3 (cf. [18]). Let us assume we are working with Hilbert spaces of distributions where one has a natural conjugation $(\langle \bar{T}, \varphi\rangle = \overline{\langle T, \bar{\varphi}\rangle}$ for $T \in D'$ and $\varphi \in D$ where $\bar{\varphi}$ denotes ordinary complex conjugation). Thus we assume $\bar{h} \in H$ when $h \in H$ without loss of generality since with more elaborate notation one can develop all this in terms of an arbitrary conjugation (or in terms of kernels and antispaces). Let antilinear action of $h \in H$ on $h' \in H'$ be defined by linear action of h on $\bar{h}'$ (cf. [61]). Let $\{v_i\}$ be an orthonormal basis in V and note that the antilinear map $L : H' \to H$ can be written

$$L = \sum v_i \otimes \bar{v}_i. \tag{4.1}$$

Indeed, for $h' \in H$ one has $Lh' \in V$ so that

$$(4.2) \qquad Lh' = \sum ((Lh', v_i))v_i = \sum \langle \overline{h', v_i} \rangle v_i,$$

which is the same as (4.1). We note also that $\theta = \sum v_i \otimes \bar{v}_i$ as an antilinear map $V' \to V$ since then

$$(4.3) \qquad \begin{aligned} ((w, \theta w')) &= ((w, \textstyle\sum v_i \langle \bar{v}_i, \bar{w}' \rangle)) = \\ \textstyle\sum \langle v_i, w' \rangle ((w, v_i)) &= \langle w', \textstyle\sum ((w, v_i))v_i \rangle \\ &= \langle w', w \rangle. \end{aligned}$$

This illustrates the fact that L and θ are really the same thing but referred to different spaces. This expression $\sum v_i \otimes \bar{v}_i$ is of course also the classical form of a reproducing kernel in V (cf. [2; 55]); this is usually expressed by $((v, \sum v_i \otimes \bar{v}_i)) = \sum ((v, v_i))v_i = v$ (we could also act on the second terms since $\bar{v}_i$ is also an orthonormal basis but it is convenient here to retain the classical action). Thus the second terms involve the y variable of § 3 which was there in the first position. We have shown that L or θ corresponds to h_1 of § 3 when thought of as a kernel and one must scrupulously distinguish L as a kernel or as an operator.

Similarly $\hat{\theta} = \sum h_i \otimes \bar{h}_i$ when $\{h_i\}$ is an orthonormal basis in H. Now we take $S_0 = \tilde{S} = S_1$ for convenience where $V = D(\tilde{S}) = H_1$ with graph Hilbert structure. To find the kernel associated with $\tilde{S}^{-1}$ in H consider

$$(4.4) \qquad \begin{aligned} &(\tilde{S}v, \textstyle\sum (\tilde{S}^*)^{-1} h_i \otimes \bar{h}_i) = \\ &\qquad \textstyle\sum (\tilde{S}v, (\tilde{S}^*)^{-1} h_i) h_i = \sum (v, h_i) h_i = v. \end{aligned}$$

Thus $\tilde{S}^{-1}$ has kernel $K = (\tilde{S}^*)^{-1}\hat{\theta}$ with action on the first variable where we regard $\hat{\theta}$ here as a reproducing kernel in H (this is in fact a special case of a formula in [2; 55]). Now one shows easily (cf. [18; 19]) that for $V = D(\tilde{S}) = H_1$ as indicated, $T = (1 + \tilde{S}^*\tilde{S})$. We have an operator equation $L = T^{-1}\hat{\theta}$, so T^{-1} acts on the first variable since $\hat{\theta}$ as an operator acts as in (4.3) on the second variable. Therefore considered as a kernel in H, $T^{-1}\hat{\theta}$ is the kernel associated with T^{-1}. Hence thinking of L as a kernel we have

$$(4.5) \qquad K = [(\tilde{S}^*)^{-1} + \tilde{S}]\, L.$$

This yields a somewhat expanded version of the formula (3.9) since the kernels g_2 and h_1 will only be present for certain spaces H and H_1.

THEOREM 4.1. *If L is thought of as a kernel and K is the kernel for $\tilde{S}^{-1}$, then* (4.5) *expresses their relation.*

We remark that this improves the presentation of [18] where some confusion arises in the role of L as operator or kernel.

References

1. S. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space,* Comm. Pure Appl. Math. **16** (1963), 121–239.

2. N. Aronszajn, *Theory of reproducing kernels,* Trans. Amer. Math. Soc. **68** (1950), 337–404.

3. ———, *Green's functions and reproducing kernels,* Proc. Symp. Spectral Theory and Diff. Probs., Oklahoma, 1955, pp. 355–411.

4. N. Aronszajn and K. Smith, *Characterization of positive reproducing kernels. Applications to Green's functions,* Amer. Jour. Math. **79** (1957), 611–622.

5. J. P. Aubin, *Abstract boundary value operators and their adjoints,* Rend. Sem. Mat. Padova **43** (1970), 1–33.

6. C. Baiocchi, *Regolarità e unicità della soluzione di una equazione differenziale astratta,* Redn. Sem. Mat. Padova **35** (1965), 380–417.

7. R. Beals, *On the abstract Cauchy problem,* to appear.

8. ———, *Semigroups and abstract Gevrey spaces,* to appear.

9. S. Bergman and M. Schiffer, *Kernel functions and elliptic partial differential equations in mathematical physics,* Academic Press, New York, New York, 1953.

10. S. Bezgman, *Integral Operators in the Theory of Linear Partial Differential Equations,* Springer, Berlin (1961).

11. F. Browder, *Nonlinear equations of evolution,* Annals Math. **80** (1964), 485–523.

12. ———, *Nonlinear initial value problems,* Annals Math. **82** (1965), 51–87.

13. ———, *Functional analysis and partial differential equations,* Math. Annalen **138** (1959), 55–79.

14. R. Carroll, *On the structure of the Green's operator,* Proc. Amer. Math. Soc. **15** (1964), 225–230.

15. ———, *Problems in linked operators I and II,* Math. Annalen **151** (1963), 272–282 and **160** (1965), 233–256.

16. ———, *Some remarks on the propagator equation,* Jour. London Math. Soc. **42** (1967), 737–743.

17. ———, *On the propagator equation,* Illinois Jour. Math. **11** (1967), 506–527.

18. ———, *On the structure of some abstract differential problems, I and II,* Annali di Mat. **72** (1966), 305–318 and **81** (1969), 93–110.

19. ———, *Abstract methods in partial differential equations,* Harper-Row, New York, 1969.

20. R. Carroll and J. Cooper, *Remarks on some variable domain problems in abstract evolution equations,* Math. Annalen **188** (1970), 143–164.

21. R. Carroll and J. Neuwirth, *Some uniqueness theorems for differential equations with operator coefficients,* Trans. Amer. Math. Soc. **110** (1964), 459–472.

22. R. Carroll, *Some variable domain problems in abstract evolution equations,* Proc. Conf. Eqs. Evolution and Nonlinear Semigroups, Univ. Kentucky, 1969, 11–24.

23. R. Carroll and E. State, *Existence theorems for some weak abstract variable domain hyperbolic problems,* Canad. Jour. Math. **23** (1971), 611–626.

24. R. Carroll and T. Mazumdar, *Solutions of some possibly noncoercive evolution problems with regular data,* Jour. Applicable Anal. **1** (1972), 381–395.

25. J. Cooper, *Some aspects of abstract linear evolution equations in Hilbert space,* Thesis, Univ. Illinois, 1967.

26. ———, *Evolution equations in Banach space with variable domain,* Jour. Math. Anal. Appl. **36** (1971), 151–171.

27. H. Cordes, *On maximal first order partial differential operators,* Amer. Jour. Math. **82** (1960), 63–91.

28. Yu. Daletskij and M. Krein, *The stability of solutions of differential equations in Banach spaces,* Izd. Nauka, Moscow, 1970.

29. G. Da Prato, *Somma di generatori infinitesimali di semigruppi di contrazioni e equazioni di evoluzione in spazi di Banach,* Annali di Mat. **78** (1968), 131–158.

30. N. Dunford and I. Schwartz, *Linear operators, Parts* 1 *and* 2, Wiley-Interscience, New York, 1958 and 1963.

31. C. Foias, G. Gussi, and U. Poenaru, *L'étude de l'équation $u' = A(t)u$ pour certaines classes d'opérateurs non bornés dans l'espace de Hilbert,* Trans. Amer. Math. Soc. **86** (1957), 335–347.

32. A. Friedman, *Generalized functions and partial differential equations,* Prentice-Hall, Englewood Cliffs, N.J., 1963.

33. I. Gelfand and G. Šilov, *Generalized functions, Vol.* 3, Gos. Izd. Fiz-Mat. Lit., Moscow, 1958.

34. J. Goldstein, *Abstract evolution equations,* Trans. Amer. Math. Soc. **141** (1969), 159–185.

35. P. Grisvard, *Equations différentielles abstraites,* Ann. Scient. Ecole Norm. Sup. **2** (1969), 311–395.

36. M. Hackman, *The abstract time dependent Cauchy problem,* Trans. Amer. Math. Soc. **133** (1968), 1–50.

37. E. Hille and R. Phillips, *Functional analysis and semigroups,* Amer. Math. Soc. Colloq. Pub. Vol. **31**, 1957.

38. E. Hille, *The abstract Cauchy problem,* Jour. d'Anal. Math., Jerusalem 3 (1953–54), 81–196.

39. L. Hormander, *On the theory of general partial differential operators,* Acta Math. **94** (1955), 161–248.

39a. T. Ichinose, *Operators on tensor products of Banach spaces,* to appear.

40. T. Kato, *Nonlinear evolution equations in Banach spaces,* Proc. Symp. Appl. Math., AMS, **17** (1965), 50–67.

41. T. Kato and H. Tanabe, *On the abstract evolution equation,* Osaka Math. Jour. **14** (1962), 107–133.

42. T. Kato, *Integration of the equation of evolution in a Banach space,* Jour. Math. Soc. Japan **5** (1953), 208–234.

42a. ———, *Linear evolution equations of "hyperbolic" type,* Jour. Fac. Sci. Univ. Tokyo **17** (1970), 241–258.

43. J. Kisynski, *Sur les opérateurs de Green des problèmes de Cauchy abstraits,* Studia Math. **23** (1964), 285–328.

44. H. Komatsu, *Semigroups of operators in locally convex spaces,* Jour. Math. Soc. Japan **16** (1964), 230–262.

45. T. Komura, *Semigroups of operators in locally convex spaces,* Jour. Funct. Anal. **2** (1968), 258–296.

45a. ———, *On linear evolution operators in reflexive Banach spaces,* Jour. Fac. Sci. Univ. Tokyo **17** (1970), 529–542.

46. S. Krein, *Linear differential equations in Banach spaces,* Izd. Nauka, Moscow, 1967.

47. J. Lions, *Equations différentielles opérationnelles,* Springer, Berlin, 1961.

48. J. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications,* Vol. 1-3, Dunod, Paris, 1968 and 1970.

49. J. Lions, *Sur les semi-groupes distributions,* Port. Math. **19** (1960), 141-164.

50. Yu. Lyubič, *The classical and local Laplace transformation in an abstract Cauchy problem,* Uspekhi Mat. Nauk. **21** (1966), 3-51.

51. ———, *On conditions for density of the initial value manifold for the abstract Cauchy problem,* Doklady Akad. Nauk. SSR **155** (1964), 262-265.

52. J. Massera and J. Schaeffer, *Linear differential equations and function spaces.* Academic Press, New York, 1966.

53. T. Mazumdar, *Existence theorems for noncoercive variable domain evolution problems,* Thesis, Univ. Illinois, 1971.

54. ———, *Generalized projection theorem with application to linear noncoercive equations and some nonlinear situations,* Jour. Math. Anal. Appl. 43 (1973), 72-100.

54a. ———, *Generalized projection theorem and weak noncoercive evolution problems in Hilbert space,* Jour. Math. Anal. Appl. **46** (1974), 143-168.

55. H. Meschkowski, *Hilbertsche Räume mit Kernfunktion,* Springer, Berlin, 1962.

56. V. Millionščikov, *On the theory of differential equations in locally convex spaces,* Mat. Sbornik **57** (1961), 385-406.

57. D. Milman, *The formulation and methods of solving a general boundary value problem of operator theory from the aspect of functional analysis.* Problems of Cauchy and Dirichlet type. Dokl. Akad. Nauk SSR **161** (1965), 1276-1281.

58. R. Phillips, *Dissipative operators and hyperbolic systems of parital differential equations,* Trans. Amer. Math. Soc. **90** (1959), 193-254.

59. E. Poulsen, *Evolutions-gleichungen in Banach-Räumen,* Math. Zeit. **90** (1965), 286-309.

60. G. Pozzi, *Problemi di Cauchy e problemi ai limiti per equazioni di evoluzione del tipo di Schroedinger lineari e non lineari,* Annali di Mat. **78** (1968), 197-258 and **81** (1969), 205-248.

61. L. Schwartz, *Sousespaces Hilbertiens et antinoyaux associés,* Sém. Bourbaki, 1961-62, exposé 238, pp. 1-18.

62. ———, *Matematica y fisica quantica,* Sem. Univ. Buenos Aires, 1958.

63. ———, *Applications of distributions to the study of elementary particles in relativistic quantum mechanics,* Lectures, Univ. Calif. Berkeley, 1961.

64. ———, *Sous espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés,* Jour. Anal. Math., Jerusalem **13** (1964), 115-256.

65. ———, *Les équations d'évolution liées au produit de composition,* Annales Inst. Fourier **2** (1950), 19-49.

66. R. Showalter, *The Sobolev equation, I and II,* Jour. Applicable Analysis, to appear.

67. P. Sobolevskij, *On equations of parabolic type in a Banach space,* Trudy Mosk. Mat. Obšč. **10** (1961), 297-350.

68. E. State, *Weak hyperbolic problems with variable domains,* Thesis, Univ. Illinois, 1971.

69. H. Tanabe, *On the equations of evolution in a Banach space,* Osaka Math. Jour. **11** (1959), 121-145; **12** (1960), 145-166 and 363-376.

70. F. Treves, *Ovcyannikov theorem and hyperdifferential operators,* Notas de Mat., 46, Inst. Mat. Pura Applic. Rio de Janeiro, 1968.

71. M. Višik, *On general boundary value problems for elliptic differential equations,* Trudy Mosk. Mat. Obšč. 1 (1952), 187–246.

72. O. Wyler, *Green's operators,* Annali di Mat. **66** (1964), 251–263.

73. K. Yosida, *Functional analysis,* Springer, Berlin, 1965.

74. S. Zaidman, *Equations différentielles abstraites,* Sem. Mat. Sup. Univ. Montréal 1965, Presses de l'Univ. Montréal, 1966.

University of Illinois, Urbana, Illinois 61801