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TREES AND ISOLS, PART I

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1. **Introduction.** Let N denote the set $\{0, 1, 2, 3, \cdots\}$ of all natural numbers. By a *tree* we shall here mean the graph of a partial number-theoretic function f having the properties: (1) $\rho f \subseteq \delta f$, where ρf denotes the range of f and δf denotes the domain of f; and (2) $(\forall x)\ [x \in \rho f \Rightarrow (\exists y)(f^y(x) = f^{y+1}(x))]$, where we define $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all $n \in N$ and all $x \in \delta f$. If $f^{-1}(x)$ is finite for all $x \in \rho f$, we say that the tree is *finite-branching.* The reader should note that other authors generally use the term *tree* in senses slightly different from the above. We shall say that a tree is *semicomputable* if it is the graph of a partial recursive function f having properties (1) and (2). (Thus, we call a tree *semicomputable* in place of calling it *recursively enumerable.*) If f is a partial number-theoretic function with properties (1) and (2) and if T is the graph of f, then T is said to be a *non-trivial* tree just in case T has at least one infinite branch; i.e., T is non-trivial provided there exists a one-to-one function $g : N \to N$ such that $f(g(0)) = g(0)$ & $(\forall n)\ [f(g(n+1)) = g(n)]$. (We shall usually identify a branch of T with the set of "nodes" occurring along it; i.e., with ρg where g is a one-to-one function such that $\{\langle g(n), f(g(n))\rangle \mid n \in \delta g\}$ = the branch in question.) Thus, a non-trivial semicomputable tree is the graph of what has elsewhere been called a *special regressing function* (see, for example, [1]).

By a Δ_2 *function* we mean a function $f : N \to N$ such that the graph of f is explicitly definable in both 2-quantifier forms in the arithmetical hierarchy; equivalently, in view of the Kleene-Post theorem, f is $\Delta_2 \Leftrightarrow f$ is recursive in the degree 0^1 of the halting problem. Let T_0 and T_1 be trees, corresponding respectively to partial number-theoretic functions f_0 and f_1; then we say that T_0 is a T_1-*skeleton,* and write $T_0 \dashv T_1$, provided that (a) if α is any infinite set regressed by f_1 then α has exactly one infinite subset β regressed by f_0, and (b) $\delta f_0 \subseteq \delta f_1$ & $(\forall x)[x \in \delta f_0 \Rightarrow f_0(x) \in \hat{f}_1(x)]$, where $\hat{f}_1(x) = \{x, f_1(x), f_1(f_1(x)), \cdots\}$. (In the context of this definition, the words "regressed by f_1" do not carry the requirement that f_1 be partial recursive. Thus, here, "α is an infinite set regressed by f_1" simply means that there is a one-to-one function $g : N \to N$ such that $\rho g = \alpha$ & $f_1(g(0)) = g(0)$ & $(\forall n)[f_1(g(n+1)) = g(n)]$, and similarly for f_0 and β.) If , given $T_0 \dashv T_1$, there is a fixed recursively enumerable set γ such that in (a) we always have $\alpha - \beta = \alpha \cap \gamma$, then

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we say that T_0 is a *uniformly co-enumerable* T_1-skeleton, and we write $T_0 \dashv_+ T_1$. (Naturally, we are interested in the relations $T_0 \dashv T_1$, $T_0 \dashv_+ T_1$ only in case T_1 is non-trivial.) Clearly, both $\dashv$ and $\dashv_+$ are transitive relations. Let T be a tree determined by a partial number-theoretic function f, and let a function $g : N \to N$ be given. We shall say that T is *g-dominant* $\Leftrightarrow_{df} (\forall n)[(n \in \delta f \,\&\, f(n) \neq n \,\&\, n$ lies on at least one infinite branch of $T) \Rightarrow n > g(f(n))]$.

Our basic result is that for every non-trivial semicomputable tree T and every Δ_2 function g there exists a g-dominant semicomputable tree T_g (finite-branching if T is finite-branching) such that $T_g \dashv_+ T$. Various applications to the theory of the regressive isols are exhibited in §3; in particular, the main theorem of [14] is derived as a rather easy corollary. All unexplained (and non-self-explanatory) special notation and terminology appearing in §§ 2 and 3 should be read in accordance with the conventions of [13, §1], with the following exceptions: the notations φ_e and φ_e^s of [13] are here replaced by $\varphi_e{}^1$ and $\varphi_e{}^{1,s}$ respectively, this change being made to eliminate a minor ambiguity from the notational scheme of [13]; also, we stipulate $\varphi_0{}^1 =$ the identity function.

In addition to the applications to be considered in §3 of this paper, there are a number of applications (to be presented in a subsequent article) to the class of *universal regressive isols*. These isols were first studied by Ellentuck, in some unpublished notes circulated by him around 1968. Ellentuck has since obtained a variety of interesting results about such isols, including some which have appeared in [11]. We conclude this introduction by mentioning a theorem which can, as we shall demonstrate in our later article, rather easily be derived with the aid of the procedures used in proving Theorem 2.2 below: let $\mathbf{a}$ be a recursively enumerable degree such that $\mathbf{a}^1 = \mathbf{0}^{11}$, and let $\langle P, \leqq \rangle$ be a countable partial order; then there exists a recursively enumerable family $\mathcal{U}$ of co-simple universal regressive isols, each of degree $\mathbf{a}$, such that (i) the closure of $\mathcal{U}$ under finite sums consists entirely of regressive isols and (ii) $\langle P, \leqq \rangle$ can be order-imbedded into $\mathcal{U}$. Indeed, this can be proven in such a way as to answer, at the same time, a question about $\leqq_{\Lambda^-}, <_{\Lambda^-}$ embeddings indicated on p. 636 of [11]. Ellentuck's work on universal isols contains many other interesting results less closely related to the methodology of the present paper.

2. The basic theorem: existence of g-dominant, semicomputable, uniformly co-enumerable T-skeletons, for all $g \in \Delta_2$.

2.1. Lemma (Shoenfield, [18]). *If g is a Δ_2 function then there exists a recursive function $g_0 : N \times N \to N$ such that $(\forall x)[\lim_{y \to \infty} g_0(x, y)$ exists and is equal to $g(x)]$.*

2.2. THEOREM. *Let g be a Δ_2 function, and let T_1 be a non-trivial semicomputable tree. Then there exists a g-dominant semicomputable tree T_0 such that $T_0 \dashv_+ T_1$. Moreover, T_0 is finite-branching if T_1 is.*

PROOF. Let g_0, a recursive function of two variables, be related to g as in Lemma 2.1. In all that follows, the notations $f^*(x)$ and $\hat{f}(x)$ (for an arbitrary partial number-theoretic function f satisfying (1) and (2) of §1, and for an arbitrary $x \in \delta f$) shall have the meanings assigned them in [13, §1]. Let f_1 be the partial recursive function to which T_1 corresponds as graph. We shall assume, for convenience, that $0 \notin \delta f_1$; this entails no genuine loss of generality. Since T_1 is semicomputable (i.e., since f_1 is partial recursive), there exists a recursive function $\bar{h}$ such that $(\forall n)[\delta\varphi^{1}_{\bar{h}(n)} = \{y \mid y \in \delta f_1 \;\&\; f_1{}^*(y) = n\}]$; that is to say, the "levels" of T_1 form a recursively enumerable sequence of pairwise-disjoint r.e. sets. We shall employ an infinite array $\langle \Lambda_{ij} \rangle_{i=0,j=0}^{\infty}$ of *approximation markers*, an infinite array $\langle \Sigma_{ij} \rangle_{i=0,j=0}^{\infty}$ of *reservation tags*, and (an unlimited supply of copies of) a special *barrier marker* $\mathcal{B}$. The markers Λ_{ij} are given an (sequential) ordering by the rule that $\Lambda_{k\ell}$ *has higher priority than* $\Lambda_{mn} \Leftrightarrow_{df} \pi_2(k, \ell) < \pi_2(m, n)$, where π_2 is a fixed one-to-one recursive function from $N \times N$ onto N; for reasons of technical convenience, we stipulate that $\pi_2(0, 0) = 0$ and that π_2 is increasing separately in each of its variables. Here and throughout the remainder of the paper, we shall denote by W_n the set $\delta\varphi_n{}^1$, $n \in N$. Thus W_n is the n-th recursively enumerable subset of N, in a standard enumeration of the class of all such subsets. Let $\bar{k}$ be a two-place recursive function such that $(\forall n)[\rho\bar{k}_n =_{df} \rho\bar{k}(n, y) = W_{\bar{h}(n)}]$. We are now ready to describe the stage-by-stage construction by means of which T_0 is to be obtained from T_1. In this construction, we shall define T_0 by stages as the graph of a partial recursive function p_0 having the property that $\delta p_0 \subseteq \delta f_1 \;\&\; (\forall x)[x \in \delta p_0 \Rightarrow p_0(x) \in \hat{f}_1(x)]$. We let λ^s_{ij} denote 0 if the marker Λ_{ij} is not attached to any number at the conclusion of stage s; otherwise, we take $\lambda^s_{ij} =$ the position of Λ_{ij} (i.e., the number to which Λ_{ij} is attached) at the end of Stage s. (It will be clear from the construction that no approximation marker ever occupies more than one position at a time, so that λ^s_{ij} is well-defined; moreover, it will be evident that for each fixed pair $\langle i_0, j_0 \rangle$ of numbers the value of $\lambda^s_{i_0 j_0}$ changes at most twice as $s \to \infty$ and that $(\forall s)\,(\forall i)\,(\forall j)\,[\lambda^s_{ij} > 0 \Rightarrow (\exists u)\,(\exists w)\,[\lambda^s_{ij} = \bar{k}(u, w)]]$.) Finally, we let $p_0{}^s$ denote the partial number-theoretic function whose graph is that portion $T_0{}^s$ of T_0 which has been defined by the end of Stage s. (When we have finished describing the construction, it will be obvious that $T_0{}^s$ is a finite set whose exact contents can be determined effectively from s.)

Stage 0. Attach Λ_{00} to $\mathfrak{k}(0,0)$, set $T_0{}^0 = \{\langle \mathfrak{k}(0,0), \mathfrak{k}(0,0)\rangle\}$, and go to Stage 1.

Stage $s+1$. We shall divide our procedure into a series of three steps.

Step A. First, let us suppose that $(\exists e)\,(\exists k)\,(\exists \ell)\ [e \leqq s+1\ \&\ k \leqq s+1\ \&\ \ell \leqq s+1\ \&\ 0 < \lambda^s_{ek}\ \&\ 0 < \lambda^s_{e+1,\ell}\ \&\ \langle \lambda^s_{e+1,\ell}, \lambda^s_{ek}\rangle \in T_0{}^s\ \&\ g_0(\lambda^s_{ek}, s) \geqq \lambda^s_{e+1,\ell}]$. In this event, let $\langle \Lambda_{e_0k_0}, \Lambda_{e_0+1,\ell_0}\rangle$ be the lexicographically least marker-pair $\langle \Lambda_{ek}, \Lambda_{e+1,\ell}\rangle$, in terms of the π_2-ordering of $\langle \Lambda_{ij}\rangle^{\infty}_{i=0,j=0}$, such that e, k, and ℓ satisfy the foregoing condition. (Note that *least* here means *of highest priority*. When our description of the construction is complete, it will be clear that λ^s_{ij} is a recursive function of s, i, and j, and that there exists a recursive function ζ such that $(\forall s)\,[\zeta(s) = \max\{\pi_2(i,j) \mid \lambda^s_{ij} > 0\}]$. Hence the bound $s+1$ on the quantifiers in the statement of the condition can, if we so desire, be dispensed with.) We *attack* $\Lambda_{e_0k_0}$, as follows. Give a barrier marker $\mathcal{B}$ to t_0, where $t_0 = (\mu t)\,[(\exists u)(\lambda^s_{e_0+1,\ell_0} = \mathfrak{k}(t,u))] =$ ***the unique t*** such that $(\exists u)\,[\lambda^s_{e_0+1,\ell_0} = \mathfrak{k}(t,u)]$; and give a $\mathcal{B}$ also to each number m such that either $(\exists q)\,(\exists u)\,[\mathfrak{k}(t_0,u) \in \widehat{p_0{}^s}(\mathfrak{k}(m,q))]$ or $(\exists q)\,(\exists u)\,(\exists w)\,(\exists v)$ [m bears the reservation tag $\sum_{wv}\ \&\ \mathfrak{k}(t_0,u) \in \widehat{p_0{}^s}(\mathfrak{k}(w,q))$]. (Note that the last of these two alternatives includes the case: $(\exists v)$[m bears the reservation tag Σ_{t_0v}].) Now, if z is any one of the numbers which have just been given barrier markers and if r is a member of $\rho\mathfrak{k}_z$ such that $(\exists i)\,(\exists j)\,[r = \lambda^s_{ij}]$, then we remove from r whichever approximation marker Λ_{ij} has been found to be attached to it. Next, take h_0 to be the smallest number h such that $(\forall \ell)$[(ℓ currently bears a reservation tag or a barrier marker, or some element of $\rho\mathfrak{k}_\ell$ currently bears or has formerly borne an approximation marker) $\Rightarrow h > \ell$]. (When our description of the construction is complete, it will be evident that h_0 can be effectively computed from s.) Give h_0 the reservation tag $\Sigma_{\lambda^s_{e_0k_0}, w_0}$, where $w_0 = (\mu w)$ [the tag $\Sigma_{\lambda^s_{e_0k_0}, w_0}$ has not previously been used in the construction]. (When our description is complete, it will be clear that $(n \in \rho p_0{}^s \Rightarrow n \in \delta p_0{}^s)\ \&\ (n \in \delta p_0{}^s \Rightarrow (\exists t)\,(\exists i)\,(\exists j)\,[t \leqq s\ \&\ n = \lambda^t_{ij}])$; hence, we see that no member of $\rho\tilde{\mathfrak{k}}_{h_0}$ can be a member of $\delta p_0{}^s \cup \rho p_0{}^s \cup \{x \mid (\exists t)\,(\exists i)\,(\exists j)\,[t \leqq s\ \&\ i \in N\ \&\ j \in N\ \&\ x = \lambda^t_{ij}]\}$.) Now proceed to Step B.

If, on the other hand, no such triple $\langle e, k, \ell\rangle$ exists, go directly to Step B without altering the situations of any markers or tags.

Step B. If there is no number j such that $(\exists y)\,[\langle j, y\rangle \in T_0{}^s]\ \&\ \neg\,(\exists z)\,[j \neq z\ \&\ \langle z, j\rangle \in T_0{}^s]\ \&\ f_1{}^*(j)$ does not currently bear a $\mathcal{B}$ and no number currently bears a reservation tag of the form Σ_{jv}, we proceed directly to Step C. Otherwise, let j_0 be the smallest j such that $(\exists y)\ [\langle j, y\rangle \in T_0{}^s]\ \&\ \neg\ (\exists z)\ [j \neq z\ \&\ \langle z, j\rangle \in T_0{}^s]\ \&\ f_1{}^*(j)$ does not currently bear a $\mathcal{B}$ and no number currently bears a reservation tag of the form Σ_{jv}. (When the construction has been fully described, it will be clear that neither $\mathcal{B}$'s nor reservation tags are ever *removed*; hence, it will be seen that the word "currently" is not essential in the specification of j_0.) Let m_0 be the smallest number m such that $(\forall n)$ [(n currently bears a reservation tag or a barrier marker, or some element of ρk_n currently bears or has formerly borne an approximation marker) $\Rightarrow m > n$]. (Thus, we here specify m_0 exactly as we did h_0 in Step A.) Give m_0 the reservation tag $\Sigma_{j_0 v_0}$ where $v_0 = (\mu v)\ [\Sigma_{j_0 v}$ has not previously been used in the construction]; then proceed to Step C.

Step C. We consider $\rho k_{(s+1)_0}$. If $(s + 1)_0$ currently bears a $\mathcal{B}$ or if $(s + 1)_0$ does not currently bear any reservation tag, then, provided $(s + 1)_0 > 0$, we set $T_0{}^{s+1} = T_0{}^s$ and proceed to Stage $s + 2$. If $(s + 1)_0 = 0\ \&\ \neg\ (\exists w)\ [w \leqq s + 1\ \&\ k(0, w) \notin \delta p_0{}^s]$, then again we set $T_0{}^{s+1} = T_0{}^s$ and go on to Stage $s + 2$. If $(s + 1)_0 = 0\ \&\ (\exists w)\ [w \leqq s + 1\ \&\ k(0, w) \notin \delta p_0{}^s]$, we let $w_1 = (\mu w)\ [w \leqq s + 1\ \&\ k(0, w) \notin \delta p_0{}^s]$; then we define $T_0{}^{s+1} = T_0{}^s \cup \{\langle k(0, w_1), k(0, w_1)\rangle\}$, attach Λ_{0w_1} to $k(0, w_1)$, and proceed to Stage $s + 2$. Finally, suppose that $(s + 1)_0 > 0$ and $(s + 1)_0$ does not currently bear a $\mathcal{B}$, but $(s + 1)_0$ *does* currently bear some reservation tag. Then, as will be clear when our account of the construction is complete, there exists *just one* number y, say y_0, such that $(\exists u)\ [(s + 1)_0$ bears the tag $\Sigma_{yu}]$. If there is no number $n \leqq s + 1$ such that $k((s + 1)_0, n) \notin \delta p_0{}^s\ \&\ y_0 \in \hat{f}_1(k((s + 1)_0, n))$, we set $T_0{}^{s+1} = T_0{}^s$ and go on to Stage $s + 2$. Otherwise, let $n_0 = (\mu n)[n \leqq s + 1\ \&\ k((s + 1)_0, n) \notin \delta p_0{}^s\ \&\ y_0 \in \hat{f}_1(k((s + 1)_0, n))]$. Let z_0 be the uniquely determined number z such that $(\exists v)\ [y_0 = \lambda^s_{zv}]$; when we have concluded our description of the construction, it will be plain that there exists exactly one such z corresponding to y_0. (The existence of *at least one* such z is assured, since otherwise either $(s + 1)_0$ would currently bear a $\mathcal{B}$ in virtue of an application of Step A or else no reservation tag of the form $\Sigma_{y_0 u}$ would as yet have been given to $(s + 1)_0$; *uniqueness* is simply a matter of no two approximation markers ever occupying the same position.) Attach Λ_{z_0+1, q_0} to $k((s + 1)_0, n_0)$, where $q_0 = (\mu q)\ [\Lambda_{z_0+1, q}$ has not previously been used in the construction]. *Set* $T_0{}^{s+1} = T_0{}^s \cup \{\langle k((s + 1)_0, n_0), y_0\rangle\}$ and go to Stage $s + 2$.

That completes our description of the construction of the sequence $\langle T_0{}^s\rangle_{s=0}^{\infty}$; the finishing touch is to set $T_0 = \bigcup_s T_0{}^s$. It is obvious from the construction that T_0, so defined, is the graph of a partial recursive function p_0 such that $\delta p_0 \subseteq \delta f_1$; moreover, it is clear that $(\forall x)$ $[x \in \delta p_0 \Rightarrow p_0(x) \in \hat{f}_1(x)]$ & $\rho p_0 \subseteq \delta p_0$. Thus T_0 is a semicomputable tree, since T_1 is. Furthermore, from what has just been said we see that T_0 satisfies condition (b) for being a T_1-skeleton. Next, we observe that $\lim_{s\to\infty}\lambda^s_{ij}$ exists for all pairs $\langle i,j\rangle$. Indeed, since an approximation marker other than Λ_{00} can be attached only during Step C of Stage $s+1$ (for some s) and then *only if it has not previously been used in the construction*, and since (obviously) Λ_{00} is never detached after Stage 0, we readily see that the set of all markers Λ_{ij} separates into three mutually exclusive subsets: $M_0 =_{df}\{\Lambda_{ij} \mid \Lambda_{ij}$ is never attached during the construction$\}$; $M_1 =_{df}\{\Lambda_{ij} \mid \Lambda_{ij} = \Lambda_{00}$ or Λ_{ij} becomes permanently attached during Step C of some stage $s+1\}$; and, finally, $M_2 =_{df}\{\Lambda_{ij} \mid \Lambda_{ij}$ is (necessarily *permanently*) removed during Step A of Stage $s+1$, for some $s\}$. Thus, for any fixed pair $\langle i_0,j_0\rangle \neq \langle 0,0\rangle$, the total number of values which $\lambda^s_{i_0j_0}$ assumes as $s\to\infty$ is *one* if $\Lambda_{i_0j_0} \in M_0$ and *two* if $\Lambda_{i_0j_0} \in M_1 \cup M_2$; while the total number of *changes of value* undergone by $\lambda^s_{i_0j_0}$ as $s\to\infty$ is *zero* if $\Lambda_{i_0j_0} \in M_0$, *one* if $\Lambda_{i_0j_0} \in M_1$, and *two* if $\Lambda_{i_0j_0} \in M_2$. Hence certainly $\lim_{s\to\infty}\lambda^s_{ij}$ exists for all pairs $\langle i,j\rangle$. We shall henceforth denote $\lim_{s\to\infty}\lambda^s_{ij}$ by λ_{ij}. Now, it is clear from the manner in which the barrier marker $\mathscr{B}$ is used to "kill" potentially infinite branches of T_0 that every branch of T_0 is either finite or of the form $\{\lambda_{ny_n} \mid n \in N\}$ for some sequence $y_0, y_1, y_2, \cdots$. Hence, if $x \in \delta p_0$ & $x \neq p_0(x)$ & x belongs to at least one infinite branch of T_0 then there must be numbers k_0, j_0, and ℓ_0 such that $x = \lambda_{k_0+1,j_0}$ & $p_0(x) = \lambda_{k_0\ell_0}$. Now observe that, since $(\forall n)$ $(\forall m)$ $[n \neq m \Rightarrow \rho k_n \cap \rho k_m = \phi]$ & $(\forall x)$ $[\lim_{s\to\infty}g_0(x,s)$ exists$]$, a given marker Λ_{ek} can be *attacked* only finitely many times. Therefore $x > g(p_0(x))$, since otherwise Step A would eventually force Λ_{k_0+1,j_0} to be permanently removed from x. Thus T_0 is g-dominant. Somewhat more tedious is the proof that if f_1 regresses an infinite set α then p_0 regresses β for some infinite set $\beta \subseteq \alpha$. Suppose that $a_0, a_1, a_2, \cdots$ is a sequence of numbers for which $f_1(a_0) = a_0$ & $(\forall n)$ $[f_1(a_{n+1}) = a_n]$; i.e., suppose that $\{a_i \mid i \in N\}$ is an infinite branch of T_1. Then, in view of Stage 0, Step C of Stage $s+1$, and the fact that $(s+1)_0 = 0$ for infinitely many numbers s, we have $(\exists s)$ $(\exists y)$ $[a_0 = \lambda^s_{0y} = \lambda_{0y}$ & $\langle a_0, a_0\rangle \in T_0{}^s]$. Let s_0 be the smallest such s, and y_0 the (uniquely determined) corresponding y. Now assume, for the sake of an induction, that there exist a number s_m, a finite sequence $y_0, \cdots, y_m$ (reducing to a one-term sequence y_0 in case $m = 0$), and a corresponding subsequence $a_{i_0}, \cdots, a_{i_m}$ of $\{a_i \mid i \in N\}$ such that $a_{i_0} = a_0$ & $(\forall j)$ $[0 \leqq j \leqq m \Rightarrow a_{i_j} = \lambda^{s_m}_{jy_j} = \lambda_{jy_j}]$

& $\langle a_0, a_0\rangle \in T_0^{s_m}$ & $(\forall j)$ $[0 < j \leqq m \Rightarrow \langle a_{i_j}, a_{i_{j-1}}\rangle \in T_0^{s_m}]$. Then, in view of Steps B and C, there must exist some number $\bar{s}_1 > s_m$ such that at the conclusion of Stage $\bar{s}_1$ we have $(\exists k)$ $[\Lambda_{m+1,k}$ is attached to some number w for which $a_{i_m} \in \hat{f}_1(w)$ & $p_0^{\bar{s}_1}(w) = a_{i_m}]$. Now, it may happen that, on account of Step A, $f_1^*(w)$ receives a $\mathcal{B}$ at some stage $\bar{s}_2$ in the construction; however, if this occurs then some number $h > f_1^*(w)$ receives a reservation tag $\Sigma_{a_{i_m}\ell}$, for some ℓ, at Stage $\bar{s}_2$. Clearly, we have $\bar{s}_2 \geqq \bar{s}_1$ (since attaching a $\mathcal{B}$ to x forces removal of all approximation markers which happen to be attached to members of ρk_x, and since no barrier marker is ever *removed*); so, there will be a stage $\bar{s}_3, \bar{s}_3 \geqq \bar{s}_2 \geqq \bar{s}_1$, at which Step C compels us to attach a marker of the form $\Lambda_{m+1,y}$ to some number $z \in \rho k_h$ and place the pair $\langle z, a_{i_m}\rangle$ in $T_0^{\bar{s}_3}$. ρk_h may later cease, in virtue of the exigencies of Step A, to be a source of new p_0-preimages of a_{i_m}; if so, however, then it is subsequently replaced (in the manner just indicated for $\rho k_{f_1^*(w)}$) by some *new* source, ρk_r, of such preimages. This process of "changing our minds about the set of desirable p_0-preimages of a_{i_m}" can be iterated only finitely often, since (i) $\lim_{s\to\infty} g_0(a_{i_m}, s)$ exists and (ii) the sets $\rho k_x, x \in N$, are mutually disjoint. Hence, there must be a stage $s' > s_m$ such that $(\exists j)$ $[\lambda^{s'}_{m+1,j} = \lambda_{m+1,j} \in \delta f_1$ & $a_{i_m} = p_0^{s'}(\lambda^{s'}_{m+1,j})]$. It is easily seen by inspection of the construction that there is in fact a *uniquely determined* j, say j_1, such that $\lambda^{s'}_{m+1,j} = \lambda_{m+1,j} \in \delta f_1$ & $a_{i_m} = p_0^{s'}(\lambda^{s'}_{m+1,j})$. Moreover, there is a uniquely determined number ℓ, say ℓ_1, such that $\lambda_{m+1,j_1} \in \rho k_\ell$. Now, since $\{a_i \mid i \in N\}$ is an infinite branch of T_1, there exists a uniquely determined index t, say t_1, such that $a_t \in \rho k_{\ell_1}$. Since Λ_{m+1,j_1} is permanently attached to a member of ρk_{ℓ_1}, ℓ_1 cannot ever bear a $\mathcal{B}$; hence, in view of Step C, we have $\langle a_{t_1}, a_{i_m}\rangle \in T^{s_{m+1}}$ for some number $s_{m+1} \geqq s'$. So, if we set $i_{m+1} = t_1$ then the conjunction $a_{i_0} = a_0$ & $(\forall j)$ $[0 \leqq j \leqq m+1 \Rightarrow a_{i_j} = \lambda^{s_{m+1}}_{jy_j} = \lambda_{jy_j}]$ & $\langle a_0, a_0\rangle \in T_0^{s_{m+1}}$ & $(\forall j)$ $[0 < j \leqq m+1 \Rightarrow \langle a_{i_j}, a_{i_{j-1}}\rangle \in T_0^{s_{m+1}}]$ holds for a suitably specified sequence $y_0, y_1, \cdots, y_{m+1}$. It follows by induction that some infinite subset of $\{a_i \mid i \in N\}$ is a branch of T_0. It now remains only to be shown that T_0 is finite-branching if T_1 is finite-branching and that T_0 is uniformly co-enumerable relative to T_1, i.e., that there is a fixed r.e. set γ such that $(\forall\alpha)$ $(\forall\beta)$ $[(\alpha$ an infinite branch of T_1 & β an infinite branch of T_0 & $\beta \subseteq \alpha \Rightarrow \alpha - \beta = \alpha \cap \gamma]$ (for if such a γ exists then, obviously, it is also the case that each infinite branch α of T_1 can include *at most one* infinite branch β of T_0). But to verify the uniform co-enumerability condition, we need only note that (because of the "largeness criterion" according to which h_0 is determined in Step A) p_0 satisfies the condition: $(\forall m)$ $(\forall n)$ $(\forall q)$

$[(m \in \delta p_0 \ \&\ n = p_0(m) \ \&\ q \in \hat{f}_1(m) - \{m\} \ \&\ n \in \hat{f}_1(q) - \{q\}) \Rightarrow (q \in \delta p_0 \Rightarrow f_1{}^*(q)$ eventually bears a $\mathcal{B}$ and hence no member of $\rho k_{f_1{}^*(q)}$ belongs to an infinite branch of $T_0)]$. For, from this special feature of p_0 it follows easily that if $\gamma = \{n \mid n \in \delta f_1 \ \&\ (\exists w)\ (\exists z)\ [n \in \hat{f}_1(w) - \{w\} \ \&\ z \in \hat{f}_1(n) - \{n\} \ \&\ p_0(w) = z]\}$ then γ is a recursively enumerable set such that (α an infinite branch of T_1 & β an infinite branch of T_0 & $\beta \subseteq \alpha) \Rightarrow \alpha - \beta = \alpha \cap \gamma$. Finally, T_0 is finite-branching if T_1 is. For, given $x_0 \in \rho p_0$, it is clear from the construction that if x_0 is drawing new p_0-preimages from ρk_{w_0} at stage s then it continues to draw them solely from ρk_{w_0} at all stages $t > s$ *unless* $g_0(x_0, t) \neq g_0(x_0, t-1)$ for some $t > s$. Thus, all the p_0-preimages of x_0 lie in the union of just finitely many sets ρk_w. The proof of Theroem 2.2 is therewith complete.

2.3. Remarks. (1): Our technique in proving Theorem 2.2 derives from Yates' proof of [**20**, Theorem 3]. In the latter proof, Yates showed that a cohesive Π_1^0 set can be so constructed as to bound the principal function of a given (infinite) Σ_2^0 set. We can analogously strengthen Theorem 2.2, replacing the arbitrary Δ_2 function in the statement of the theorem by the principal function of an arbitrarily given infinite Σ_2^0 set. All that is required for such a strengthening is to observe that every infinite Σ_2^0 set has an infinite subset recursive in 0^1 and that such a subset can be enumerated in increasing order by a Δ_2 function. (2): After seeing a pre-print of this article, Alfred Manaster devised a proof of Theorem 2.2 which is similar to but shorter than our above proof. We are inclined to prefer the proof as given above, however, since it seems to us that it affords a slightly more dynamic view of the construction of T_0 from T_1 than does Manaster's more concise version.

We further observe at this point that A. N. Degtev has given an independent construction for a particular co-r.e. t-retraceable set, in terms of the "deficiency sets" of recursive functions (cf. [**19**]). (For the definition of "t-retraceable" see §3 below.) His procedure can be adapted, without very much difficulty, to the problem of proving Theorem 2.2 and its corollaries in the special case of regressive sets *with r.e. complements*; for regressive sets in general, however, there would seem to be no routine extension of his technique which would lead to the full Theorem 2.2. Thus, from a general point of view, it would appear better to work with trees in a direct manner, as in the present article.

Let $T_0 \dashv_* T_1$ mean that $T_0 \dashv T_1$ & $(\forall \alpha)\ (\forall \beta)\ [(\alpha$ an infinite branch of T_1 & β an infinite branch of T_0 & $\beta \subseteq \alpha) \Rightarrow \alpha - \beta$ is recursively enumerable in $\alpha]$. (Recall that a set τ_1 is said to be *recursively enumer-*

able in a set τ_2 just in case $\tau_1 = \rho f$ where f is a function recursive in τ_2.) The ease with which we were able to build uniform co-enumerability into our construction of T_0 in the proof of Theorem 2.2 might cause the reader to wonder just how much is added to T_1-skeletonhood by requiring specifically that T_0 be a *uniformly co-enumerable* T_1-skeleton. Before proceeding to §3, we pause to prove a theorem which shows that there is, in fact, a considerable gap even between the relations $T_0 \dashv_* T_1$ and $T_0 \dashv_+ T_1$.

2.4. THEOREM. *Let β be an arithmetical subset of N, and let $\{W_n{}^\beta\}_{n=0}^\infty$ be an enumeration of all sets recursively enumerable in β. Then there exist semicomputable trees T_0 and T_1 such that*

(2.4i) *T_1 has a unique infinite branch*;

(2.4ii) *$T_0 \dashv_* T_1$; and*

(2.4iii) $(\forall \alpha_0)$ $(\forall \alpha_1)$ $[(\alpha_0 =$ *an infinite branch of* T_0 & $\alpha_1 =$ *an infinite branch of* T_1 & $\alpha_0 \subseteq \alpha_1) \Rightarrow (\forall m)$ $[\alpha_1 - \alpha_0 \neq \alpha_1 \cap W_m{}^\beta]]$.

PROOF. Since β is arithmetical, there exists a number n_0 such that $\boldsymbol{\beta} \leqq \mathbf{0}^{(n_0)}$. (Following the (standard) practice of [13], we denote by $\boldsymbol{\alpha}$ the degree (of unsolvability) of a given set $\alpha \subseteq N$ and by $\mathbf{0}^{(m)}$ the degree of the m-th jump of the empty set; $\leqq$ denotes less-than-or-equal-to in the sense of the upper semilattice of degrees of unsolvability.) Now, for any $n \in N$ there exist a Π^0_{n+1} set α_1 and a Π^0_{n+2} set γ such that α_1 is the unique infinite branch of a semicomputable tree T_1 satisfying $(\forall x)$ $(\forall y)$ $[\langle x, y\rangle \in T_1 \Rightarrow x \geqq y]$, γ is the unique infinite branch of a semicomputable tree T satisfying $(\forall x)$ $(\forall y)$ $[\langle x, y\rangle \in T \Rightarrow x \geqq y]$, $\boldsymbol{\alpha}_1 = \mathbf{0}^{(n+1)}$, and $\boldsymbol{\gamma} = \mathbf{0}^{(n+2)}$; in case $n \geqq 1$ this follows from [13, Theorem 4.14 (2)], while for $n = 0$ we appeal also to [7, Theorem T3] and [19, Theorem 2]. Let α_1, γ be so chosen, with $n = n_0$. Let p, q be the partial recursive functions defining T_1, T, respectively. Following the procedure used in the proof of [7, Proposition P4], we define a partial recursive function ω by the condition that $\langle x, y\rangle \in$ the graph of $\omega \Leftrightarrow [x \in \delta p$ & $p^*(x) \in \delta q$ & $y \in \hat{p}(x)$ & $p^*(y) = q(p^*(x))]$. It is easily checked that ω defines a tree T_0 whose unique infinite branch α_0 is the range of the composite function $p_{\alpha_1}(p_\gamma(x))$, where p_{α_1} enumerates α_1 in natural order and p_γ enumerates γ in natural order. Clearly $\alpha_0 \subseteq \alpha_1$, and so (noting that $\delta\omega \subseteq \delta p$ & $(\forall x)$ $[x \in \delta\omega \Rightarrow \omega(x) \in \hat{p}(x)]$) we have $T_0 \dashv T_1$. Moreover, since $(\forall x)$ $[x \in \alpha_0 \Leftrightarrow (x \in \alpha_1$ & $p^*(x) \in \gamma)]$ we have that α_0 is a $\Pi^0_{n_0+2}$ set. Thus $N - \alpha_0$ is $\Sigma^0_{n_0+2}$, so that $N - \alpha_0$ is recursively enumerable in any set of degree $\mathbf{0}^{(n_0+1)}$. In particular, then, $N - \alpha_0$

is recursively enumerable in α_1. It follows that $(N - \alpha_0) \cap \alpha_1$, i.e., $\alpha_1 - \alpha_0$, is recursively enumerable in α_1; hence $T_0 \dashv_* T_1$. Finally, if $(\exists m)\ [\alpha_1 - \alpha_0 = \alpha_1 \cap W_m{}^\beta]$ then (since $(\forall m)\ \{W_m{}^\beta$ is recursive in $0^{(n_0+1)}])$ we have $\alpha_0 \leqq 0^{(n_0+1)}$. But it is easy to see that $\alpha_0 \leqq 0^{(n_0+1)} \Rightarrow \gamma \leqq \alpha_1$. Hence, since γ is strictly larger than α_1, we conclude that $(\forall m)\ [\alpha_1 - \alpha_0 \neq \alpha_1 \cap W_m{}^\beta]$, and the proof is complete.

2.5. REMARK. Theorem 2.4 is by no means a comprehensive survey of the discrepancy between $\dashv_*$ and $\dashv_+$ for semicomputable trees; in particular, one might inquire about a precise violation of *uniformity*, in the sense of the existence of semicomputable trees T_0 and T_1 such that $T_0 \dashv T_1$ & $(\forall \alpha_0)\ (\forall \alpha_1)\ [(\alpha_0 =$ an infinite branch of T_0 & $\alpha_1 =$ an infinite branch of T_1 & $\alpha_0 \subseteq \alpha_1) \Rightarrow (\exists n)\ [\alpha_1 - \alpha_0 = \alpha_1 \cap W_n]]$ & $\neg\ (\exists n)(\forall \alpha_0)(\forall \alpha_1)\ [(\alpha_0 =$ an infinite branch of T_0 & $\alpha_1 =$ an infinite branch of T_1 & $\alpha_0 \subseteq \alpha_1) \Rightarrow \alpha_1 - \alpha_0 = \alpha_1 \cap W_n]$. We think that such pairs of trees exist, but do not at present have any clear ideas about proving it. Also left open are the questions whether Theorem 2.4 can be extended into the hyperarithmetical hierarchy, and whether an example can be found in which (2.4i) is replaced by the condition that T_1 have uncountably many branches. Finally, we mention that the proof of Theorem 4.14 (2) of [13] is sufficiently effective to allow us to prove the following, where $\dashv_+{}^A$ is defined exactly as is $\dashv_+$ except that γ is allowed to be any fixed *arithmetical* subset of N: There are semicomputable trees T_1 and T_0 such that (i) T_1 has $\aleph_0$ infinite branches, each of which is arithmetical, (ii) $T_0 \dashv_* T_1$, and (iii) $T_0 \not\dashv_+{}^A T_1$.

3. **Some applications to the class of regressive isols.** In this section, we shall indicate a few applications of Theorem 2.2 within its most obvious domain of relevance: the theory of regressive isols. (For an excellent survey of the basic results on regressive isols, the reader is referred to Dekker's paper [10]; more recent results can be found, for example, in [3], [4], [5], [6], [12], and [17]. Some of these articles, as well as some others less recent (e.g. [2]), contain results which can be immediately sharpened by applying the "thinning" technique of the present paper.)

Following well-established notational practice, we let $\Lambda_R(\Lambda_{ZR})$ denote the class of regressive (co-simple regressive) isols ([10]); and we let Λ^* denote the ring of isolic integers, defined in [8, Chapter XI]. $\Lambda_R{}^*(\Lambda_{ZR}^*)$ will denote the subring of Λ^* generated by Λ_R (by Λ_{ZR}). Since it is customary to include the *finite* isols (i.e., the natural numbers) among the members of both Λ_R and Λ_{ZR}, we shall use $\Lambda_R{}^\infty$ (respectively Λ_{ZR}^∞) to denote the class of *infinite* regressive (*infinite* co-simple regressive) isols; and similarly for the notations $\Lambda_R^{*\infty}$, $\Lambda_{ZR}^{*\infty}$.

As shown in [9], each member of Λ_R has a retraceable representative. We shall say that an isol $\mathbf{A}$ is *invariantly retraceable* provided that $(\forall\alpha)$ $[\alpha \in \mathbf{A} \Rightarrow \alpha$ is retraceable$]$. (Our use of boldface capital roman letters for isols will not clash with our boldface notation for degrees, since we never use capital roman letters for degrees.) Let $\mathbf{A}$ be called *hereditarily invariantly retraceable* $\Leftrightarrow_{df} (\forall \mathbf{B})$ $[\mathbf{B} \leqq \mathbf{A} \Rightarrow \mathbf{B}$ is invariantly retraceable$]$. (Here, of course, $\leqq$ means *isolic* less-than-or-equal-to as defined in [8].) It is a simple exercise to show that invariant retraceability implies hereditary invariant retraceability, so that the two concepts are in fact equivalent. By a *totally retraceable* isol we shall mean an isol $\mathbf{A}$ such that for every regressive isol $\mathbf{B}$ and every pair of sets α, β, we have $(\alpha \in \mathbf{A}$ & $\beta \in \mathbf{B}$ & $\beta \subseteq \alpha) \Rightarrow \mathbf{B}$ is invariantly retraceable. Let $\mathcal{TR}$ denote the class of all totally retraceable isols. Theorem 1 of [14], which we shall here obtain (in strengthened form) as Theorem 3.4, implies the existence of a rich supply of isols belonging to the class $\mathcal{TR}$. In preparation for this result, we shall now derive an appropriate specialization of Theorem 2.2.

3.1 Theorem. *Let T_1 be a non-trivial semicomputable tree, and let f_1 be the partial recursive function whose graph is T_1; we assume* $0 \notin \rho f_1$. (The latter assumption is inessential; it is only a matter of convenience.) *Then there exists a semicomputable tree T_0 such that $T_0 \dashv_+ T_1$, T_0 is finite-branching if T_1 is, and T_0 has in addition the following property, where p_0 is the partial recursive function whose graph is T_0:*

$$(*) \qquad (\forall x)\ [x \in \delta p_0 \Rightarrow p_0(x) \leqq x]\ \&$$

$(\forall\alpha)$ $(\forall d)$ $(\forall e)$ $(\forall n)$ $(\forall m)$ $[(\alpha \subseteq$ *a branch of* T_0 & α *infinite* & $p_\alpha =$ *the function from N into N which enumerates α in order of magnitude* & $\max\{e, d\} \leqq n < m$ & $\varphi_d{}^1$ *is one-to-one on* $\delta\varphi_d{}^1$ & $\{p_\alpha(n), p_\alpha(m)\} \subseteq \delta\varphi_d{}^1$ & $\varphi_d{}^1(p_\alpha(n)) \in \delta\varphi_e{}^1) \Rightarrow \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n))) < \varphi_d{}^1(p_\alpha(m))]$.

Proof. By Theorem 2.2, there exists a semicomputable tree T_2 defined by a partial recursive function p_2 such that $T_2 \dashv_+ T_1$ & T_2 is finite-branching if T_1 is & T_2 is ξ-dominant where ξ is the *identity function*: $\xi(x) = x$ for all x. Then, as is easily seen, p_2 *retraces* each infinite branch of T_2. There is no loss of generality in assuming, in fact, that $(\forall x)[x \in \delta p_2 \Rightarrow p_2(x) \leqq x]$; for any such modification of p_2 as may be needed to actually achieve this can affect only the *finite* branches of T_2, and we can easily insure retention of property (b) of

§1. We shall next define a Δ_2 function $\Psi : N \rightarrow N$ such that if T_0 is any Ψ-dominant uniformly co-enumerable T_2-skeleton (which is finite-branching provided T_2 is) then T_0 meets the demands of the theorem. The definition of Ψ in terms of p_2 is as follows:

$$\Psi(x) = \begin{cases} 0, \quad \text{if } x = 0; \\ 1 + \Psi(x-1), \text{if } x > 0 \;\&\; x \notin \delta p_2; \\ (\mu z)[z > \Psi(x-1) \;\& \\ (\forall d)(\forall e)(\forall k)(\forall \ell)(\forall j)\,[(d \leqq x \;\&\; e \leqq x \;\& \\ \varphi_d{}^1 \text{ is one-to-one on } \delta\varphi_d{}^1 \;\&\; k \in \hat{p}_2(x) \cap \delta\varphi_d{}^1 \;\& \\ \ell = \varphi_d{}^1(k) \;\&\; \ell \in \delta\varphi_e{}^1 \;\&\; j = \varphi_e{}^1(\ell)) \Rightarrow \\ (\forall w)\,[(w \in \delta\varphi_d{}^1 \;\&\; j \geqq \varphi_d{}^1(w)) \Rightarrow z > w]] \;\& \\ (\forall d)(\forall k)(\forall \ell)\,[(d \leqq x \;\&\; \varphi_d{}^1 \text{ is} \\ \text{one-to-one on } \delta\varphi_d{}^1 \;\&\; k \in \hat{p}_2(x) \cap \delta\varphi_d{}^1 \;\&\; \ell = \\ \varphi_d{}^1(k)) \Rightarrow (\forall w)[(w \in \delta\varphi_d{}^1 \;\&\; \ell \geqq \varphi_d{}^1(w)) \Rightarrow z > w]], \\ \qquad\qquad \text{if } x > 0 \;\&\; x \in \delta p_2. \end{cases}$$

It is easily verified that Ψ is a strictly increasing Δ_2 function. Hence, by Theorem 2.2, T_2 has a uniformly co-enumerable T_2-skeleton T_0 such that T_0 is Ψ-dominant and is finite-branching if T_2 is. Let p_0 be the partial recursive function whose graph is T_0. We note that (by the definition of the relation $\dashv$) we have $\delta p_0 \subseteq \delta p_2$ & $(\forall x)[x \in \delta p_0 \Rightarrow p_0(x) \in \hat{p}_2(x)]$; moreover, we have $0 \notin \rho p_2$, since $\rho p_2 \subseteq \rho f_1 \;\&\; 0 \notin \rho f_1$. These conditions imply that $0 \notin \rho p_0$; and they further imply that every infinite branch of T_0 is *retraced* by p_0. Now let α be an infinite subset of some branch τ of T_0; and let p_α be the function which enumerates α in order of magnitude. Let d be a number such that $\varphi_d{}^1$ is one-to-one on $\delta\varphi_d{}^1$, and let $e \in N$. Suppose $n \geqq \max\{d, e\}$, $m > n$, $\{p_\alpha(n), p_\alpha(m)\} \subseteq \delta\varphi_d{}^1$, and $\varphi_d{}^1(p_\alpha(n)) \in \delta\varphi_e{}^1$. Since $\rho p_0 \subseteq \delta p_0 \subseteq \delta p_2$ & $(\forall x)[x \in \delta p_0 \Rightarrow p_0(x) \in \hat{p}_2(x)]$, we see that $p_\alpha(n) \in \hat{p}_2(p_0(p_\alpha(m)))$; also, $p_0(p_\alpha(m)) > 0$ since $0 \notin \rho p_0$. Hence, by the definition of Ψ, $\Psi(p_0(p_\alpha(m)))$ is larger than w for every number w such that $w \in \delta\varphi_d{}^1 \;\&\; \varphi_d{}^1(w) \leqq \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n)))$. But, since T_0 is Ψ-dominant, we therefore have that $p_\alpha(m)$ is larger than w for every w satisfying $w \in \delta\varphi_d{}^1 \;\&\; \varphi_d{}^1(w) \leqq \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n)))$. Thus $\varphi_d{}^1(p_\alpha(m)) > \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n)))$, and we have completed the proof. (Note that the condition $\delta p_0 \subseteq \delta p_2$ & $(\forall x)[x \in \delta p_0 \Rightarrow p_0(x) \in \hat{p}_2(x)]$ guarantees $(\forall x)[x \in \delta p_0 \Rightarrow p_0(x) \leqq x]$ and so insures the initial clause in $(*)$.)

Our next theorem singles out for special notice two related, useful properties of (branches of) those trees T_0 which can be obtained from given trees by applying Theorem 3.1. Let us say, for want of better terminology, that a non-trivial semicomputable tree T_0 defined by a partial recursive function p_0 is *uniformly thin-branched* $\Leftrightarrow_{df}$ the pair $\langle T_0, p_0 \rangle$ satisfies the condition $(*)$ in the statement of Theorem 3.1. Property (2) below is one which various other authors (e.g., Ellentuck, J. Gersting, and A. N. Degtev) have studied under a variety of labels (but not in the present conceptual framework of *trees*). Property (1) is a standard domination property.

3.2. THEOREM. *Let T_0 be a uniformly thin branched, non-trivial, semicomputable tree; and let p_0 be the partial recursive function whose graph is T_0. If τ is an infinite branch of T_0 and p_τ is the function which enumerates τ in order of magnitude and $\varphi_e{}^1$ is any partial recursive function, then*

(1) $(\exists m_1)(\forall n)[(n \geqq m_1 \ \& \ n \in \delta\varphi_e{}^1) \Rightarrow p_\tau(n) \geqq \varphi_e{}^1(n)]$, and

(2) $(\exists m_2)(\forall n)[(n \geqq m_2 \ \& \ n \in \tau \ \& \ p_0(n) \in \delta\varphi_e{}^1) \Rightarrow n > \varphi_e{}^1(p_0(n))]$.

PROOF. Letting T_1 be any semicomputable tree such that $T_0 \dashv_+ T_1$ (e.g., take $T_1 = T_0$), take $\varphi_d{}^1 =$ the identity function and apply Theorem 3.1; since $p_0(n) < n$ for all those $n \in \delta p_0$ such that $p_0{}^*(n) > 0$, part (2) follows at once. For part (1), let a partial recursive function p_1 be defined as follows: $\langle x, y \rangle \in$ the graph of $p_1 \Leftrightarrow_{df} x \in \delta p_0$ & $p_0{}^*(x) + 1 \in \delta\varphi_e{}^1$ & $y = \varphi_e{}^1(p_0{}^*(x) + 1)$. Let u be a number such that $\varphi_u{}^1 = p_1$. By part (2), there is a number m_2 such that $(n \geqq m_2$ & $n \in \tau$ & $p_0(n) \in \delta\varphi_u{}^1) \Rightarrow n > \varphi_u{}^1(p_0(n))$. Now, clearly, $p_0{}^*(p_\tau(n)) = n$ for all n, since p_0 *retraces* τ. We set $m_1 = m_2 + 1$. Suppose $n \geqq m_1$ & $n \in \delta\varphi_e{}^1$. Then $p_0{}^*(p_\tau(n-1)) = n - 1$. Since $(n-1) + 1 = n \in \delta\varphi_e{}^1$, we have that $\varphi_u{}^1(p_\tau(n-1))$ is defined and $= \varphi_e{}^1(n)$. But therefore, since $p_\tau(n) \geqq n > m_2$ & $p_\tau(n) \in \tau$ & $p_0(p_\tau(n)) = p_\tau(n-1) \in \delta\varphi_u{}^1$, we get $p_\tau(n) > \varphi_u{}^1(p_0(p_\tau(n))) = \varphi_u{}^1(p_\tau(n-1)) = \varphi_e{}^1(n)$. (1) is thus verified, taking $m_1 = m_2 + 1$ as just indicated.

3.3 REMARK. Our proof of Theorem 3.2 shows that the implication (2) $\Rightarrow$ (1) holds for all retraceable sets. Regarding the converse, it can rather easily be shown that (1) $\Rightarrow$ (2) *does not* hold for all retraceable sets. In fact, while it can easily be shown that $\rho p_{\tau_2} p_{\tau_1}$ has property (1) if τ_1 does (τ_1, τ_2 infinite), one can construct infinite co-r.e. retraceable sets τ_1 and τ_2 such that τ_1 has property (2) while $\rho p_{\tau_2} p_{\tau_1}$ does not.

An infinite set τ which satisfies condition (2) in the statement of Theorem 3.2 (relative to a partial recursive function p_0 such that p_0

retraces τ) is termed *T-retraceable* in [11]. (As noted in [11], the terminology originated with Judith Gersting.) To avoid any possible confusion with our notation for trees, we shall here change to lower case and refer to such sets as being *t-retraceable*. By the obvious extension of terminology, an isol A is said to be *t-retraceable* provided it contains at least one t-retraceable set. That each infinite branch τ of a uniformly thin-branched tree is t-retraceable is just part (2) of Theorem 3.2 above; that each such τ is *totally t-retraceable* is an immediate consequence of Theorem 3.4 below.

From now on, we shall use $[\alpha]$ as an alternative form of notation for the isol A satisfying $\alpha \in A$; if α is represented as (the range of) a sequence $\langle a_i \rangle_{i=0}^{\infty}$, then $[\langle a_i \rangle_{i=0}^{\infty}]$ is also admitted as a notation for the isol containing α. We shall say that an infinite isol $[\alpha]$ is *totally t-retraceable* $\Leftrightarrow_{df} (\forall B)\ (\forall \beta)\ (\forall \gamma)\ (\forall \tau)\ [(\tau \in [\alpha]\ \&\ \beta \in B\ \&\ \gamma \in B\ \&\ B$ is infinite and regressive $\&\ \beta \subseteq \tau) \Rightarrow \gamma$ is a t-retraceable set$]$. It is obvious that $[\alpha]$ totally t-retraceable $\Rightarrow$ $[\alpha]$ totally retraceable. We are now ready to derive our main corollary to Theorem 2.2.

3.4 THEOREM. (cf. [14, Theorem 1]). *Let T_0 be any uniformly thin-branched, non-trivial, semicomputable tree. Then* $(\forall \tau)\ [(\tau =$ *an infinite branch of* $T_0) \Rightarrow [\tau]$ *is totally t-retraceable*$]$. *Thus every non-trivial semicomputable tree T_1 admits a semicomputable, uniformly co-enumerable T_1-skeleton T_0 such that T_0 is finite-branching if T_1 is and each infinite branch of T_0 determines a totally t-retraceable isol. In particular, if W_n is a recursively enumerable set whose complement is infinite, immune, and regressive, then there exists a recursively enumerable set W_m such that $W_n \subseteq W_m$ & the complement of W_m represents an infinite, totally t-retraceable isol.*

PROOF. Suppose τ is an infinite branch of T_0; and let α be an infinite subset of τ. Let $\varphi_d{}^1$ be a one-to-one partial recursive function such that $\alpha \subseteq \delta\varphi_d{}^1$; and suppose $\varphi_e{}^1$ is a partial recursive function which regresses $\varphi_d{}^1(\alpha)$. By $(*)$, $m > n \geqq \max\{d, e\} \Rightarrow \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n))) < \varphi_d{}^1(p_\alpha(m))$. But also, if $m > n \geqq \max\{e, d\} \geqq d$, $(*)$ yields $\varphi_d{}^1(p_\alpha(n)) < \varphi_d{}^1(p_\alpha(m))$. (Simply take $e = 0$ so that $\varphi_e{}^1 =$ the identity function.) Thus, there exists a fixed *finite* set F such that $(n \geqq \max\{e, d\}\ \&\ \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n))) > \varphi_d{}^1(p_\alpha(n))) \Rightarrow \varphi_e{}^1(\varphi_d{}^1(p_\alpha(n))) \in F$. Thus, $\varphi_e{}^1$ maps elements of $\varphi_d{}^1(\alpha)$ to *smaller* elements of $\varphi_d{}^1(\alpha)$ *with at most finitely many exceptions.* Therefore $\varphi_d{}^1(\alpha)$ is retraced by some finite modification, $\varphi_{e'}{}^1$, of $\varphi_e{}^1$. But, as already noted, $\varphi_d{}^1(p_\alpha(m)) > \varphi_d{}^1(p_\alpha(n))$ whenever $m > n \geqq d$. Hence there exists a number $n_0 \geqq d$ such that $n > n_0 \Rightarrow \varphi_{e'}{}^1(\varphi_d{}^1(p_\alpha(n))) = \varphi_d{}^1(p_\alpha(n-1))$. Taking $n > \max\{n_0, k\}$, then, we have: $\varphi_k{}^1(\varphi_{e'}{}^1(\varphi_d{}^1(p_\alpha(n+1)))) = \varphi_k{}^1(\varphi_d{}^1(p_\alpha(n)))$

$< \varphi_d{}^1(p_\alpha(n+1))$. It is thus proven that $\varphi_d{}^1(\alpha)$ is, in fact, *t-retraceable*. To get the last assertion in the statement of Theorem 3.4 as a special case of what has just been proved, simply note that if $T_0 \dashv_+ T_1$ and τ_1 is an infinite *co-r.e.* branch of T_1, then τ_0 is also co-r.e. where $\tau_0 =$ the unique infinite branch τ of T_0 such that $\tau \subseteq \tau_1$.

As a second application, we get a virtually instantaneous proof of a stronger version of [13, Lemma 4.21] (the proof of which was merely sketched in [13]).

3.5 THEOREM. *Let T_1 be a non-trivial semicomputable tree. Then there exists a semicomputable tree T_0, defined by a partial recursive function p_0, such that $T_0 \dashv_+ T_1$ & T_0 is finite-branching if T_1 is finite-branching* & $(\forall\tau)[(\tau =$ *an infinite branch of* $T_0) \Rightarrow (p_0$ *retraces* τ & $\tau \geqq 0')]$.

PROOF. Applying Theorem 3.1, let T_0 be a uniformly thin-branched tree such that $T_0 \dashv_+ T_1$ & T_0 is finite-branching provided T_1 is. Then each infinite branch τ of T_0 is retraced by p_0, where p_0 is the partial recursive function defining T_0. Moreover, each such τ has property (1) of Theorem 3.2; i.e., if p_τ is the function which enumerates τ in order of magnitude then p_τ eventually bounds any given partial recursive function. But this latter property is well known to imply that $\tau \geqq \mathbf{0}^1$, and the proof is complete.

It is natural to inquire whether t-retraceability is an isolic invariant; i.e., whether $[\alpha]$ t-retracable $\Rightarrow \alpha$ t-retraceable. Since obviously any t-retraceable set is hyperimmune, the following theorem shows that the answer is in the negative.

3.6 THEOREM. *There exists a nonrecursive, recursively enumerable set W and a 1-1 partial recursive function p such that $N - W$ is a t-retraceable set* & $N - W \subseteq \delta p$ & $p(N - W)$ *is not hyperimmune.*

PROOF. The proof is a virtual repetition of the proof of [**16**, Theorem 1]; so, since it is possible to prove a stronger theorem (see Theorem 3.7 below), we shall merely outline the procedure. We replace Lemma 1 of [**16**] by the following *Lemma* 1′. There exists a co-infinite r.e. set α having the following property: $N - \alpha$ is t-retraced by a partial recursive function f such that there is a total recursive function r satisfying $(\forall n)[r(n) > \text{card}(\{x \mid x \in \delta f \;\&\; f^*(x) = n\})]$. The proof of this modified version of [**16**, Lemma 1] proceeds in exactly the same way as the proof of [**16**, Lemma 1] itself, except that in the construction we move markers so as to secure t-retraceability (of the set of final marker-positions) rather than mere eventual bounding of arbitrary partial recursive functions;

it remains easy to display a recursive bound $r(m)$ on the total number of different positions held by the marker Λ_m during the course of the construction. Having proved *Lemma* 1′, we finish by repeating *verbatim* the argument for [**16**, Theorem 1] given on p. 83 of [**16**].

By suitably combining the constructions of [**15**] and [**16**, proof of Theorem 1], we can establish the following stronger result:

3.7 THEOREM. *There exists a nonrecursive recursively enumerable set W and a 1-1 partial recursive function p such that $N - W$ is t-retraceable & $N - W \subseteq \delta p$ & $p(N - W)$ is neither retraceable nor hyperimmune.*

We shall omit the proof of Theorem 3.7, since it is both cumbersome and *ad hoc.*

In conclusion, as further illustrations of the uses of Theorem 2.2, we state without proof three theorems (representing refinements of known results) which are simple corollaries to Theorems 3.1 and 3.2. (For the definition of the relation $\leqq^*$ mentioned in 3.9, see [2], [4], or [**10**]; as Barback has shown in [4], $\leqq^*$ restricted to $\Lambda_R \times \Lambda_R$ coincides with $\Lambda_{\leqq} |_{\Lambda_R \times \Lambda_R}$, where $\Lambda_{\leqq}$ is the canonical extension to isols of the ordinary $\leqq$ relation in $N \times N$.)

3.8 THEOREM. *If $A \in \Lambda_R{}^\infty$ and $\alpha \in A$, then there is a $\Pi_1{}^0$ set β such that $[\alpha \cap \beta] \in \Lambda_R{}^\infty$ & $[\alpha \cap \beta]$ is multiple-free* ([8]); *whence $[\alpha \cap \beta]$ is neither an odd nor an even isol; whence (see* [**10**] *for a discussion of the Φ_f- operators and their connection with idempotency) the isols $\Phi_{2n}([\alpha \cap \beta])$ and $\Phi_{2n+1}([\alpha \cap \beta])$ have non-trivial idempotent difference in $\Lambda_R{}^*$.*

3.9 THEOREM. *Let $A \in \Lambda_R{}^\infty$, $\alpha \in A$. Then there exist sets β, γ, and τ such that β is $\Pi_1{}^0$ & $\{[\gamma], [\tau]\} \subseteq \Lambda_R{}^\infty$ & $[\alpha \cap \beta] = [\gamma] + [\tau]$ & $[\gamma] \not\leqq^* [\tau]$ & $[\tau] \not\leqq^* [\gamma]$. (In the co-simple case, if α is a retraceable $\Pi_1{}^0$ set having degree a where $a^1 = 0^{11}$, then we can require $\beta = \gamma = \tau =$ a; this, however, follows not from Theorem 3.1 but from a variant of Theorem 2.2.*

3.10 THEOREM (refining to the co-simple case an improvement due to Gersting of a theorem of Hassett). *There exist isols $A \in \Lambda_{ZR}^\infty$ and $B \in \Lambda_{ZR}^\infty$ such that B is an isolic summand of A but $\Phi_f(A) \leqq^*$ B fails for all strictly increasing recursive functions f.*

We remark that an even stronger result than Theorem 3.10 is obtainable via the techniques of the present paper: if **a** is a given r.e. degree satisfying $\boldsymbol{a}^1 = \boldsymbol{0}^{11}$ and γ is a $\Pi_1{}^0$ regressive set of degree $\boldsymbol{a}$, then, in Theorem 3.10, we can require that A be a multiple-free isol represented by a $\Pi_1{}^0$ subset α of γ such that $\boldsymbol{\alpha} = \mathbf{a}$. (Note that this simultaneously strengthens 3.8 in the co-simple case.)

4. **Relativization.** It is easily seen that the results of the preceding sections can be relativized routinely to non-semicomputable trees. For instance, when thus relativized Theorem 2.2 becomes

THEOREM 2.2*. *Let* **d** *be any fixed degree of unsolvability; and let* T_1 *be a non-trivial tree whose defining function* p_1 *is partial recursive in* **d**. *Let g be a function from N into N such that g is* Δ_2 *in* **d**. *Then, there exists a tree* T_0 *(finite-branching if* T_1 *is) such that*

(2.2*i) *the function* p_0 *whose graph is* T_0 *is partial recursive in d;*

(2.2*ii) $T_0 \dashv T_1$;

(2.2*iii) $(\exists n)\ (\forall \alpha_0)\ (\forall \alpha_1)\ [(\alpha_0 =$ *an infinite branch of* $T_0\ \&\ \alpha_1 =$ *an infinite branch of* $T_1\ \&\ \alpha_0 \subseteq \alpha_1) \Rightarrow \alpha_1 - \alpha_0 = \alpha_1 \cap W_n{}^d]$, *where* $W_n{}^d =$ *the n-th set recursively enumerable in d; and*

(2.2*iv) T_0 *is g-dominant.*

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