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ON POLYNOMIAL AND POWER SERIES RINGS OVER A COMMUTATIVE RING

ROBERT GILMER [1]

1. **Introduction.** Let R be a commutative ring with identity and let X be an indeterminate over R. The polynomial ring $R[X]$ and the ring $R[[X]]$ of formal power series over R are basic objects of study in the theory of commutative rings. One of the reasons that the polynomial ring is important is the following fact: If S is a commutative ring containing R as a subring, and if $s \in S$, then the "substitution mapping" $f(X) \to f(s)$ is a homomorphism of $R[X]$ into S. By extension of this result to the case of polynomial rings in finitely many indeterminates over R, we see that each commutative finitely generated ring extension $R[s_1, s_2, \cdots, s_n]$ of R is isomorphic to a residue class ring of $R[X_1, \cdots, X_n]$. The power series ring plays an important role in the theory of topological rings. For example, under suitable topological restrictions on S, a "substitution mapping" $f(X) \to f(s)$ of $R[[X]]$ into S can be defined that is a homomorphism; here $f(X) = \sum_{i=0}^{\infty} f_i X^i$, and $f(s)$ is the limit of the sequence $\{\sum_{i=0}^{n} f_i s^i\}_{n=1}^{\infty}$ in S.

We consider here the following questions concerning polynomial and power series rings.

1A. What are the zero divisors of $R[X]$?
2A. What are the nilpotent elements of $R[X]$?
3A. What are the units of $R[X]$?
4A. Determine the set of R-automorphisms of $R[X]$.
5A. What is the (Krull) dimension of $R[X]$?

Questions 1B, 2B, $\cdots$ are the analogues of the preceding questions for the power series ring $R[[X]]$. A word of explanation concerning our point of view is in order. We seek answers to the preceding questions in terms of the coefficient ring R; thus in questions 1A, 2A, 3A, our answers will be given in terms of the coefficients of the polynomial in $R[X]$, and on question 5A, we relate the dimension of $R[X]$ to the dimension of R. Questions 1–3 can be regarded primarily as tools for considering other questions, not as ends in themselves, whereas results on questions 4 and 5 play more of a dual role — they are useful in considering other problems, but they are also of independent interest.

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2. **Zero divisors, nilpotent elements, and units.** We continue to use the notation of the preceding section; R denotes a commutative ring with identity and X is an indeterminate over R.

If ϕ is a homomorphism of R onto a ring S, then ϕ induces a homomorphism ϕ^* of $R[X]$ onto $S[X]$ defined by $\phi^*(\sum_{i=0}^n r_iX^i) = \sum_{i=0}^n \phi(r_i)X^i$. If A is the kernel of ϕ, then the kernel of ϕ^* is the set of polynomials $\sum_0^n r_iX^i$ such that $r_i \in A$ for each i; we denote this ideal by $A[X]$. Hence

$$R[X]/A[X] \cong (R/A)[X],$$

and from this isomorphism, we draw several conclusions: (1) ϕ is one-to-one if and only if ϕ^* is one-to-one, (2) A is prime in R if and only if $A[X]$ is prime in $R[X]$, and (3) if A is a maximal ideal of R, then $A[X]$ is not maximal in $R[X]$, but each proper prime ideal of $R[X]$ properly containing $A[X]$ is maximal in $R[X]$ (in other terminology, $A[X]$ has *depth one* [59, p. 240]). Similar remarks apply to $R[[X]]$; ϕ induces a homomorphism $\phi^{**}: R[[X]] \to S[[X]]$ defined by $\phi^{**}(\sum_0^\infty r_iX^i) = \sum_0^\infty \phi(r_i)X^i$. The kernel of ϕ^{**} is the ideal $A[[X]]$ of $R[[X]]$ consisting of all power series $\sum r_iX^i$, with $r_i \in A$ for each i. Therefore $R[[X]]/A[[X]] \cong (R/A)[[X]]$, ϕ^{**} is an isomorphism if and only if ϕ is an isomorphism, $A[[X]]$ is prime in $R[[X]]$ if and only if A is prime in R, and if A is maximal in R, then $A[[X]] + (X)$ is the unique maximal ideal of $R[[X]]$ containing $A[[X]]$.

(2.1) ANSWER TO QUESTION 1A. *If $f = \sum_{i=0}^n f_iX^i \in R[X]$, then f is a zero divisor in $R[X]$ if and only if there is a nonzero element r in R such that $rf_i = 0$ for each i between 0 and n.*

This result is due to N. H. McCoy [42]. Half of the result is obvious; f is a zero divisor in $R[X]$ if the condition given is satisfied. W. R. Scott [50] has given an elegant inductive proof to establish necessity of this condition. It is interesting to note that a proof of McCoy's theorem can be based on a result established independently by R. Dedekind [14] and F. Mertens [43] in 1892; we hasten to add, however, that this result, which Krull in [37, p. 128] calls the *Hilfssatz von Dedekind-Mertens*, is a much deeper theorem than McCoy's theorem on zero divisors. The following statement is a general form of the Dedekind-Mertens Lemma given by J. Arnold and the author in [8, p. 559] (see also [46], [22], and [24, §24]).

(2.2) *Assume that S is a subring of the commutative ring T, and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over T. If $h \in T[\{X_\lambda\}]$, denote by A_h the S-submodule of T generated by the coefficients of h. If $f, g \in T[\{X_\lambda\}]$, then there is a positive integer k such that*

$$A_f^{k+1}A_g = A_f^k A_{fg}.$$

To obtain a proof of McCoy's Theorem from the Dedekind-Mertens Lemma, we consider the case where $S = T = R$. If g is a nonzero element of $R[X]$ such that $fg = 0$, then either $A_fA_g = (0)$ or $A_fA_g \neq (0)$. In the first case, each nonzero coefficient of g annihilates each coefficient of f. In the second, we choose $t > 1$ such that $A_f^tA_g = (0)$, while $A_f^{t-1}A_g \neq (0)$; each nonzero element of $A_f^{t-1}A_g$ annihilates each coefficient of f, and hence the proof is complete.

If D is the ring of algebraic integers in a finite algebraic number field and if $f \in D[X]$, then the ideal A_f of D generated by the coefficients of f has classically been called *the content of f*. A well known theorem in algebraic number theory [41, p. 68] states that if D is the ring of algebraic integers in a finite algebraic number field, then the content of the product of two elements of $D[X]$ is the product of their contents — that is, $A_{fg} = A_fA_g$ for all $f, g \in D[X]$. The author [22] and H. Tsang [56] have proved independently that among integral domains J with identity, Prüfer domains (the reader unfamiliar with the theory of Prüfer domains may consult Chapter IV of [24]) are characterized by the property that $A_{fg} = A_fA_g$ for all $f, g \in J[X]$. Tsang has also investigated the condition $A_{fg} = A_fA_g$ over a commutative ring R with identity; she calls a ring R with this property a *Gaussian ring*.

If $r \in R$, we say that r is *regular* if r is not a zero divisor in R. An ideal A of R is *regular* if A contains a regular element of R. One consequence of (2.1) is that f is not a zero divisor in $R[X]$ if the content of f is a regular ideal of R. In particular, a *primitive polynomial* over R — that is, a polynomial with content R — is not a zero divisor in $R[X]$. (The reader is warned that the terms *regular element* and *primitive polynomial* are used in several different ways in ring theory; see, for example, [33, p. 7] and [57, p. 91].) The sets $S_p = \{f \in R[X] \mid A_f = R\}$ and $S_r = \{f \in R[X] \mid A_f$ is a regular ideal of $R\}$ are regular multiplicative systems in $R[X]$, and $R[X]_{S_p} = R[X]_{S_r}$. The ring $R[X]_{S_p}$, which was introduced by Krull in [39], and which M. Nagata in [44, p. 18] denotes by $R(X)$, has proved to be of interest in several different areas of commutative algebra; see [13, p. 62], [3], [32, p. 218], [31, p. 757].

Another consequence of (2.1) is the following result.

(2.3) *If Q is a P-primary ideal of R, then $Q[X]$ is $P[X]$-primary in* $R[X]$.

Proof. By definition, $Q[X]$ is a primary ideal of $R[X]$ if and only if each zero divisor of $R[X]/Q[X]$ is nilpotent. Since $R[X]/Q[X] \cong (R/Q)[X]$, it is therefore sufficient to prove (2.3) in the case where

$Q = 0$. In this case, if $f = \sum_0^n f_i X^i$ is a zero divisor in $R[X]$, then there is a nonzero element r of R such that $rf_i = 0$ for each i. Hence each f_i is a zero divisor in R, and because (0) is a primary ideal of R, each f_i is nilpotent so that $f = \sum_0^n f_i X^i$ is also nilpotent. This proves that $Q[X]$ is a primary ideal of $R[X]$. Since $Q[X]$ is contained in $P[X]$, a prime ideal, it follows that $\sqrt{Q[X]}$ is contained in $P[X]$. But since $\sqrt{Q} = P$, it is clear that $P[X]$ is contained in $\sqrt{Q[X]}$, so that $P[X] = \sqrt{Q[X]}$, and $Q[X]$ is $P[X]$-primary.

(2.4) Partial Answer to Question 1B. *If $f = \sum_{i=0}^{\infty} f_i X^i \in R[[X]]$, then f may be a zero divisor in $R[[X]]$ although one of the coefficients of f is a unit of R. If R is Noetherian, then the analogue of* (2.1) *is valid; namely, f is a zero divisor in $R[[X]]$ if and only if there is a nonzero element r in R such that $rf_i = 0$ for each i.*

The results of (2.4) were proved by D. Fields in [18]. To establish the first assertion of (2.4), Fields takes R to be $S[Y, \{X_i\}_{i=0}^{\infty}]/A$, where S is a commutative ring with identity, $\{Y\} \cup \{X_i\}_0^{\infty}$ is a set of indeterminates over S, and A is the ideal of $S[Y, \{X_i\}_0^{\infty}]$ generated by the set $\{X_0 Y\} \cup \{X_i - X_{i+1} Y\}_{i=0}^{\infty}$. If $y = Y + A$ and if $x_i = X_i + A$ for each i, then the polynomial $f = y - X \in R[X]$ has a unit coefficient, but f is a zero divisor in $R[[X]]$ since $fg = 0$, where $g = \sum_{i=0}^{\infty} x_i X^i \neq 0$. To prove the assertion of (2.4) concerning Noetherian rings, we need some information on Question 2B, and hence we delay our further remarks on the remainder of (2.4).

It is easy to prove that an element $f = \sum_0^n f_i X^i$ of $R[X]$ is nilpotent if and only if each f_i is nilpotent in R. A rather sophisticated proof of this result uses the fact that f is nilpotent if and only if $f \in \sqrt{(0)} = \bigcap_\alpha P_\alpha$, where $\{P_\alpha\}$ is the family of prime ideals of $R[X]$. Moreover, since $P_\alpha \supseteq (P_\alpha \cap R)[X]$, where $P_\alpha \cap R$ is prime in R and $(P_\alpha \cap R)[X]$ is prime in $R[X]$, it follows that $\sqrt{(0)} = \bigcap_\beta (Q_\beta[X]) = (\bigcap_\beta Q_\beta)[X] = N[X]$, where $\{Q_\beta\}$ is the set of prime ideals of R, and hence $N = \bigcap_\beta Q_\beta$ is the set of nilpotent elements of R. (We remark that S. A. Amitsur in [2] has investigated the topic of radicals of polynomial rings over a noncommutative ring.)

It is interesting to examine the proof in the preceding paragraph in seeking an answer to Question 2B. Thus, if $\{Q_\beta\}$ is the family of prime ideals of R, then $\{Q_\beta[[X]]\}$ is a family of prime ideals of $R[[X]]$ and $\bigcap_\beta (Q_\beta[[X]]) = (\bigcap_\beta Q_\beta)[[X]] = N[[X]]$. Hence $\sqrt{(0)} \subseteq N[[X]]$ — that is, if $f = \sum_0^{\infty} f_i X^i$ is nilpotent, then each f_i is nilpotent. In attempting to prove the converse, we encounter the fact that for an ideal A of $R[[X]]$, A need not contain $(A \cap R)[[X]]$, although A does contain $(A \cap R) \circ R[[X]]$, the extension of $A \cap R$ to $R[[X]]$. Thus, if B is an ideal of R, the containment $B \circ R[[X]] \subseteq B[[X]]$ may be proper. In fact, $B \circ R[[X]] = \{f \in R[[X]] \mid A_f$, the ideal of R generated by the coefficients of f, is contained in a finitely generated ideal contained in $B\}$. Therefore, $B \circ R[[X]] = B[[X]]$ if B is finitely generated. On the other hand, if B is countably generated, say $B = (\{b_i\}_0^\infty)$, but not finitely generated, then $\sum_0^\infty b_i X^i$ is in $B[[X]]$, but not in $B \circ R[[X]]$. Upon examination of our determination of the set of nilpotent elements of $R[X]$ we see, however, that the possible proper containment $(Q_\beta \cap R) \circ R[[X]] \subset (Q_\beta \cap R)[[X]]$ is no real problem to us in answering Question 2B *if* the ideal $(Q_\beta \cap R) \circ R[[X]]$ is prime in $R[[X]]$ for each prime ideal Q_β of $R[[X]]$. Unfortunately (from one point of view), this condition fails, and in fact, $f = \sum_0^\infty f_i X^i$ need not be nilpotent if each f_i is nilpotent. (In Example 2 of [18], Fields shows that f need not be nilpotent even if there is a positive integer m such that $f_i^m = (0)$ for each i.) We state in (2.5) the most complete answer we know to Question 2B; (2.5) contains as a special case the answer to Question 2A that we have already derived.

(2.5) Partial Answer to Question 2B. *If $f = \sum_0^\infty f_i X^i$ is a nilpotent element of $R[[X]]$, then each f_i is nilpotent — that is, $A_f \subseteq N$, the nil radical of R. If A_f is nilpotent of order k, then $f^k = 0$; in particular, if A_f is finitely generated, then f is nilpotent. If R has nonzero characteristic, then f is nilpotent if and only if there is a positive integer m such that $f_i^m = 0$ for each i. In general, nilpotence of f does not imply nilpotence of A_f, and the condition $f_i^m = 0$, for a fixed positive integer m and for each i, does not imply that f is nilpotent.*

Proofs of the assertions of (2.5) can be found in [18]. In [5], J. Arnold proves that f is nilpotent if there is a positive integer k such that $b^k = 0$ for each element b in A_f; this statement generalizes the result that f is nilpotent if A_f is nilpotent. As corollaries to (2.5), we obtain the following results.

(2.6) *If A is an ideal of R and if $B = \sqrt{A}$, then $\sqrt{(A[[X]])} \subseteq B[[X]]$; if A contains a power of B, then $\sqrt{(A[[X]])} = B[[X]]$. If A is B-primary and if A contains a power of B, then $A[[X]]$ is $B[[X]]$-primary.*

(2.7) *Assume that the ring R is Noetherian. An element f of $R[[X]]$ is nilpotent if and only if each coefficient of f is nilpotent. If A is an ideal of R with radical B, then $B[[X]]$ is the radical of $A[[X]]$; if A is B-primary, then $A[[X]]$ is $B[[X]]$-primary.*

We prove the assertion in (2.6) concerning primary ideals. By passage to the residue class ring $R[[X]]/A[[X]]$, it suffices to prove that if (0) is B-primary in R, where B is nilpotent, then the zero ideal of $R[[X]]$ is $B[[X]]$-primary. To do so, we prove that each element f of $R[[X]] - B[[X]]$ is not a zero divisor in $R[[X]]$. Some coefficient of f is not in B, and hence is a regular element of R. We choose the integer i so that $f_0, \cdots, f_i$ are in B, while f_{i+1} is not in B. Then $f_0, \cdots, f_i$ are nilpotent, and hence $g = f_0 + f_1X + \cdots + f_iX^i$ is nilpotent—say $g^r = 0$. We write f as $g + X^{i+1}h$, where the constant term of h is f_{i+1}, a regular element of R. It is clear that h is a regular element of $R[[X]]$, and hence h^r is also regular. But f divides $g^r + (X^{i+1}h)^r = X^{(i+1)r}h^r$, and consequently, f is regular in $R[[X]]$. This completes the proof of (2.6). We observe that the hypothesis 'A contains a power of B' has been used in the proof only to conclude that $B[[X]] = \sqrt{(A[[X]])}$; the conclusion that no element of $R[[X]] - B[[X]]$ is a zero divisor with respect to $A[[X]]$ (that is, $A[[X]] : (f) = A[[X]]$ for each f in $R[[X]] - B[[X]]$) depends only upon the assumption that A is B-primary.

We return to (2.4) and to Question 1B. We first prove the assertion of (2.4) concerning the analogue, in $R[[X]]$, of McCoy's Theorem for a Noetherian ring R. Thus we let $(0) = \bigcap_1{}^n Q_i$ be an irredundant primary decomposition of (0) in R, where Q_i is P_i-primary. By (2.7), $(0) = \bigcap_1{}^n Q_i[[X]]$ is a primary decomposition of the zero ideal of $R[[X]]$, where $Q_i[[X]]$ is $P_i[[X]]$-primary; it is clear that this intersection is irredundant. Since the rings R and $R[[X]]$ are Noetherian, $\bigcup_1{}^n P_i$ is the set of zero divisors of R and $\bigcup_1{}^n P_i[[X]]$ is the set of zero divisors of $R[[X]]$ [**59**, p. 214]. Thus, if f is a zero divisor in $R[[X]]$, then $f \in P_i[[X]]$ for some i. Since $(0) : P_i \supset (0)$ [**58**, p. 132], there is a nonzero element r of R such that $rP_i = (0)$, and hence $rf = 0$ also.

Much more is known about Question 1B than is included in (2.4). Although McCoy's Theorem does not carry over to arbitrary power series rings, some results about zero divisors of $R[[X]]$ can be obtained by imposing special conditions on the coefficient ring or on the elements of $R[[X]]$ under consideration. For example, our proof of (2.6) shows that if $f = \sum_{j=0}^{\infty} f_jX^j$, where $f_0, \cdots, f_i$ are nilpotent and f_{i+1} is regular, then f is not a zero divisor in $R[[X]]$. More generally, M. O'Malley in [**48**, Thm. 2.1] has proved that if f_t is a unit of R and if $\bigcap_{n=1}^{\infty} (f_0, f_1, \cdots, f_{t-1})^n = (0)$, then f is not a zero divisor in $R[[X]]$.

In [**30**], Gilmer, A. Grams, and T. Parker have conducted an investigation in depth of the area of zero divisors of $R[[X]]$. Their considerations center around the problem of determining sufficient conditions on R in order that McCoy's theorem should be valid in $R[[X]]$; one such condition given in [30] is that R should have zero nil radical. We have already observed that McCoy's theorem in $R[X]$ follows from the Dedekind-Mertens Lemma. Hence the Dedekind-Mertens Lemma does not generalize completely to $R[[X]]$. On the other hand, a special case of Theorem 3.6 of [30] is the following result.

(2.8) *If* $f \in R[[X]]$, *if* $g \in R[X]$, *and if* k *is the number of nonzero coefficients of* g, *then* $A_f^{k-1}A_{fg} = A_f^k A_g$, *where for* $h \in R[[X]]$, A_h *is the abelian group generated by the coefficients of* h.

As an immediate corollary to (2.8), we have the following result on zero divisors in $R[[X]]$.

(2.9) *If* $f \in R[[X]]$, *if* $g \in R[X]$, *and if* $fg = 0$, *then there is a nonzero element* r *in* R *such that* $rf = 0$.

As a final result concerning Question 1B, we cite part of Proposition 2.1 of [30].

(2.10) *If* R *is zero-dimensional and if* $A_f = R$, *then* f *is not a zero divisor in* $R[[X]]$.

The reader should not be misled into the erroneous conclusion that questions concerning power series rings are inherently more difficult than the corresponding questions for polynomial rings. Our Question 3 is a case in point. It is quite easy to prove that the element f = $\sum_{i=0}^{\infty} f_i X^i$ is a unit of $R[[X]]$ if and only if f_0 is a unit of R. The units of $R[X]$ are a bit more difficult to determine, but they too have been completely described. The following result is due to E. Snapper [55 I, p. 683]. (In the introduction to [55 I], Snapper makes the following statement. "The author wishes to express here his obvious indebtedness to Krull's beautiful work. All ideas which can be found in CPI through CPIV have their origin in the papers [3], [4], and [5] of Krull." In the preceding sentence, 'CPI through CPIV' refers to the series of four papers [55] published by Snapper under the titles *Completely Primary Rings* I, II, III, IV; the references [3], [4], [5] are to Snapper's bibliography, not ours. The papers of Krull in question are those listed in our bibliography as [35]. These papers deal principally with primary rings, and indeed Satz 6, page 92, of [35I] is the special case of (2.11) in which R is a primary ring.

(2.11) ANSWER TO QUESTION 3A. *The element* $f = f_0 + f_1X + \cdots + f_nX^n$ *is a unit of* $R[X]$ *if and only if* f_0 *is a unit of* R *and* $f_1, \cdots, f_n$ *are nilpotent.*

Because the proof of (2.11) illustrates a useful technique, we give a proof here. Since the sum of a unit and a nilpotent element is a unit, f is a unit of $R[X]$ if f_0 is a unit of R and $f_1, \cdots, f_n$ are nilpotent. Conversely, if $fg = 1$, where $g \in R[X]$, then we observe that $\phi_P(f)\phi_P(g) = 1$ for each prime ideal P of R; here ϕ_P is the canonical homomorphism of $R[X]$ onto $(R/P)[X]$. Since $(R/P)[X]$ is an integral domain, it follows that $\phi_P(f)$ is a unit of R/P for each prime ideal P of R—thus $f_0 \notin P$, while $f_i \in P$ for each $i \geqq 1$. It follows that f_0 is in no proper prime ideal of R, and hence f_0 is a unit of R, while f_i, for $i \geqq 1$, is in each prime ideal of R, and is therefore nilpotent.

3. **Endomorphisms and automorphisms.** In discussing endomorphisms of $R[X]$ and $R[[X]]$ in this section, we restrict to R-endomorphisms; that is, to endomorphisms ϕ such that $\phi(r) = r$ for each element r in R (for more general considerations, see [23, Th. 4]). An R-endomorphism ϕ of $R[X]$ is uniquely determined by $\phi(X)$—if $\phi(X) = t$, then $\phi(f(X)) = f(t)$ for each $f(X)$ in $R[X]$; moreover, if $t = \sum_{i=0}^{n} t_iX^i$ is an element of $R[X]$, then the map $\phi_t : R[X] \to R[X]$ defined by $\phi_t(f) = f(t)$ is an R-endomorphism of $R[X]$ that sends X onto t. To determine the R-automorphisms of $R[X]$, it therefore suffices to solve the following problems.

(1) Determine necessary and sufficient conditions on t in order that ϕ_t should be onto.
(2) Determine necessary and sufficient conditions on t in order that ϕ_t should be one-to-one.

Since $R[t]$ is the range of ϕ_t, ϕ_t is onto if and only if $R[t] = R[X]$—that is, if and only if $X \in R[t]$. In [23, p. 329], Gilmer proves that ϕ_t is onto if and only if t_1 is a unit of R and t_i, for $i \geqq 2$, is nilpotent. Gilmer's proof that the preceding conditions are sufficient in order that ϕ_t be onto is too complicated to repeat here, but the proof of necessity of these conditions uses the same method that we employed in the proof of (2.11), namely: from the relation $X \in R[t]$, conclude that $\phi_P(X) \in (R/P)[\phi_P(t)]$ for each prime ideal P of R, then use the (well known) fact that the conditions are necessary in the case where the coefficient ring is an integral domain [**59**, p. 30].

Gilmer answers (2) in Theorem 2 of [**23**]: ϕ_t is one-to-one if and only if $t - t_0$ is a regular element of R. We consider this condition to be satisfactory because of McCoy's Theorem referred to previously.

We observe that if ϕ_t is onto, then ϕ_t is one-to-one, for t_1 is a unit of R, and hence $t - t_0$ is a regular element of $R[X]$ by (2.1). By use of (2.11) and the fact that X is a regular element of $R[X]$, we can state our observations in the following form.

(3.1) ANSWER TO QUESTION 4A. *Let $t = \sum_{i=0}^{n} t_i X^i$ be an element of $R[X]$, and let ϕ_t be the R-endomorphism of $R[X]$ which maps $f(X)$ onto $f(t)$ for each element $f(X) \in R[X]$. Then ϕ_t is onto if and only if $(t - t_0)/X$ is a unit of $R[X]$; ϕ_t is one-to-one if and only if $(t - t_0)/X$ is a regular element of $R[X]$. The set of R-automorphisms of $R[X]$ is $\{\phi_t \mid t_1$ is a unit of R and t_i, $i \geqq 2$, is nilpotent$\}$.*

In considering Question 4B, one encounters numerous difficulties that are not present in the case of polynomial rings. For example, if $t = \sum_{i=0}^{\infty} t_i X^i \in R[[X]]$, then it isn't clear that there exists an R-endomorphism of $R[[X]]$ that maps X onto t, and if such an R-endomorphism exists, it isn't clear that such an endomorphism is unique. It is true, of course, that if ϕ is an R-endomorphism of $R[[X]]$ such that $\phi(X) = t$, then $\phi(f(X)) = f(t)$ for each *polynomial* $f(X) \in R[X]$. This leads to the problem of defining, in some "natural" way, $f(t)$, where $f(X) = \sum_0^{\infty} f_i X^i$ is in $R[[X]]$. There certainly is an obvious way to begin.

$$
\begin{aligned}
f_0 &= f_0 \\
f_1 t &= f_1 t_0 + f_1 t_1 X + f_1 t_2 X^2 + \cdots \\
f_2 t^2 &= f_2 t_0^2 + 2 f_2 t_0 t_1 X + f_2(2 t_0 t_2 + t_1^2) X^2 + \cdots \\
&\vdots
\end{aligned}
$$

A straightforward calculation shows that for each positive integer j, the coefficient of X^i, for $i \leqq j$, has the form $a_{j0} + a_{j1} t_0 + \cdots + a_{j,j-i} t_0^{j-i}$, where each a_{jk} is in R. Moreover, if $i \leqq j < m$, then $a_{jk} = a_{mk}$ for $0 \leqq k \leqq j - i$. It follows that there is a natural way to define $f_0 + f_1 t + f_2 t^2 + \cdots$ if, for example, t_0 is nilpotent. More generally, what seems to be needed in order to be able to define the sum $f_0 + f_1 t + f_2 t^2 + \cdots$ is a topology on R in which each sequence of the form $r_0, r_0 + r_1 t_0, r_0 + r_1 t_0 + r_2 t_0^2, \cdots$ converges. This leads us to a consideration of a topological ring — that is, a ring R with a topology $\mathcal{T}$ such that the operations of addition and multiplication, when considered as functions from $R \times R$ into R, are continuous (here $R \times R$ has the product topology $\mathcal{T} \times \mathcal{T}$). Because we wish to focus our attention upon Question 4B, we shall be very utilitarian and quite restrictive in our treatment of topological rings at this point; for more general treatments, see [11] or [**60**, Chap. VIII] or for a treatment directly related to Question 4B, [47, §3]. If $b \in R$, then the family $\mathcal{G} = \{\phi\} \cup \{r + (b^n) \mid r \in R, n \in Z\}$ of subsets of R is closed under finite intersection, and hence is a basis for a topology $\mathcal{T}_b$ on R; moreover, $\mathcal{T}_b$ satisfies the first axiom of countability. Under the topology $\mathcal{T}_b$, R is a topological ring; $\mathcal{T}_b$ is a Hausdorff topology if and only if $\bigcap_{n=1}^{\infty} (b^n) = (0)$. Each sequence $\{r_0, r_0 + r_1 b, r_0 + r_1 b + r_2 b^2, \cdots\}$ is a Cauchy sequence in the (b)-adic topology, and hence if (R, $\mathcal{T}_b$) is a complete Haudorff space, then the sequence $\{r_0 + r_1 b + \cdots + r_i b^i\}_{i=0}^{\infty}$ has a unique limit in R, which we denote by $\sum_{i=0}^{\infty} r_i b^i$. Returning to Question 4B, we see that if R is a complete Hausdorff space in the (t_0)-adic topology, then for each element $f = \sum_0^\infty f_i X^i$ in $R[[X]]$, we can define the sum $f_0 + f_1 t + \cdots$ to be a uniquely determined element of $R[[X]]$; we denote this element by $f(t)$. Is the mapping ϕ_t: $f(X) \to f(t)$ an R-endomorphism of $R[[X]]$? If so, is it the unique R-endomorphism of $R[[X]]$ mapping X onto t? O'Malley proved [47, pp. 66–67] that the answer to each of these questions is affirmative. (We emphasize that ϕ_t is defined only if R is a complete Hausdorff space in the (t_0)-adic topology.) That brings us to the problem of determining conditions under which ϕ_t is onto and/or one-to-one. Once more, a solution (complete for onto, and partial for one-to-one) is contained in [47, p. 74]: The mapping ϕ_t is onto if and only if t_1 is a unit of R; if the initial coefficient of $t - t_0$ is regular in R, then ϕ_t is one-to-one; in particular, ϕ_t is one-to-one if ϕ_t is onto, and hence ϕ_t is an automorphism of $R[[X]]$ if and only if t_1 is a unit of R.

The topological considerations of the previous paragraph lead to the following question which, at first glance, seems a bit ambitious.

(*) *If there exists an R-endomorphism ϕ of $R[[X]]$ mapping X onto $t = \sum_0^\infty t_i X^i$, must R be a complete Hausdorff space in the (t_0)-adic topology?*

Under the hypothesis of (*), O'Malley [47] proved that t_0 belongs to the Jacobson radical of R and that R is complete in the (t_0)-adic topology *if* the (t_0)-adic topology on R is a Hausdorff topology. Therefore, we modify (*) to (**).

(**) *If there exists an R-endomorphism ϕ of $R[[X]]$ mapping X onto $t = \sum_0^\infty t_i X^i$, does it follow that $\bigcap_{n=1}^{\infty} (t_0{}^n) = (0)$?*

Since t_0 belongs to the Jacobson radical J of R, the answer to (**) is affirmative if $\bigcap_{n=1}^{\infty} J^n = (0)$; this condition is satisfied, for example, if R is Noetherian [44, p. 12]. O'Malley proved that the answer to (**) is also affirmative if t_0 is regular in R; moreover, if ϕ is an automorphism; then $\bigcap_1{}^\infty(t_0{}^n) = (0)$ if and only if $(t_0)[\bigcap_1{}^\infty(t_0{}^n)] = \bigcap_1{}^\infty(t_0{}^n)$. This leads, of course, to a third form of (*):

(***) *If there is an R-automorphism of* $R[[X]]$ *that maps* X *onto* $\sum_0^\infty t_iX^i$, *does it follow that* $(t_0)[\bigcap_1{}^\infty(t_0{}^n)] = \bigcap_1{}^\infty(t_0{}^n)$?

In [29, p. 18], Gilmer gives an example of a commutative ring R with identity containing an element b such that $(b)[\bigcap_1{}^\infty (b^n)] \subset \bigcap_1{}^\infty (b^n)$. Then by applying Theorem 2.3 of [29], which we presently cite at (3.2), we obtain a negative answer to (***).

(3.2) *Assume that the commutative ring R with identity contains an element* a_0 *such that* $a_0[\bigcap_1{}^\infty(a_0{}^n)] \subset \bigcap_1{}^\infty(a_0{}^n)$. *If* $\alpha = \sum_0^\infty a_iX^i$, *where* a_1 *is a unit of R, then the ring* $S = R[[X]]/(\alpha)$ *has the property that* $S[[Y]]$ *admits an S-automorphism mapping Y onto an element* $s_0 - Y$, *where* $\bigcap_1{}^\infty s_0{}^nS \neq (0)$.

We give (3.3) as a summary statement of our results on Question 4B.

(3.3) PARTIAL ANSWER TO QUESTION 4B. *Let R be such that either R is Noetherian or R is an integral domain or* $\bigcap_{n=1}^\infty (r^n) = (0)$ *for each element r in the Jacobson radical of R. If* $t = \sum_0^\infty t_iX^i$ *is an element of* $R[[X]]$, *then there exists an R-endomorphism* ϕ *of* $R[[X]]$ *sending X onto t if and only if R is a complete Hausdorff space in the* (t_0)*-adic topology. If such a* ϕ *exists, it is unique; moreover,* ϕ *is an automorphism of* $R[[X]]$ *if and only if* t_1 *is a unit of R.*

In Theorem 3.2 of [29], Gilmer proves that there is an R-automorphism ϕ of $R[[X]]$ mapping X onto t if and only if $R[[X]] = R \oplus tR[[X]]$. While this result is easy to state and to understand, it is not easy to apply. O'Malley and C. Wood [49] have also given some equivalent topological conditions for the existence of such an R-automorphism ϕ.

4. **Dimension theory.** Let S be a commutative ring. If $P_0 \subset P_1 \subset \cdots \subset P_n$ is a finite chain of proper prime ideals of S, we say that this chain has length n. If S has no proper prime ideal, we say that S *has dimension* -1; otherwise, *the* (Krull) *dimension of* S, which we write as dim S, is defined to be the supremum of the set of lengths of finite chains of proper prime ideals of S. Thus, a field has dimension 0, $Z/(n)$ has dimension 0 for each integer $n > 1$, and a principal ideal domain has dimension 0 or 1.

If $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of proper prime ideals of the commutative ring R with identity, then $P_0[X] \subset P_1[X] \subset \cdots \subset P_n[X] \subset$

$P_n[X] + (X)$ is a chain of proper prime ideals of $R[X]$ and $P_0[[X]] \subset \cdots \subset P_n[[X]] \subset P_n[[X]] + (X)$ is a chain of proper ideals of $R[[X]]$. Therefore, $\dim R[X] \geqq \dim R + 1$ and $\dim R[[X]] \geqq \dim R + 1$, and in particular, $R[X]$ and $R[[X]]$ are infinite-dimensional if R is infinite-dimensional. Is the converse also true? Equivalently, if R is finite-dimensional, are $R[X]$ and $R[[X]]$ finite-dimensional? We discuss the polynomial and power series cases of these questions separately.

If $\dim R = n < \infty$, then it is clear that the dimension of $R[X]$ is finite if and only if there is a uniform bound on the length of a chain of prime ideals of $R[X]$ lying over a fixed prime ideal of R. We observe that such a bound must be at least one, for if P is a proper prime ideal of R, then $P[X] \subset P[X] + (X)$ is a chain of prime ideals of $R[X]$, each lying over P in R. A result that is fundamental in relating properties of R to those of $R[X]$ is that a chain of prime ideals of $R[X]$ lying over a fixed proper prime P of R has length at most one. To prove this statement, we observe that since each prime ideal of $R[X]$ lying over P contains the prime ideal $P[X]$, is suffices, by passage to R/P and $R[X]/P[X]$, to prove the following statement.

($\checkmark$) *If D is an integral domain with identity, and if P and A are nonzero ideals of $D[X]$ such that P is prime, $P \subset A$, and $P \cap D = (0)$, then $A \cap D \neq (0)$.*

To prove ($\checkmark$), we let $N = D - \{0\}$. The ideal P extends to a nonzero proper prime ideal $P \circ D[X]_N$ of the quotient ring $(D[X])_N = D_N[X]$, where $D_N = K$ is the quotient field of D. But $K[X]$ is a Euclidean domain, hence a principal ideal domain, and therefore $P \circ D[X]_N$ is a maximal ideal of $D[X]_N$. Since $P = P \circ D[X]_N \cap D$, it then follows that $P \circ D[X]_N \subset A \circ D[X]_N$, $A \circ D[X]_N = D[X]_N$, and hence A meets N nontrivially—that is, $A \cap D \neq (0)$ [59, Chap. IV, § 8].

As we have already observed, the following statement ($\checkmark\checkmark$) follows from ($\checkmark$); cf. [24, Prop. 25.1, p. 340].

($\checkmark\checkmark$) *If $P_1 \subset P_2 \subset P_3$ is a chain of three proper prime ideals of $R[X]$, then $P_1 \cap R \subset P_3 \cap R$.*

From ($\checkmark\checkmark$), it follows easily that $\dim R[X] \leqq 2(\dim R) + 1$, and consequently, if $\dim R = n$, then $n + 1 \leqq \dim R[X] \leqq 2n + 1$. Can these bounds be improved upon? In general, no. In [53, p. 605], A. Seidenberg proves that if n and k are nonnegative integers such that $n + 1 \leqq k \leqq 2n + 1$, then there is an integral domain D with identity such that $\dim D = n$ and $\dim D[X] = k$. On the other hand, for some special classes of rings R, $\dim R[X] = \dim R + 1$. For example, Krull [**40**, Satz 13, p. 376] establishes this equality for a Noetherian ring R, and Seidenberg [**53**, Th. 4, p. 606] proves that $\dim R[X] = \dim R + 1$ if R is a Prüfer domain. An easy proof that $\dim R[X] = \dim R + 1$ if R is Noetherian can be based on an extension of Krull's principal ideal theorem [**36**, pp. 11, 12], [**38**, p. 220]; see, for example, [**24**, p. 343]. Moreover, this proof easily extends to power series rings over a Noetherian ring [**19**, p. 603]. We summarize our results on the dimension of $R[X]$ in the following result.

(4.1) Partial Answer to Question 5A. *The rings R and $R[X]$ are simultaneously finite-dimensional. If* $\dim R = n < \infty$, *then* $n + 1 \leqq \dim R[X] \leqq 2n + 1$; *these bounds are, in general, the best possible. If the ring R is Noetherian or a Prüfer domain, then* $\dim R[X] = \dim R + 1$.

Although we have labeled (4.1) as a "partial" answer to Question 5A, this is primarily due to the fact that more is known about the question than we have stated in (4.1) (see, for example, § 25 of [**24**]); no one seems to be up in arms about the incomplete state of knowledge concerning Question 5A.

For several years, the question as to whether finite-dimensionality of R implies finite-dimensionality of $R[[X]]$ was open. The question has recently been answered in the negative by Arnold [**5**]. In fact, Arnold proves that $V[[X]]$ is infinite-dimensional if V is a rank one nondiscrete valuation ring. Arnold's work follows an earlier paper [**19**] of Fields, who proved that $\dim V[[X]] = \dim V + 1$ if V is a discrete valuation ring of finite rank. Before Arnold's paper [**5**] appeared, Fields had proved that $\dim V[[X]] \geqq 3$ if V is a rank one nondiscrete valuation ring, and under the same hypothesis on V, Arnold and J. Brewer [**7**] had proved that $\dim V[[X]] \geqq 4$. A reason, of course, for considering the case of valuation rings (or Prüfer domains) is the fact, already mentioned, that $\dim D[X] = \dim D + 1$ if D is a Prüfer domain. Incidentally, Arnold has proved that Fields' result concerning discrete valuation rings does not carry over to the global case. More specifically, if D is an almost Dedekind domain that is not Dedekind (an *almost Dedekind domain* is an integral domain J with identity such that J_M is a discrete valuation ring of rank at most 1 for each maximal ideal M of J [**20**], [**24**, § 29]), then $\dim D = 1$ and $\dim D[[X]] = \infty$ [**5**, Example 2].

An important condition that has arisen in Arnold's work on the dimension of $R[[X]]$ is what he calls the *SFT-condition,* which we proceed to define. If A is an ideal of the ring S, then A is *an ideal of strong finite type* (an SFT-ideal) if there is a finitely generated ideal B contained in A and a positive integer k such that $a^k \in B$ for each element a in A; S is *a ring of strong finite type* (an SFT-ring) if each ideal of S is an SFT-ideal. In [**5**], Arnold proves that $\dim R[[X]] = \infty$ if R is not an SFT-ring; for R a Prüfer domain, he proves the converse in [**6**] ([**6**] is a beautiful paper, but it is far from elementary)—in fact, if R is a Prüfer domain that is an SFT-ring, then $\dim R[[X]] = \dim R + 1$. Once again we summarize our results in a single statement.

(4.2) Partial Answer to Question 5B. *If R is infinite-dimensional, then so is $R[[X]]$. If* $\dim R = n < \infty$, *then* $\dim R[[X]] = n + 1$ *if R is Noetherian or if R is a Prüfer domain of strong finite type. If R is not of strong finite type, then* $\dim R[[X]] = \infty$.

Several questions concerning the dimension of $R[[X]]$ remain open. Among these, we mention the following.

(1) If R is a finite-dimensional SFT-ring, is $R[[X]]$ finite-dimensional?

(2) Is it possible for the dimension of $R[[X]]$ to be finite, but distinct from $\dim R + 1$?

(3) If R is a finite-dimensional Krull domain (see [**12**] and [**24**, § 35]), is $R[[X]]$ finite-dimensional?

Before leaving Question 5, we ask: Why does the proof that finite-dimensionality of R implies finite-dimensionality of $R[X]$ fail to generalize to power series rings? If we examine the material immediately preceding and following ($\vee$), we see that a couple of problems arise. One is the fact, already encountered, that for a prime ideal P of R, $P \circ R[[X]]$ need not be prime in $R[[X]]$. The second problem is that $R[[X]]_N$ may be properly contained in $R_N[[X]]$ for a regular multiplicative system N in R. In fact, if R is an integral domain and if $N = R - \{0\}$, then Gilmer in [21] proves that $R[[X]]_N = R_N[[X]]$ if and only if each sequence of nonzero ideals of R has nonzero intersection. In [**54**], P. Sheldon has extended this result to more general multiplicative systems. In particular, $(Z[[X]])_{Z-\{0\}}$ is properly contained in $Q[[X]]$, where Q is the field of rational numbers, for $1 + (1/2)X + (1/4)X^2 + \cdots + (1/2^n)X^n + \cdots$ is in the second set, but not in the first.

5. **Extensions and generalizations.** Extensions of the results discussed in the three preceding sections—at least for commutative rings—are obtained by considering polynomial and power series rings in an arbitrary set $\{X_\lambda\}$ of indeterminates over R and/or dropping the

assumption that R contains an identity element. For power series rings, one immediately comes upon the fact that three different "rings of formal power series" in an infinite set of indeterminates over R have been considered in the literature [28, p. 543]; moreover, [45] contains a variant of each of the three rings mentioned in [28]. Actually, this causes very few problems in regard to Questions 1–5, but for different reasons on different questions.

McCoy's theorem on zero divisors carries over to arbitrary polynomial rings over an arbitrary commutative ring [24, p. 337]; one proof can be obtained from an extension of the Dedekind-Mertens Lemma to this case. The analogue of McCoy's theorem is valid for power series rings $S[[\{X_\lambda\}]]$, where S is an Noetherian ring (not necessarily with identity) and $\{X_\lambda\}$ is finite. The paper [30] previously referred to contains some contribution to Question 1 in the case of power series rings in more than one variable over a non-Noetherian ring; we do not elaborate on those results here.

A polynomial $f \in S[\{X_\lambda\}]$ is nilpotent if and only if each coefficient of f is nilpotent. The first two conclusions in (2.5) do not depend on the hypothesis that R contains an identity or that the set of indeterminates in question has cardinality 1. Moreover, an examination of the proof of Theorem 1 of [18] yields a partial generalization of the statement in (2.5) concerning a ring of nonzero characteristic.

It is true in general that a polynomial $f \in R[\{X_\lambda\}]$ over a commutative ring with identity is a unit if and only if the constant term of f is a unit of R and each other coefficient of f is nilpotent; our proof of (2.11) establishes this result. The answer to Question 3B also carries over to arbitrary power series rings, namely: If $f \in R[[\{X_\lambda\}]]$, then f is a unit if and only if the constant term of f is a unit of R.

Gilmer determined the set of S-automorphisms of $S[X]$, for S a commutative ring containing a regular element, in [23]; we do not give the results of [23] here. The problem of determining the R-automorphisms of $R[\{X_\lambda\}]$, even for $|\{X_\lambda\}| = 2$ and R an algebraically closed field (even the field of complex numbers), is very difficult. A consultation of [1], [16], [17], and [51, § 130] will indicate the difficulties involved. No work on an extension of the results of (3.3) has apparently been attempted.

If $\{X_\lambda\}$ is infinite, then $S[\{X_\lambda\}]$ and $S[[\{X_\lambda\}]]$ are infinite-dimensional. On the other hand, if $|\{X_\lambda\}| = m < \infty$, then except for a change in bounds in (4.1) (in the second sentence, $n + m \leqq \dim R[X_1, \cdots, X_m] \leqq (n+1)(m+1) - 1$, and in the third, $\dim R[X_1, \cdots, X_m] = n + m$), that result carries over to polynomial rings in finitely many indeterminates (see [34, p. 17], [24, § 25], and [53, p. 606]). For R finite-dimensional, there are numerous results and existence theorems (see [34, Chap. III], [52], [53], [4, p. 325], and [10, § 5]) concerning the sequences of integers $n_0 = \dim R$, $n_1 = \dim R[X_1]$, $\cdots, n_k = \dim R[X_1, \cdots, X_k], \cdots$ and $n_1 - n_0, n_2 - n_1, \cdots$. In [10], E. Bastida and Gilmer call the first of these sequences *the dimension sequence of R*; the second is *the difference sequence of R*. P. Jaffard [34, p. 42] has proved that the difference sequence is eventually a constant less than or equal to $n_0 + 1$. For a zero-dimensional ring, only the sequence $0, 1, 2, \cdots$ can be realized as a dimension sequence. For $n_0 = 1$, the possibilities are:

$$1, 2, 3, \cdots$$

$$1, 3, 5, \cdots, 2k+1, 2k+2, 2k+3, \cdots \quad k \geqq 1$$

$$1, 3, 5, \cdots$$

Arnold and Gilmer [9] have recently determined all sequences of positive integers that can be realized as the dimension sequence of a ring; their main result is the following. Let $\mathcal{S}$ be the set of strictly increasing sequences $\{a_i\}_0^\infty$ such that the difference sequence $\{b_i\}_1^\infty$, where $b_i = a_i - a_{i-1}$, is nonincreasing, is bounded above by $a_0 + 1$, and is eventually constant — that is, $a_0 + 1 \geqq b_1 \geqq b_2 \geqq \cdots \geqq b_k = b_{k+1} = \cdots \geqq 1$ for some integer k. For $s_1 = \{a_{1i}\}_0^\infty, \cdots, s_m = \{a_{mi}\}_0^\infty$, define $t = \{t_i\}_0^\infty$ to be the supremum of the finite set $\{s_1, \cdots, s_m\}$ in the cardinal order — that is, $t_i = \max\{a_{1i}, a_{2i}, \cdots, a_{mi}\}$ for each i — and let $\mathfrak{D}$ be the set of all such sequences t, as $\{s_1, \cdots, s_m\}$ ranges over all finite subsets of $\mathcal{S}$. Then $\mathfrak{D}$ is the set of dimension sequences of commutative rings; moreover, each element of $\mathfrak{D}$ is, in fact, the dimension sequence of an integral domain.

No work, *per se*, seems to have been done on the dimension of $S[X_1, \cdots, X_n]$, where S is a commutative ring without identity. But some results on this topic are inherent from other considerations (see, for example, § 3 of [25]).

Other than the result $\dim R[[X_1, \cdots, X_m]] = \dim R + m$, if R is a Noetherian ring with identity, which follows by induction from (4.2), there seems to have been no work to date on the dimension theory of $R[[X_1, \cdots, X_m]]$ for $m > 1$. Of course, there is one trivial observation in this connection — $R[[X_1, \cdots, X_m]]$ is infinite-dimensional if $R[[X_1]]$ is infinite-dimensional.

There are, of course, generalizations of polynomial and power series rings to consider — semigroup rings, graded rings, Rees rings, etc., — but for one survey article, we seem to have said enough already.

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Florida State University, Tallahassee, Florida 32306