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# SOME REMARKS ON THE SLICE ALGEBRA FOR $H^\infty$

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1. **Introduction.** For $N \geqq 1$ let $U^N$ denote the open unit polydisc in $N$ complex variables and $H^\infty(U^N)$ the usual Hardy class of bounded holomorphic functions on $U^N$. If $M(U^N)$ denotes the maximal ideal space of $H^\infty(U^N)$, then by the $N$-dimensional slice algebra $S(M(U)^N)$ of $H^\infty(U)$ we mean the function algebra of all continuous functions $f$ on $M(U)^N$ for which $f(\varphi, \cdot, \psi)$ belongs to $\hat{H}^\infty(U)$ for each $\varphi$ in $M(U)^{k-1}$, $\psi$ in $M(U)^{N-k}$, and $k = 1, \cdots, N$. $\hat{H}^\infty(U^N)$ denotes the isomorphic Gelfand representation of $H^\infty(U^N)$ as a function algebra on $M(U^N)$. The purpose of this note is to give several characterizations of $S(M(U)^N)$ as a closed subalgebra of $H^\infty(U^N)$ and to relate the results to questions raised by [1].

In § 2 we obtain a description of all the functions in $U^N$ which extend continuously to $M(U)^N$. This leads in § 3 to new characterizations for $S(M(U)^N)$ as well as to straightforward proofs of two representations obtained by F. T. Birtel [1] and L. Eifler [6]. Birtel represents $S(M(U)^N)$ as those functions in $H^\infty(U^N)$ whose boundary function on the $N$-torus belongs to $\otimes_{k=1}^N L^\infty(T)$. Furthermore, both obtain $S(M(U)^N)$ as functions in $H^\infty(U^N)$ having continuous extensions to $M(U)^N$.

Also in § 3 we use a recent result of R. G. Douglas and W. Rudin [4] to show that $f$ in $H^\infty(U^N)$ belongs to $S(M(U)^N)$ if and only if there exists a sequence of functions $B_n$ in $H^\infty(U^N)$, each $B_n$ a tensor product of $N$ Blaschke products, such that the distance from $B_N f$ to the $N$-fold tensor product of $H^\infty(U)$ tends to zero. It follows from this that each $f$ in $S(M(U)^N)$ is constant on the fibers in $M(U^N)$ above the Šilov boundary of $S(M(U)^N)$. This answers in part the general question raised by Birtel in [1] of whether $S(M(U)^N)$ is precisely those functions in $H^\infty(U^N)$ for which $\hat{f}$ is constant on all fibers above $M(U)^N$. We do show that this general question is equivalent to answering the corona conjecture for $S(M(U)^N)$ or, equivalently, to showing that the maximal ideal space of $S(M(U)^N)$ is $M(U)^N$.

§ 4 deals with the localization of some of the results in § 3 to a neighborhood in $U^N$ of a point $\alpha$ of $T^N$ and to the fiber in $M(U)^N$ above $\alpha$.

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In § 5 we show that the cluster set for $f$ in $S(M(U)^N)$ at $\alpha$ is the range of $f$ on the fiber above $\alpha$ in the maximal ideal space of $S(M(U)^N)$. Similarly the range of $f$ in $H^\infty(U^N)$ on the fiber above $\alpha$ in $M(U^N)$ is the cluster set of $f$ at $\alpha$. As a corollary we show that these fibers are connected.

2. **Preliminaries.** Let $CB(U^N)$ denote all continuous complex-valued bounded functions on $U^N$. If $\vartheta_k$ is a subset of $CB(U)$, then by $\otimes_{k=1}^N \vartheta_k$ we mean the smallest closed subalgebra of $CB(U^N)$ which contains all functions of the form $F(\xi_1, \cdots, \xi_N) = f(\xi_k)$ for some $k$ and some choice of $f$ in $\vartheta_k$. Similarly we define $\otimes_{k=1}^N L^\infty(T)$ as a closed subalgebra of $L^\infty(T^N)$. By $h^\infty(U)$ we denote the subset of $CB(U)$ of all bounded real-valued harmonic functions on $U$.

It is well known [3] that $M(U)^N$ is the maximal ideal space of $\otimes_{k=1}^N H^\infty(U)$ and that $X^N$ is its Šilov boundary where $X$ is the Šilov boundary of $H^\infty(U)$. We also need the fact that $U^N$ is dense in $M(U)^N$, which follows from Carleson's corona theorem [5].

The following lemma is the key to several characterizations for $S(M(U)^N)$.

Lemma 1. *The following complex algebras are isometrically isomorphic to* $C(M(U)^N)$:

(i) *functions in* $U^N$ *which extend continuously to* $M(U)^N$.

(ii) $\otimes_{k=1}^N H^\infty(U) \cup \overline{H^\infty}(U)$.

(iii) $\otimes_{k=1}^N h^\infty(U)$.

Proof. Since $U^N$ is dense in $M(U)^N$, supremum norms on $U^N$ and $M(U)^N$ agree. Hence, (i) is clearly $C(M(U)^N)$ restricted to $U^N$. The Stone-Weierstrass theorem shows that (ii) is $C(M(U)^N)$. Clearly (ii) is contained in (iii). If $u$ belongs to $h^\infty(U)$ with harmonic conjugate $v$, then $F = \exp(u + iv)$ belongs to $H^\infty(U)$ and $\log|\hat{F}|$ is a continuous extension of $U$ to $M(U)$. It follows that each element of (iii) extends continuously to $M(U)^N$ and (i) then shows that (iii) is contained in (ii).

3. **Several Characterizations and Corollaries.** If $f$ belongs to $H^\infty(U^N)$, it is well known (see [9]) that the radial limits $f^*(w) = \lim_{r\to 1} f(rw)$ exist for almost all $w$ in $T^N$, where $rw = (rw_1, \cdots, rw_N)$. This gives an isometry of $H^\infty(U^N)$ onto a closed subspace of $L^\infty(T^N)$ with $f$ the $n$-dimensional Poisson integral of $f^*$. We will denote this closed subalgebra of $L^\infty(T^N)$ by $H^\infty(T^N)$.

If $\varphi$ belongs to $M(U^N)$ then $\pi(\varphi)$ will denote the restriction of $\varphi$ to a complex homomorphism on $\otimes_{k=1}^N H^\infty(U)$.

THEOREM 1. *The following complex algebras are isometrically isomorphic to* $S(M(U)^N)$:

(i) *functions in* $H^\infty(U^N)$ *which admit continuous extensions to* $M(U)^N$.

(ii) $\otimes_{k=1}^N [H^\infty(U) \cup \overline{H^\infty}(U)] \cap H^\infty(U^N)$.

(iii) $\otimes_{k=1}^N h^\infty(U) \cap H^\infty(U^N)$.

(iv) $\otimes_{k=1}^N L^\infty(T) \cap H^\infty(T^N)$.

(v) *functions* $f$ *in* $H^\infty(U^N)$ *such that given* $\epsilon > 0$ *there exists* $B = g_1 \otimes g_2 \otimes \cdots \otimes g_N$, *each* $g_k$ *a Blaschke product, such that* $\operatorname{dist}(Bf, \otimes_{k=1}^N H^\infty(U)) < \epsilon$.

(vi) *functions* $f$ *in* $H^\infty(U^N)$ *for which* $\hat{f}$ *is constant on* $\pi^{-1}(m)$ *for all* $m$ *in* $X^N$.

PROOF. In view of Lemma 1, to show that (i), (ii), and (iii) are $S(M(U)^N)$ it suffices to show that (i) and $S(M(U)^N)$ are identical. By Hartog's theorem each $f$ in $S(M(U)^N)$ restricted to $U^N$ belongs to $H^\infty(U^N)$ and sup norms on $U^N$ and $M(U)^N$ agree. Hence, $S(M(U)^N)$ is isometrically isomorphic to a closed subalgebra of (i). If $f$ in $H^\infty(U^N)$ extends continuously to $M(U)^N$, then $f(\alpha_\lambda, \cdot, \beta_\lambda)$ converges uniformly on compact subsets of $U$ to $f(\varphi, \cdot, \psi)$ as $(\alpha_\lambda, \beta_\lambda)$ converges to $(\varphi, \psi)$. It follows that $f(\varphi, \cdot, \psi)$ belongs to $\hat{H}^\infty(U)$ and the isometry is onto.

The equivalence of (iii) and (iv) follows directly from the isometry between $H^\infty(U^N)$ and $H^\infty(T^N)$. Each $f$ in $\otimes_{k=1}^N h^\infty(U)$ has boundary function $f^*$ in $\otimes_{k=1}^N L^\infty(T)$ and the Poisson integral of each function in (iv) clearly belongs to (iii).

Assume $f$ in $H^\infty(U^N)$ satisfies the conditions stated in (v). To show that $f$ belongs to $S(M(U)^N)$ it suffices by [1] to show that $f$ extends continuously to $X^N$. Let $\{\alpha_\lambda\}$ and $\{\beta_\lambda\}$ be two nets in $U^N$ converging to $m$ in $X^N$ with $\lim_\lambda f(\alpha_\lambda) = \alpha$ and $\lim_\lambda f(\beta_\lambda) = \beta$. For $\epsilon > 0$ choose $B$ of the stated form such that $\|Bf - g\|_\infty < \epsilon/2$ for some $g$ in $\otimes_{k=1}^N H^\infty(U)$. It follows that $|m(B)\alpha - m(g)| < \epsilon/2$ and $|m(B)\beta - m(g)| < \epsilon/2$. Since $m$ belongs to $X^N$ we have $|m(B)| = 1$ (see [8, p. 179]). Then $|\alpha - \beta| \leqq |m(B)\alpha - m(g)| + |m(g) - m(B)\beta| < \epsilon$. Thus $\alpha = \beta$ and $f$ extends continuously to $X^N$.

For $f$ in $S(M(U)^N)$ and $\epsilon > 0$, we can choose by (iv) a sum $\sum_{i=1}^n F_i$ in $\otimes_{k=1}^N L^\infty(T)$, each $F_i$ of the form $f_1^i \otimes f_2^i \otimes \cdots \otimes f_N^i$, such that $\|f^* - \sum_{i=1}^n F_i\|_\infty < \epsilon/2$. By Theorem 7 of [4] there exist singular inner functions $p_j^i$ and finite linear combinations $\chi_j^i$ of inner functions such that $\| F_i - \otimes_{j=1}^N \chi_j^i/p_j^i\|_\infty < \epsilon/2n$ for each $i = 1, \cdots, n$. It follows that

$$\left\| \prod_{i=1}^{n} (\otimes_{j=1}^{N} p_j^i) f^* - \sum_{i=1}^{n} \otimes_{j=1}^{N} \chi_j^i \otimes_{J \neq i} p_j^i \right\|_\infty < \epsilon.$$

Now extend to $U^N$ and use the result that every inner function is a uniform limit of Blaschke products (see [8, p. 175]). This gives (i) equivalent to ($v$).

If $f$ belongs to $H^\infty(U^N)$ with $\hat{f}$ constant on $\pi^{-1}(m)$ for every $m$ in $X^N$, then $f$ extends continuously to $X^N$; otherwise, there would exist two nets converging to $\varphi$ and $\psi$ respectively with $\pi(\varphi) = \pi(\psi) = m$ but $\varphi(f) \neq \psi(f)$. For the converse, suppose $f$ belongs to $S(M(U)^N)$, $\varphi$ and $\psi$ belong to $\pi^{-1}(m)$, $m$ belongs to $X^N$, and $\epsilon > 0$. Again by ($v$) we can choose $B$ and $g$ in $\otimes_{k=1}^{N} H^\infty(U)$ such that

$$\begin{aligned} |\hat{f}(\varphi) - \hat{f}(\psi)| &= |\varphi(B)\hat{f}(\varphi) - \varphi(B)\hat{f}(\psi)| \\ &\leqq |\varphi(B)\hat{f}(\varphi) - \varphi(g)| + |\psi(g) - \psi(B)\hat{f}(\psi)| < \epsilon, \end{aligned}$$

since $\varphi$ and $\psi$ agree on $\otimes_{k=1}^{N} H^\infty(U)$. This shows (i) and (vi) equivalent and the theorem is proved.

REMARKS. F. T. Birtel [1] has also obtained a representation of $S(M(U)^N)$ as the compact linear operators from $L^1/H_0{}^1$ to $H^\infty$. It still remains an open question whether $S(M(U)^N)$ is precisely $\otimes_{k=1}^{N} H^\infty(U)$. The equivalence of $S(M(U)^N)$ with (vi) provides a converse of Theorem 4 in [1]. Furthermore, the argument can easily be extended to give the following corollary.

COROLLARY 1. *The natural projection map $\tau$ of the maximal ideal space $\Sigma(S^N)$ of $S(M(U)^N)$ onto $M(U)^N$ is a homeomorphism over $X^N$.*

COROLLARY 2. *The following are equivalent:*
(i) *$\tau$ is a homeomorphism.*
(ii) *$U^N$ is dense in $\Sigma(S^N)$.*
(iii) *$S(M(U)^N)$ is the subalgebra of $H^\infty(U^N)$ for which $\hat{f}$ is constant on $\pi^{-1}(m)$ for all $m$ in $M(U)^N$.*

PROOF. It is clear that (i) implies (ii) since $U^N$ is dense in $M(U)^N$. If we assume that $U^N$ is dense in $\Sigma(S^N)$, then each $f$ in $S(M(U)^N)$ has a unique continuous extension to $\Sigma(S^N)$. Let $\epsilon > 0$ with $\varphi$ and $\psi$ belonging to $\tau^{-1}(m)$. Each function in $\otimes_{k=1}^{N} h^\infty(U)$ must extend continuously to $\Sigma(S^N)$ since $U^N$ is dense in $\Sigma(S^N)$. Then point evaluation at $\varphi$ and $\psi$ are continuous linear functionals $\hat{\varphi}$ and $\hat{\psi}$ on $\otimes_{k=1}^{N} h^\infty(U)$ which must agree with $\varphi$ and $\psi$ on $S(M(U)^N)$. If $g$ is chosen in $\otimes_{k=1}^{N} h^\infty(U)$ such that $\|f - g\|_\infty < (\epsilon/2)\max\{\|\hat{\varphi}\|, \|\hat{\psi}\|\}$, then

$|\hat{f}(\varphi) - \hat{f}(\psi)| \leqq \|\hat{\varphi}\| \; \|f - g\|_\infty + \|\hat{\psi}\| \; \|g - f\|_\infty < \epsilon$. Thus $\varphi = \psi$ and $\tau$ is one-to-one.

Let $\eta$ be the natural projection map of $M(U^N)$ onto $\Sigma(S^N)$. If $\varphi(f) \neq \psi(f)$ for $f$ in $S(M(U)^N)$ and $\pi(\varphi) = \pi(\psi) = m$, then $\eta(\varphi) \neq \eta(\psi)$ but $\tau(\eta(\varphi)) = \tau(\eta(\psi)) = m$. It follows that $\tau$ is not a homeomorphism and (i) implies (iii). Conversely, if $\tau(\varphi) = \tau(\psi)$ but $\varphi \neq \psi$, then there exists $f$ in $S(M(U)^N)$ such that $\hat{f}(\varphi) \neq \hat{f}(\psi)$. But if $\eta(\varphi^*) = \varphi$ and $\eta(\psi^*) = \psi$, then $\pi(\varphi^*) = \pi(\psi^*)$ but $\hat{f}(\varphi^*) = \hat{f}(\varphi) \neq \hat{f}(\psi) = \hat{f}(\psi^*)$ and (iii) implies (i).

4. **Localization.** In this section we show how to localize some of the preceding sections. For $\alpha = (\alpha_1, \cdots, \alpha_N)$ in $T^N$ the fiber in $M(U)^N$ above $\alpha$ is the subset $M(\alpha)$ containing all $\varphi$ such that $\varphi(z_k) = \alpha_k$ where $z_k$ is the $k$th coordinate function on $\mathbb{C}^N$ for $k = 1, \cdots, N$. Similarly, $\Gamma(\alpha)$ denotes the fiber in $M(U^N)$ above $\alpha$.

LEMMA 2. *Let $g$ belong to $\otimes_{k=1}^N h^\infty(N_k(\alpha_k) \cap U)$ with $N(\alpha) = N_1(\alpha_1) \times \cdots \times N_N(\alpha_N)$ a neighborhood in $\mathbb{C}^N$ of $\alpha = (\alpha_1, \cdots, \alpha_N)$ in $T^N$. Then there exists $h$ in $\otimes_{k=1}^N h^\infty(U)$ such that $g - h$ extends continuously to $\alpha$ with value* 0.

PROOF. It suffices to assume that each $N_k(\alpha_k)$ is a circular neighborhood of $\alpha_k$ and that $g$ belongs to $h^\infty(N_k(\alpha_k) \cap U)$ for some $k$. Let $v$ be a harmonic conjugate of $g$ on $N_k(\alpha_k) \cap U$. Then $F = e^{g+iv}$ belongs to $H^\infty(N_k(\alpha_k) \cap U)$. Now using Vitushkin's localization operator [7] we construct $H$ in $H^\infty(U)$ such that $H - F$ extends analytically across $\alpha_k$ with value 0 at $\alpha_k$. Since $H$ extends continuously to $m(\alpha_k)$ and $\log|F| = g$, we have that $g$ extends continuously to $M(\alpha_k)$.

Let $h$ be the restriction of the continuous extension of $g$ to $M(\alpha_k)$. Then $h$ extends continuously to $M(U)$ and by Lemma 1 must belong to the smallest closed subalgebra of $CB(U)$ containing $h^\infty(U)$. Then $g$ and $h$ satisfy the lemma.

THEOREM 2. *For $\alpha$ in $T^N$ the following complex algebras are identical:*

(i) *functions in $H^\infty(U^N)$ which extend continuously to $M(\alpha)$.*

(ii) *functions in $H^\infty(U^N)$ such that given $\epsilon > 0$, there exists a neighborhood $N(\alpha)$ of $\alpha$ and $g$ in $\otimes_{k=1}^N h^\infty(N_k(\alpha_k) \cap U)$ such that $\|f - g\|_\infty < \epsilon$ on $U^N \cap N(\alpha)$.*

(iii) *same as* (ii) *except with $g$ in $\otimes_{k=1}^N h^\infty(U)$, or in $\otimes_{k=1}^N [H^\infty(N_k(\alpha_k) \cap U) \cup \overline{H^\infty}(N_k(\alpha_k) \cap U)]$, or in $\otimes_{k=1}^N H^\infty(U) \cup \overline{H^\infty}(U)$.*

PROOF. Assume $f$ belongs to $H^\infty(U^N)$ and extends to a continuous function $g$ on $M(\alpha)$. Then $g$ extends continuously to $M(U)^N$ and by Lemma 1 must belong to $\otimes_{k=1}^N h^\infty(U)$. Then $f$ and $g$ satisfy (ii) since otherwise $\|f - g\| \geqq \epsilon$ on $M(\alpha)$. For the converse choose $g$ in $\otimes_{k=1}^{N} h^{\infty}(N_k(\alpha_k) \cap U)$ such that $\|f - g\|_{\infty} < \epsilon$ on $U^N \cap N(\alpha)$. By Lemma 2 there exists $G$ in $\otimes_{k=1}^{N} h^{\infty}(U)$ such that $g - G$ extends continuously to $\alpha$ with value 0. Then for some neighborhood $K(\alpha)$ of $\alpha$ we have $\|f - G\|_{\infty} < \epsilon$ on $K(\alpha) \cap U^N$. It follows that the cluster set of $f$ at any $\varphi$ in $M(\alpha)$ has diameter less than $\epsilon$. The other equivalences follow in a similar fashion using Lemma 1.

5. **Cluster Values.** Although we cannot answer the corona conjecture for $H^{\infty}(U^N)$ or $S(M(U)^N)$, we can prove a cluster value theorem which gives the range of $f$ in $H^{\infty}(U^N)$ on $\Gamma(\alpha)$ and the range of $f$ in $S(M(U)^N)$ on the fiber $S(\alpha)$ in $\Sigma(S^N)$ above $\alpha$.

We begin with a lemma on extending bounded holomorphic functions.

LEMMA 3. *Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ belong to $T^N$ and $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$ be a neighborhood in $U^N$ of $\alpha$. Then given $f$ in $H^{\infty}(\Omega)$ there exists $F$ in $H^{\infty}(U^N)$ such that $F - f$ extends continuously to $\alpha$ with value* 0.

PROOF. We employ a vector-valued version of Vitushkin's localization operator. Define $T_1 : \Omega_1 \to H^{\infty}(\Omega_2 \times \cdots \times \Omega_N)$ by $T_1(z) = f(z, \cdot)$. Then $T$ belongs to $H^{\infty}(\Omega_1, H^{\infty}(\Omega_2 \times \cdots \times \Omega_N))$ which is isometrically isomorphic to $H^{\infty}(\Omega)$ (see [2]). Let $\varphi$ be a continuously differentiable function on $\mathbb{C}$ with compact support in $\Delta(\alpha_1; \delta) = \{z : |z - \alpha_1| < \delta\}$ which is contained in $\Omega_1$ and such that $|\partial\varphi/\partial\bar{z}| \leqq 4/\delta$. Define

$$\hat{T}_1(\xi) = \left(\frac{1}{\pi}\right) \int_{\mathbb{C}} \frac{T(z) - T(\xi)}{z - \xi} \frac{\partial \varphi}{\partial \bar{z}} \, dxdy.$$

Then $\hat{T}_1$ beongs to $H^{\infty}(U, H^{\infty}(\Omega_2 \times \cdots \times \Omega_N))$. We can adjust $\hat{T}_1$ by a constant such that $T_1 - \hat{T}_1$ has value 0 at $\alpha_1$. Let $f_1(z, w) = \hat{T}_1(z)(w)$. Then $f_1$ belongs to $H^{\infty}(U \times \Omega_2 \times \cdots \times \Omega_N)$ and $f - f_1$ extends continuously to $\alpha$ with value 0. Now proceed by induction on $n$.

THEOREM 3. *Let $\alpha = (\alpha_1, \cdots, \alpha_N)$ belong to $T^N$. Then the range of $f$ in $H^{\infty}(U^N)$ on $\Gamma(\alpha)$ and the range of $f$ in $S(M(U)^N)$ on $S(\alpha)$ consists of all complex numbers $\lambda$ for which there is a sequence $\{\lambda_n\}$ in $U^N$ with* $\lim \lambda_n = \alpha$ *and* $\lim f(\lambda_n) = \lambda$.

PROOF. Assume $\varphi$ belongs to $\Gamma(\alpha)$, $\varphi(f) = 0$ and let $\Omega = \Omega_1 \times \cdots \times \Omega_N$ be a neighborhood of $\alpha$ in $\bar{U}^N$ such that on $\Omega \cap U^N$, $f$ is bounded away from 0. It follows that $g = 1/f$ belongs to $H^{\infty}(\Omega \cap U^N)$, and by Lemma 3 there exists $F$ in $H^{\infty}(U^N)$ such that $g - F$ tends to 0 at $\alpha$. Thus $fF$ belongs to $H^{\infty}(U^N)$ and extends continuously to $\alpha$ with value 1. It follows by [8, p. 161] (with $h(z_1, \cdots, z_N) = \prod_{n=1}^{N} (1/2)^n(1 + \bar{\alpha}_n z_n))$ that $\varphi(fF) = \varphi(f)\varphi(F) = 1$, so $\varphi(f) \neq 0$. If $\varphi$ belongs to $S(\alpha)$ and $f$ belongs to $S(M(U)^N)$, choose $\psi$ in $\Gamma(\alpha)$ such that $\eta(\psi) = \varphi$. Then $\psi(fF) = \varphi(f)\psi(F) = 1$ implies $\varphi(f) \neq 0$.

COROLLARY 3. *The fibers* $\Gamma(\alpha)$ *and* $S(\alpha)$ *are connected.*

PROOF. This follows as in [8, p. 188] from Theorem 3.

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