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NONLINEAR PERTURBATION OF A LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH A REGULAR SINGULAR POINT

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ABSTRACT. The asymptotic behavior of the solutions of a nonlinear perturbation of a linear system of ordinary differential equations with a regular singular point is discussed. Properties of the solutions are considered both for large values of the independent variable and for values close to the singular point.

1. **Introduction.** If $A = A(t)$ is a continuous $n \times n$ matrix function defined on $I = (0, \infty)$ and the linear system of differential equations

$$dy/dt = A(t)y \tag{1}$$

has a regular singular point at $t = 0$, then an important problem for (1) is the determination of the asymptotic behavior of the solutions of (1) at each endpoint of I. Frequently mathematical models are perturbations of the linear equation (1) and are of the form

$$dx/dt = A(t)x + f(t, x). \tag{2}$$

In general, (2) is a nonlinear equation. A standard problem is to utilize knowledge about the solutions of (1) to investigate the behavior of the solutions of (2); this article makes a contribution in this direction.

Our work is motivated by the research of S. Faedo [2], T. G. Hallam [3] and A. F. Izé [4]. The asymptotic results in [2] and [3] are developed for scalar linear differential equations. Those in [4] are valid for a linear nonhomogeneous system of differential equations. Our results, while being of the same character as Izé's, are extensions of his work to the nonlinear equation (2). We also allow more general hypotheses upon the linear unperturbed equation than is permitted in [4].

2. **Preliminaries.** In this section, preliminary notation and results are given. Many of the omitted details may be found in [4]. In equations (1) and (2), the symbols y and x denote n-vectors. The function f is continuous from $I \times R^n$ to R^n. We require, as is done in

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[4], that A have the form

$$A(t) = (c_{ij}t^{-p_{ij}}) \tag{3}$$

where c_{ij} and p_{ij} are constants, $i, j = 1, 2, \cdots, n$.

Employing (3), equations (1) and (2) can be written in the form

$$dy_i/dt = \sum_{j=1}^{n} c_{ij}t^{-p_{ij}}y_j, \quad i = 1, 2, \cdots, n; \tag{1'}$$

$$dx_i/dt = \sum_{j=1}^{n} c_{ij}t^{-p_{ij}}x_j + f_i(t, x_1, x_2, \cdots, x_n), \tag{2'}$$
$$i = 1, 2, \cdots, n.$$

The following lemmas are proved in [4]. The first lemma indicates how the well known Cauchy-Euler scalar differential equation may be extended to system form.

LEMMA. 1. *Let the powers* p_{ij} *be of the form* $p_{ij} = \alpha_j - \alpha_i + 1$ *when* $c_{ij} \neq 0$, $i, j = 1, 2, \cdots, n$ *for some vector* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. *A necessary and sufficient condition for the existence of a solution* $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ *of equation* (1′) *such that* $y_i(t) = a_i t^{\alpha_i}$ *with* $a = (a_1, a_2, \cdots, a_n) \neq 0$ *is that* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *be a solution of the indicial equation*

$$\det(C - \Gamma) = 0 \tag{4}$$

where $C = (c_{ij})$ *and* $\Gamma = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

LEMMA 2. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be real or complex square matrices with* B *nonsingular. If* $C = AB^{-1}$ *where* $C = (c_{ij})$, *then* $c_{ij} = \det C'_{ij}/\det B$ *where* C'_{ij} *is the matrix obtained by the substitution of the ith row of A into the jth row of B.*

Another lemma which we will use can be found in [3].

LEMMA 3. *Let* $\delta > 0$, $t_0 \geqq 1$, $h(t) \geqq 0$ *on* $[t_0, \infty)$, *and suppose that* $\int_{t_0}^{\infty} h(t)\,dt < \infty$. *Then,* $\lim_{t\to\infty} t^{-\delta} \int_{t_0}^{t} s^{\delta}h(s)\,ds = 0$.

Assuming the hypotheses of Lemma 1, the change of variables

$$s = \log t, \quad y_i e^{-\alpha_i s} = z_i \tag{5}$$

transforms the system (1′) into

$$dz_i/dt = (c_{ii} - \alpha_i)z_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} c_{ij}z_i, \quad i = 1, 2, \cdots, n. \tag{6}$$

The characteristic equation of the system (6) is

$$\det((C - \Gamma) - \lambda I) = 0. \tag{7}$$

When the roots of (7), $\lambda_\ell = \beta_\ell + i\gamma_\ell$ are distinct and the only root with zero real part is identically zero, a fundamental matrix of solutions of equation (1) is

$$Y(t) = (a_{ij}t^{\alpha_i+\beta_j} g_j(t)),\ i, j = 1, 2, \cdots, n,$$

where $g_j(t) = \cos(\gamma_j \ln t) + i \sin(\gamma_j \ln t)$.

We assume that $Y(t)$ has been ordered in such a manner that $\beta_j < 0$ for $1 \leqq j \leqq m - 1$, $\beta_m = 0$, and $\beta_j > 0$ for $m + 1 \leqq j \leqq n$. Define the projection matrices P_1 and P_2 where P_1 has a one on the main diagonal in its first m columns and zeros elsewhere and P_2 has ones on its main diagonal in its last $n - m$ columns and zeros elsewhere.

Applying Lemma 2, we obtain

$$Y(t)P_1Y^{-1}(s) = (K^{-1}t^{\alpha_i}s^{-\alpha_j} \sum_{\ell=1}^{m} t^{\beta_\ell} s^{-\beta_\ell} a_{i\ell} g_\ell(t) G_{j\ell}(s)), \quad i,j = 1, 2, \cdots, n; \tag{8}$$

$$Y(t)P_2Y^{-1}(s) = (K^{-1}t^{\alpha_i}s^{-\alpha_j} \sum_{\ell=m+1}^{n} t^{\beta_\ell} s^{-\beta_\ell} a_{i\ell} \cdot g_\ell(t) G_{j\ell}(s)), \quad i,j = 1, 2, \cdots, n; \tag{9}$$

where $G_{j\ell}(s) = (-1)^{j+\ell} F_{j\ell}(s) s^{-i\gamma}$ with $F_{j\ell}(s)$ bounded on $(0, \infty)$ and $K \equiv \det Y(t_0) t_0{}^{-\alpha-\beta-i\gamma}$ with $\alpha = \sum_{\ell=1}^n \alpha_\ell$, $\beta = \sum_{\ell=1}^n \beta_\ell$, and $\gamma = \sum_{\ell=1}^n \gamma_\ell$.

3. **Asymptotic Behavior of (2) as $t \to \infty$.** In the main result of this section, we require a constant B defined as follows. Let

$$B_{ij} = \sup_{s,t \in [t_0, \infty)} |K^{-1}| \sum_{\ell=1}^{n} |a_{i\ell} g_\ell(t) G_{j\ell}(s)|$$

and $B = \sup_{i,j=1,2,\cdots,n} B_{ij}$.

THEOREM 1. *Let $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a solution of (4) and satisfy $p_{ij} = \alpha_j - \alpha_i + 1$ for every $c_{ij} \neq 0, i, j = 1, 2, \cdots, n$. Suppose that the roots of (7) are distinct, and the root $\lambda_m \equiv 0$. Let there exist constants $M > 0$ and $t_0 \geqq 1$ such that*

$$\int_{t_0}^{\infty} t^{-\alpha_i} |f_i(t, Mt^{\alpha_1}, Mt^{\alpha_2}, \cdots, Mt^{\alpha_n})| \, dt < M/2nB, \quad i = 1, 2, \cdots, n. \tag{10}$$

and assume that $f_i(t, r_1, r_2, \cdots, r_n)$ *is nondecreasing in* r_j *for each fixed* $(t, r_1, r_2, \cdots, r_{j-1}, r_{j+1}, \cdots, r_n) \in [t_0, \infty) \times I^{n-1}, j = 1, 2, \cdots, n$. *Then, corresponding to each solution* $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ *of* (1′) *such that* $\sup_{t \geqq t_0} |t^{-\alpha_i} y_i(t)| < M/2$, *there exists a solution* $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ *of* (2′) *such that* $x_i(t) = y_i(t) + a_i(t)t^{\alpha_i}$ *with* $\lim_{t\to\infty} a_i(t) = a_i, i = 1, 2, \cdots, n$.

PROOF. For the prescribed n-tuple $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we note that the space $\mathcal{B}$ of all continuous n-vector functions $x = (x_1, x_2, \cdots, x_n)$ with the property that there exist constants $k_i = k_i(x)$ such that $|t^{-\alpha_i} x_i(t)| \leqq k_i$ for all $t \in [t_0, \infty)$ and all $i = 1, 2, \cdots, n$ is a Banach space with the norm of $x \in \mathcal{B}$ given by $\|x\| = \sum_{i=1}^{n} \sup_{t \in [t_0, \infty)} |t^{-\alpha_i} x_i(t)|$.

For $\rho > 0$, define $\mathcal{B}_\rho = \{z \mid z \in \mathcal{B}, \|z\| \leqq \rho\}$. $\mathcal{B}_\rho$ is a closed convex subset of the Banach space $\mathcal{B}$. For $x \in \mathcal{B}_M$ and $y \in \mathcal{B}_{M/2}$, define the mapping T by

$$Tx(t) = y(t) + \int_{t_0}^{t} Y(t)P_1 Y^{-1}(s) f(s, x(s))\, ds - \int_{t}^{\infty} Y(t)P_2 Y^{-1}(s) f(s, x(s))\, ds.$$

We employ the Schauder-Tychonoff fixed point theorem [1, p. 9] to show that T has a fixed point.

Using the equations (8) and (9), the equalities $t^{\beta_\ell} s^{-\beta_\ell} \leqq 1$ for $s \in [t_0, t]$ and $\ell = 1, 2, \cdots, m$; $t^{\beta_\ell} s^{-\beta_\ell} \leqq 1$ for $s \in [t, \infty)$ and $\ell = m + 1, \cdots, n$; and the definition of B, we have

$$|Tx_i(t)| \leqq |y_i(t)| + B \int_{t_0}^{\infty} t^{\alpha_i} \sum_{j=1}^{n} |s^{-\alpha_j} f_j(s, x_1(s), x_2(s), \cdots, x_n(s))|\, ds.$$

Since $x \in \mathcal{B}_M$, we have

$$\begin{aligned} |t^{-\alpha_i} Tx_i(t)| &\leqq |t^{-\alpha_i} y_i(t)| \\ &\quad + B \sum_{j=1}^{n} \int_{t_0}^{\infty} |s^{-\alpha_j} f_j(s, Ms^{\alpha_1}, Ms^{\alpha_2}, \cdots, Ms^{\alpha_n})|\, ds \\ &\leqq M/2 + M/2 = M; \end{aligned}$$

therefore, T maps $\mathcal{B}_M$ into itself.

Next, we indicate how the continuity of T is established. Let x^0, $\{x^k\}_{k=1}^{\infty}$ belong to $\mathcal{B}_M$ and suppose that $\{x^k\}$ converges uniformly to x^0 on compact subintervals of I. Since each x^k, $k = 0, 1, \cdots$, belongs to $\mathcal{B}_M$, condition (10) implies that $\int^\infty t^{-\alpha_i} |f_i(t, x_1{}^k(t), \cdots, x_n{}^k(t))|\, dt < \infty$, $k = 0, 1, \cdots$, hence, given any interval $[t_0, t_1]$ and any $\epsilon > 0$, there exists $t_2 \geqq t_1$ such that

$$\int_{t_2}^{\infty} t^{-\alpha_i} |f_j(s, x_1^0(s), \cdots, x_n^0(s)) - f_j(s, x_1^k(s), \cdots, x_n^k(s))| \, ds < \epsilon/3nB.$$

The continuity of f implies the existence of N with the property that whenever $k \geqq N$

$$\int_{t_0}^{t_2} \sum_{j=1}^{n} s^{-\alpha_j} |f_j(s, x_1^0(s), \cdots, x_n^0(s)) - f_j(s, x_1^k(s), \cdots, x_n^k(s))| \, ds < \epsilon/3B.$$

From the definition of T, we obtain $|Tx_i^0(t) - Tx_i^k(t)| < \epsilon t^{-\alpha_i}$ for $t \in [t_0, t_1]$ and $k \geqq N$. This shows that T is continuous in the compact open topology.

The functions in the image space $T\mathcal{B}_M$ are uniformly bounded in $\mathcal{B}$-norm for each $t \in [t_0, \infty)$ since $T\mathcal{B}_M$ is a subset of $\mathcal{B}_M$. They form an equicontinuous family on every finite subinterval of $[t_0, \infty)$ as the function $z = Tx$ for $x \in \mathcal{B}_M$ satisfies the linear nonhomogeneous differential equation $z' = A(t)z + f(t, x(t))$; and hence they have uniformly bounded derivatives on each finite subinterval of $[t_0, \infty)$. The compactness of $\overline{T\mathcal{B}_M}$ now follows from Ascoli's Theorem.

An application of the Schauder-Tychonoff fixed point theorem yields a function $x, x(t) = (x_1(t), \cdots, x_n(t))$, in $\mathcal{B}_M$. This x satisfies the integral equations

$$\begin{aligned} x_i(t) = y_i(t) &+ K^{-1} \int_{t_0}^{t} \sum_{j=1}^{n} \sum_{\ell=1}^{m-1} t^{\alpha_i+\beta_\ell} s^{-(\alpha_j+\beta_\ell)} \\ &\qquad \cdot a_{i\ell} g_\ell(t) G_{j\ell}(s) f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds \\ &+ K^{-1} \int_{t_0}^{t} \sum_{j=1}^{n} t^{\alpha_i} s^{-\alpha_j} a_{im} G_{jm}(s) \\ &\qquad \cdot f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds \\ &- K^{-1} \int_{t}^{\infty} \sum_{j=1}^{n} \sum_{\ell=m+1}^{n} t^{\alpha_i+\beta_\ell} s^{-(\alpha_j+\beta_\ell)} a_{i\ell} g_\ell(t) \\ &\qquad \cdot G_{j\ell}(s) f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds, \end{aligned} \tag{11}$$

and is a solution of (2′). In (11), we have used the hypothesis $\lambda_m = 0$ and, hence, $g_m = 1$.

Since $\beta_\ell < 0, \ell = 1, 2, \cdots, m-1$, and the integral

$$\int^{\infty} s^{-\alpha_j} f_j(s, x_1(s), \cdots, x_n(s))\, ds \tag{12}$$

converges for each $j = 1, 2, \cdots, n$, Lemma 3 implies

$$\lim_{t\to\infty} K^{-1} \int_{t_0}^{t} \sum_{j=1}^{n} \sum_{\ell=1}^{m-1} t^{\beta_\ell} s^{-\beta_\ell} a_{i\ell} g_\ell(t) G_{j\ell}(s) \cdot s^{-\alpha_j} f_j(s, x_1(s), x_2(s), \cdots, x_n(s))\, ds = 0.$$

For $s \in [t, \infty)$, $t^{\beta_\ell} \leqq s^{\beta_\ell}$, $\ell = m+1, \cdots, n$, and the convergence of the integral in (12) yields

$$\lim_{t\to\infty} \int_{t}^{\infty} \sum_{j=1}^{n} \sum_{\ell=m+1}^{n} t^{\beta_\ell} s^{-\beta_\ell} a_{i\ell} g_\ell(t) G_{j\ell}(s) s^{-\alpha_j} \cdot f_j(s, x_1(s), x_2(s), \cdots, x_n(s))\, ds = 0.$$

The convergence of the integral (12) also implies the existence of the limit

$$\lim_{t\to\infty} K^{-1} \int_{t_0}^{t} \sum_{j=1}^{n} a_{im} G_{jm}(s) s^{-\alpha_j} f_i(s, x_1(s), x_2(s), \cdots, x_n(s))\, ds = A_i.$$

Combining these results, we obtain $\lim_{t\to\infty} t^{-\alpha_i} [x_i(t) - y_i(t)] = A_i$.

REMARK. The above result improves Theorem 2 of [4] in that we allow the real parts β_i of the characteristic roots λ_i to be positive.

4. **Asymptotic Behavior of (2) as $t \to 0^+$.** In this section, we consider the behavior of the solutions of the perturbed equation (2) near the singular point $t = 0$.

The following analogue of Lemma 3 for the case $t = 0$ is used in our arguments. The proof is similar to that of Lemma 3, see [3].

LEMMA 4. *Let $\delta > 0$, $0 < t_0 \leqq 1$, $h(t) \geqq 0$ for $t \in (0, t_0]$, and suppose that $\int_0^{t_0} h(t)\, dt < \infty$. Then,* $\lim_{t\to 0^+} t^{\delta} \int_t^{t_0} s^{-\delta} h(s)\, ds = 0$.

With $Y(t)$ having the same form as above, we decompose the solution space by employing the projections Q_1 and Q_2, where Q_1 has ones on the main diagonal in its first $m - 1$ columns with zeros elsewhere, and Q_2 has ones on the main diagonal in its last $n - m + 1$ columns and zeros elsewhere.

Define $B_0 = \sup_{i,j=1,2,\cdots,n} B_{ij}^0$ where

$$B_{ij}^0 = \sup_{s,t \in (0,t_0]} |K^{-1}| \sum_{\ell=1}^{n} |a_{i\ell} g_\ell(t) G_{j\ell}(s)|.$$

THEOREM 2. *Let* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ *be a solution of* (4) *and satisfy* $p_{ij} = \alpha_i - \alpha_j + 1$ *for every* $c_{ij} \neq 0$, $i, j = 1, 2, \cdots, n$. *Suppose that the roots* $\lambda_j = \beta_j + i\gamma_j$ *of* (7) *are distinct and that the root* $\lambda_m \equiv 0$. *Let there exist constants* $M > 0$ *and* t_0, $0 < t_0 \leqq 1$, *such that*

$$\int_0^{t_0} t^{-\alpha_i} |f_i(t, Mt^{\alpha_1}, Mt^{\alpha_2}, \cdots, Mt^{\alpha_n})| \, dt < M/2nB_0,$$

and assume that $f_i(t, r_1, r_2, \cdots, r_n)$ *is nondecreasing in* r_j *for each fixed* $(t, r_1, r_2, \cdots, r_{j-1}, r_{j+1}, \cdots, r_n) \in (0, t_0] \times I^{n-1}$, $j = 1, 2, \cdots, n$. *Then, corresponding to each solution* $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ *of* (1′) *such that* $\sup_{t\in(0,t_0]} |t^{-\alpha_i} y_i(t)| < M/2$, *there exists a solution* $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ *of* (2′) *such that* $x_i(t) = y_i(t) + a_i(t)t^{\alpha_i}$ *with* $\lim_{t\to 0^+} a_i(t) = a_i$, $i = 1, 2, \cdots, n$.

PROOF. The proof is the dual of that of Theorem 1 for the interval $(0, t_0]$. We consider the mapping T_0 on the space $\mathcal{B}_M{}^0 = \{z : z \in C[(0, t_0], R^n], \sup_{t\in(0,t_0]} |t^{-\alpha_i} z_i(t)| \leqq M, \ i = 1, 2, \cdots, n\}$ defined by

$$T_0 x(t) = y(t) + \int_0^t Y(t)Q_1 Y^{-1}(s) f(s, x(s)) \, ds$$
$$- \int_t^{t_0} Y(t)Q_2 Y^{-1}(s) f(s, x(s)) \, ds.$$

The Schauder-Tychonoff fixed point theorem can be used to show that T_0 has a fixed point x in $\mathcal{B}_M{}^0$. This fixed point is a solution of (2′) with the desired asymptotic behavior. The equation that corresponds to (11) is

$$x_i(t) = y_i(t) + K^{-1} \int_0^t \sum_{j=1}^{n} \sum_{\ell=1}^{m-1} t^{(\alpha_i+\beta_\ell)} s^{-(\alpha_j+\beta_\ell)}$$
$$\cdot a_{i\ell} g_\ell(t) G_{j\ell}(s) f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds$$
$$- K^{-1} \int_t^{t_0} \sum_{j=1}^{n} t^{\alpha_i} s^{-\alpha_j} a_{im} G_{jm}(x)$$
$$\cdot f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds \tag{13}$$
$$- K^{-1} \int_t^{t_0} \sum_{j=1}^{n} \sum_{\ell=m+1}^{n} t^{\alpha_i+\beta_\ell} s^{-(\alpha_j+\beta_\ell)} a_{i\ell} g_\ell(t)$$
$$\cdot G_{j\ell}(s) f_j(s, x_1(s), x_2(s), \cdots, x_n(s)) \, ds.$$

The second group of terms on the right side of (13) can be readily shown to approach zero as $t \to 0^+$. The last group of terms approaches zeros as $t \to 0^+$ by virtue of Lemma 4. The remaining expression has a limit as $t \to 0^+$.

REMARK. Our discussions were directed to polynomial-like asymptotic solutions of (2). The more general case where the characteristic roots of (1) need not be distinct can be treated by our techniques; however, in this setting, the computations and notations become almost unmanageable. The reference [2] discusses a general case for a scalar linear differential equation.

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