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CONTINUITY OF STABILITY GROUPS AND CONJUGATION

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ABSTRACT. Given a transformation group, we consider the question of when the stability groups vary continuously. There exist two notions of continuously varying stability groups which are shown to be equivalent. It is also shown that a direct product of transformation groups will have this property if and only if each factor does.

We then consider certain classes of transformation groups where the action involves the conjugation operation of the group. Specifically, we show that the natural action of a group on any of its (locally compact) coset spaces has continuously varying stability groups. On the other hand, the conjugation operation of a group on itself or on its space of closed subgroups seldom has this property. This is exhibited by different types of examples.

Introduction. Suppose $G \times X \rightarrow X$ is a transformation group where G (resp. X) is a second countable locally compact Hausdorff group (resp. space). Let the group action be denoted by $(g, x) \rightarrow gx$. Then for each x in X the stability group $S(x)$ at x is the closed subgroup of G consisting of those g for which $gx = x$. Thus, if $\mathcal{K}(G)$ denotes the set of closed subgroups of G, we have a mapping $S : X \rightarrow \mathcal{K}(G)$ given by $x \rightarrow S(x)$. In [4, § 2] J. Glimm defined the stability groups to be continuous at x in X if for every sequence $\{x_n\}$ in X converging to x and for each g in $S(x)$, there is a sequence $\{g_n\}$ such that g_n belongs to $S(x_n)$ and $g_n \rightarrow g$. Shortly thereafter, J. M. G. Fell introduced a topology [3] in $\mathcal{K}(G)$ making it a second countable compact Hausdorff space (see also [2]). Thus, there exist two possible meanings for the continuity of the mapping S. This is not so; in § 1 we show these are the same. Although the mapping S is Borel measurable in general [1, p. 69], it need not be continuous [4]. Thus, we may ask when S is continuous. There is another reason for asking this question which is related to induced representations.

Let T be a unitary representation of G and P a system of imprimitivity for T based on X [5, p. 889]. Such a pair (T, P) is called a

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representation of (G, X). In Theorem 2.1 of [5], Glimm was able to determine the topology of the space of equivalence classes of irreducible induced representations of (G, X) under the assumption that the stability groups of the given action vary continuously. It is this result which led the first author to consider the continuity question for S in the particular case where $X = \mathcal{K}(G)$ and the action is conjugation. In this case, the stability groups are the normalizers and it was originally expected that Glimm's theorem would be useful in handling some difficulties encountered in the study of representations induced from non-normal subgroups. Unfortunately, this is not so (see Theorem 5).

For convenience we will say that a transformation group (G, X) has property (S) if its stability group mapping S is continuous. Our main concern here is to investigate the extent to which this property is true. Although this paper contains new results, it is basically expository in nature. We consider property (S) for certain special cases and its behavior under the formation of products and extensions, particularly in the context of conjugation. In § 1 we show that a direct product of transformation groups has property (S) if and only if each factor does. In § 2 we consider three types of transformation group involving conjugation. First, if H is a closed subgroup of G, then we have the canonical (transitive) action $G \times G/H \to G/H$. We show that $(G, G/H)$ has property (S). Second, if we consider G by itself, we can construct two canonical transformation groups based on conjugation. Letting $X = G$ (resp. $X = \mathcal{K}(G)$), we have the action $G \times X \to X$ defined by $(g, x) \to gxg^{-1}$ (resp. $(g, K) \to gKg^{-1}$ [1, p. 68]). Here, the stability group $S(x)$ (resp. $S(K)$) is simply the centralizer $C(x)$ of x in G (resp. normalizer $N(K)$ of K in G). In this way, we obtain the centralizer mapping $C : G \to \mathcal{K}(G)$ (resp. normalizer mapping $N : \mathcal{K}(G) \to \mathcal{K}(G)$) of G. We call G a C-group (resp. N-group) if the mapping C (resp. N) is continuous. In § 2 we show that very few groups have these properties. Specifically, we exhibit a compact group which is not a C-group, a C-group extension of a C-group which is not a C-group, discrete and compact groups which are not N-groups and a finite product of N-groups which is not an N-group.

In what follows, all spaces will be second countable and locally compact Hausdorff. The first assumption is made solely for the sake of convenience and we know of no reason to doubt that it can be omitted. All group identities will be denoted by 1.

1. **Continuity of Stability Groups.** We first describe the topology of $\mathcal{K}(G)$ and establish two useful lemmas. Let Q be a compact subset of G and $\mathcal{V}$ a finite family of non-empty open subsets of G. Then the topology of $\mathcal{K}(G)$ is generated by sets of the form

$$\mathcal{U}(Q, \mathcal{V}) = \{K \in \mathcal{K}(G) : K \cap Q = \emptyset, K \cap V \neq \emptyset, V \in \mathcal{V}\}.$$

LEMMA 1. *If $\{K_n\}$ is a sequence in $\mathcal{K}(G)$, $K \in \mathcal{K}(G)$ and $K_n \to K$, then K is the set of all g in G for which there exists a subsequence $\{K_m\}$ of $\{K_n\}$ and a corresponding sequence $\{g_m\}$ such that g_m is in K_m and $g_m \to g$.*

PROOF. This follows directly from the results of [2].

LEMMA 2. *Suppose $\{x_n\}$ is a sequence in X converging to x. Then there exists K in $\mathcal{K}(G)$ and a subsequence $\{x_m\}$ of $\{x_n\}$ such that $S(x_m) \to K$ in $\mathcal{K}(G)$ and $K \subseteq S(x)$.*

PROOF. This is a consequence of the compactness of $\mathcal{K}(G)$ and Lemma 1.

Now we show that Glimm-continuity and topological continuity are the same for the stability groups.

THEOREM 1. *The mapping S is Glimm-continuous at x if and only if it is topologically continuous at x.*

PROOF. It suffices to prove the forward implication since the reverse implication follows directly from Lemma 1 above and Lemma 4 of [4]. Suppose S is Glimm-continuous at x. Let $\{x_n\}$ be a sequence in X converging to x. A basic open neighborhood of $S(x)$ in $\mathcal{K}(G)$ is then of the form $\mathcal{U}(Q, \mathcal{V})$ as above. Letting $\mathcal{V} = \{V_1, \cdots, V_r\}$, we know that $S(x) \cap Q = \emptyset$ and $S(x) \cap V_k \neq \emptyset$, $1 \leqq k \leqq r$. Let $g^k \in S(x) \cap V_k$. By hypothesis, for each $1 \leqq k \leqq r$, there exists a sequence $\{g_n{}^k\}$ such that $g_n{}^k \in S(x_n)$, all n, and $g_n{}^k \to g^k$. Since each V_k is an open neighborhood of g^k, there exists M such that $n \geqq M$ implies $g_n{}^k \in S(x_n) \cap V_k$, i.e., $S(x_n) \cap V_k \neq \emptyset$, $1 \leqq k \leqq r$. We now show that M may also be chosen so that $S(x_n) \cap Q = \emptyset$, for $n \geqq M$. If not, then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ for which $S(x_m) \cap Q \neq \emptyset$, all m. Let $h_m \in S(x_m) \cap Q$. By the compactness of Q, the sequence $\{h_m\}$ has a subsequence $\{h_j\}$ converging to some element h of Q. Since $h_j \in S(x_j)$, we have $h_j x_j = x_j$, all j, so that $hx = h$ (note that $x_j \to x$, $h_j x_j \to hx$ and X is Hausdorff). Thus, $h \in S(x) \cap Q$, which is a contradiction. We therefore see that for sufficiently large n, $S(x_n) \in \mathcal{U}(Q, \mathcal{V})$, i.e., $S(x_n) \to S(x)$ topologically in $\mathcal{K}(G)$.

The next theorem tells us how to construct non-trivial transformation groups having property (S).

THEOREM 2. *Let $G_\alpha \times X_\alpha \to X_\alpha$, $\alpha \in A$, be a family of transformation groups for which the product $G = \prod G_\alpha$ (resp. $X = \prod X_\alpha$) is a* *second countable locally compact group (resp. Hausdorff space). Then the mapping $G \times X \to X$, where $((g_\alpha), (x_\alpha)) \to (g_\alpha x_\alpha)$, defines a transformation group and (G, X) will have property (S) if and only if each (G_α, X_α) does.*

PROOF. The restrictions on G and X imply that A is essentially countable [**6**] and the first part of the theorem is routine. Suppose each (G_α, X_α) has property (S). Let $\{x^n\} = \{x_\alpha{}^n\}$ be a sequence in X converging to $x = (x_\alpha)$ and assume $S(x_n) \not\to S(x)$ in $\mathcal{K}(G)$. Then by Lemma 2, we may assume (passing to a subsequence) that there exists K in $\mathcal{K}(G)$ such that $K \subseteq S(x)$, $S(x_n) \to K$ and there exists a neighborhood $\mathcal{U}$ of $S(x)$ in $\mathcal{K}(G)$ disjoint from all the $S(x_n)$. Let B be the set of all $g = (g_\alpha)$ in $S(x)$ for which $g_\alpha = 1$, except for finitely many α. We now show that $B \subseteq K$.

Let $g \in B$ with $g_{\alpha_1}, \cdots, g_{\alpha_m}$ not equal to 1. Since $S(x) = \prod S_\alpha(x_\alpha)$ (where S_α is the stability group mapping for $(G_\alpha. X_\alpha)$), it follows that $g_\alpha \in S_\alpha(x_\alpha)$, all α. Also, since $x_\alpha{}^n \to x_\alpha$, for each α, it follows from our hypothesis that $S_{\alpha_j}(x^n_{\alpha_j}) \to S_{\alpha_j}(x_{\alpha_j})$, for $1 \leqq j \leqq m$. Applying Lemma 1 to g_{α_1} in $S_{\alpha_1}(x_{\alpha_1})$ and passing to a subsequence, we get a sequence $\{h^n_{\alpha_1}\}$ such that $h^n_{\alpha_1} \in S(x^n_{\alpha_1})$ and $h^n_{\alpha_1} \to g_{\alpha_1}$. Thus, successively applying Lemma 1 (and passing to a subsequence each time), we eventually obtain sequences $\{h^n_{\alpha_j}\}$ for which $h^n_{\alpha_j} \in S(x^n_{\alpha_j})$ and $h^n_{\alpha_j} \to g_{\alpha_j}$, for $1 \leqq j \leqq m$. Now let $h^n = (h_\alpha{}^n)$, where $h_\alpha{}^n = h^n_{\alpha_j}$, if $\alpha = \alpha_j$, and $h_\alpha{}^n = 1$, otherwise. Then $\{S(x^n)\}$ is a subsequence of the original one, each $h^n \in S(x^n)$ and $h^n \to g$. Thus, $B \subseteq K$ by Lemma 1.

Since B is dense in $S(x) = \prod S_\alpha(x_\alpha)$, we have $B^- = K = S(x)$. Hence, $S(x^n) \to S(x)$, i.e., $\{S(x^n)\}$ is eventually in $\mathcal{U}$, which is a contradiction.

Conversely, suppose (G, X) has property (S). Here it suffices to assume $A = \{1, 2\}$ and show (G_2, X_2) has property (S). Let z be a fixed arbitrary element of X_1. If $y \in X_2$, then $S(z, y) = S_1(z) \times S_2(y)$. Now let $y_n \to y$ in X_2 so that $(z, y_n) \to (z, y)$ in X and $S(z, y_n) \to S(z, y)$ in $\mathcal{K}(G)$ by hypothesis. Let $\mathcal{U}_2(Q, \mathcal{V})$ be a basic open neighborhood of $S_2(y)$ in $\mathcal{K}(G_2)$ with $\mathcal{V} = \{V_1, \cdots, V_r\}$. Define $W_j = G_1 \times V_j$, $1 \leqq j \leqq r$, and $\mathcal{W} = \{W_1, \cdots, W_r\}$. Also define $C = \{1\} \times Q$. Then $\mathcal{U}(C, \mathcal{W})$ is a basic open neighborhood of $S(z, y)$ in $\mathcal{K}(G)$, so that for sufficiently large n, $S(z, y_n) \in \mathcal{U}(C, \mathcal{W})$. This implies that $\{S_2(y_n)\}$ is eventually in $\mathcal{U}_2(Q, \mathcal{V})$ which completes the proof.

2. **Conjugation Transformation Groups.** Let H be a closed subgroup of G, G/H the space of left cosets and $p : G \to G/H$ the canonical projection. Then the mapping $G \times G/H \to G/H$ defined by $(y, r) \to yr$, where $yr = yxH$ for x in G such that $p(x) = r$, defines a transformation group. The stability group $S(r)$ is xHx^{-1}.

THEOREM 3. *The transformation group* $(G, G/H)$ *has property* (S).

PROOF. Let $r_n \to r$ in G/H and suppose $S(r_n) \not\to S(r)$. Then there exists a neighborhood $\mathcal{U}$ of $S(r)$ in $\mathcal{K}(G)$ and a subsequence $\{r_m\}$ of $\{r_n\}$ such that $S(r_m) \notin \mathcal{U}$, all m. Let $x \in G$ be such that $p(x) = r$. Then the openness of p implies that there exists a subsequence $\{r_k\}$ of $\{r_m\}$ and a corresponding sequence $\{x_k\}$ in G such that $x_k \to x$ and $p(x_k) = r_k$. Thus, $x_k H x_k^{-1} \to xHx^{-1}$, i.e., $S(r_k) \to S(r)$ which is a contradiction.

COROLLARY. *If the transformation group* $G \times X \to X$ *is transitive and either* X *is discrete or* G *is compact, then* (G, X) *has property* (S).

PROOF. In this case, (G, X) is equivalent to $(G, S(x))$, for each x in X.

COROLLARY 2. *If* K *is a closed subgroup of* G *and either* H *or* K *is open, then the canonical action* $K \times G/H \to G/H$ *has property* (S).

PROOF. If H is open then G/H is discrete. If K is open then S is continuous since $S(r) = xHx^{-1} \cap K$ (for $p(x) = r$) and intersection by an open subgroup is a continuous mapping of $\mathcal{K}(G)$ into itself.

Now let $G \times G \to G$ be the conjugation transformation group defined in the introduction and consider the centralizer mapping $C : G \to \mathcal{K}(G)$. It is obvious that discrete and abelian groups are C-groups as are direct products of C-groups.

THEOREM 4. *There exist groups of the following types which are not* C*-groups*:

(i) *Compact groups.*

(ii) C*-group extensions of* C*-groups.*

PROOF. (i) Let $\{b_n\}$ be a sequence of real numbers such that $0 < b_n < 1$ and $b_n \to 0$. Let $a_n = (1 - b_n^2)^{1/2}$ so that $0 < a_n < 1$, $a_n^2 + b_n^2 = 1$ and $a_n \to 1$. Define

$$x_n = \begin{pmatrix} a_n & -b_n & 0 \\ b_n & a_n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $x_n \to 1$ in $G = U(3)$. We may also verify directly that $C(x_n)$ is the proper subgroup of all matrices of the form

$$\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix},$$

where a, b and c are arbitrary complex numbers, i.e., $C(x_n)$ does not depend on n. However, $C(1) = G$, so that $\{C(x_n)\}$ does not converge to $C(1)$.

(ii) Consider the general linear group $GL(2, C)$ where C is the complex numbers. Let N be the closed abelian subgroup of all matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and denote by α the particular matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Then $\alpha \notin N$ and $\alpha^4 = 1$, so that the closed subgroup G of $GL(2, C)$ generated by $N \cup \{\alpha\}$ is a finite extension of abelian N. Now if we let

$$x_n = \begin{pmatrix} a_n & b_n \\ -b_n & a_n \end{pmatrix},$$

with a_n and b_n as in (i) above, then $x_n \to 1$, $C(x_n) = N$, all n, so $\{C(x_n)\}$ can't converge to $C(1) = G$.

REMARK. It follows from the results of [4] that there exists a dense open subset of $U(3)$ on which the mapping C is continuous. Thus, in view of the proof of Theorem 3(i), this is essentially the best possible general result.

Now let $G \times \mathcal{K}(G) \to \mathcal{K}(G)$ be the other conjugation transformation group defined in the introduction and consider the normalizer mapping $N : \mathcal{K}(G) \to \mathcal{K}(G)$. Clearly, finite and abelian groups are N-groups; however, this is true of very few others.

THEOREM 5. *There exist groups of the following types which are not N-groups.*

(i) *Compact groups.*
(ii) *Infinite discrete groups.*
(iii) *Finite direct products of N-groups.*

PROOF. (i) Let S_3 be the symmetric group on three elements and Z_3 the cyclic group of order three. Let a be the element (123) of S_3, c the element (12) of S_3 and b any generator for Z_3. Define $H = S_3 \times Z_3$. We may verify that the subgroup $\langle a \rangle = \{1, a, a^2\}$ of S_3 is normal and if $p \in S_3$, then $pap^{-1} = a$ if and only if $p \in \langle a \rangle$. Note that $c \notin \langle a \rangle$. Now let $G = \prod G_i$, where $G_i = H$, $i = 1, 2, \cdots$. For each $n = 1, 2, \cdots$, let $g^n = (g_i^n)$ be the element of G defined by $g_i^n = (a, 1)$, if $1 \leqq i \leqq n$, and $g_i^n = (1, b)$, if $n < i$. Denote the (closed) subgroup $\{1, g^n, (g^n)^2\}$ of G by K_n. Similarly, let $g = (g_i)$ be defined by $g_i = (a, 1)$, all i, and denote the subgroup $\{1, g, g^2\}$ by K. We may then show that $K_n \to K$ in $\mathcal{K}(G)$. However, the sequence $\{N(K_n)\}$ does not converge to $N(K)$ for the following reasons. If it does, let $h = (h_i)$ be the element of G defined by $h_i = (c, 1)$, all i. Since $\langle a \rangle$ is normal in S_3, we have $h \in N(K)$ and we may apply Lemma 1 to get a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ and a corresponding sequence $\{k^m\}$ such that $k^m \in N(K_{n_m})$, all m, and $k^m \to k$. Now let $V = \Pi V_i$ be the neighborhood of h defined by $V_1 = \{(c, 1)\}$ and $V_i = G_i = H$, otherwise. For sufficiently large m, we have $k^m \in V$, i.e., $k_1{}^m = (c, 1)$. Furthermore,

$$(k^m g^{n_m} (k^m)^{-1})_i = \begin{cases} k_i{}^m (a, 1)(k_i{}^m)^{-1}, & 1 \leqq i \leqq n_m \\ (1, b), n_m \leqq i. \end{cases}$$

Thus, in order that $k^m \in N(K_{n_m})$, it must happen that $k^m g^{n_m} (k^m)^{-1} = g^{n_m}$; in particular, for $i = 1$, we must have $(c, 1)(a, 1)(c^{-1}, 1) = (a, 1)$, i.e., $cac^{-1} = a$, for m sufficiently large. This is a contradiction.

(ii) Let G be the direct product $Z \times S_3$ with Z the integers and let a and c be as in (i) above. Define K_n to be the (closed) subgroup $\{(kn, a^k) : k \in Z\}$, $n = 1, 2, \cdots$, and let $K = \{(0, 1), (0, a), (0, a^2)\}$. Then $K_n \to K$ and $N(K) = G$, i.e., K is normal in G. Hence, in particular, $(1, c) \in N(K)$. If $N(K_n) \to N(K)$, then by Lemma 1 and the discreteness of G, there exists a subsequence $\{K_{n_m}\}$ of $\{K_n\}$ such that $(1, c) \in N(K_{n_m})$, all m. However, $(n_m, a) \in K_{n_m}$ and so $(1, c)(n_m, a)(1, c^{-1}) = (n_m, cac^{-1})$ must belong to K_{n_m}. The only way this can happen is if $cac^{-1} = a$, which is again a contradiction.

(iii) The example in (ii) above satisfies this part as well.

Remark. At first glance, Theorem 2 and Theorem 5 (iii) seem to contradict each other. However, if G_1 and G_2 are groups with corresponding normalizer transformation groups $G_i \times \mathcal{K}(G_i) \to \mathcal{K}(G_i)$, $i = 1, 2$, then the transformation space in Theorem 2 is $\mathcal{K}(G_1) \times \mathcal{K}(G_2)$ while that of Theorem 5 (iii) is $\mathcal{K}(G_1 \times G_2)$, which is of course larger. This distinction does not occur in the centralizer case.

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