ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 3, Summer 1975

ON THE SUPPORT OF REPRESENTING MEASURES FOR HARMONIC FUNCTIONS

JOHN T. KEMPER

ABSTRACT. Consider a non-negative harmonic function h in a domain D in $\mathbf{R}^n$ with representing measure μ on the minimal part of the Martin boundary. If h takes on zero boundary values on an open set $\mathcal{O}$ of the topological boundary of D, then the μ-measure of the set of minimal Martin boundary points associated to points in $\mathcal{O}$ is zero. An application to a construction of R. S. Martin provides another proof of the existence of a regular domain for the Dirichlet problem in which the solution to that problem may fail to take on prescribed continuous boundary data.

0. **Introduction.** In the well-known work [6], R. S. Martin established for a general Euclidean domain D the existence of an ideal boundary (Δ) of harmonic functions (since known as the Martin boundary) such that an arbitrary non-negative harmonic function on D can be represented uniquely by an integral over the "minimal" part of the Martin boundary (Δ_1). In general it is known that the Martin boundary is distinct from the topological boundary of D, but recent efforts have been made to relate the two boundaries in the case of special domains. The results of Hunt and Wheeden [2], for example, prove that the two boundaries coincide if D is a Lipschitz domain.

In this direction Brelot earlier established (in [1]) that every Martin boundary point which is minimal is associated to at least one topological boundary point. Precisely, this association requires that the Martin boundary function be completely determined by its values in an arbitrary neighborhood of the point on the topological boundary. We include a statement of this result in § 1.

The main result of § 2 provides a restriction on the support of the measure on Δ_1 representing a non-negative harmonic function in D which vanishes on an open subset of the boundary, ∂D. In § 3 this result is applied to shed some new light on the occurrence, first pointed out by Martin, of a Dirichlet problem for a regular domain in which continuous boundary data is not taken on by the solution.

AMS Subject Classification: Primary 31B10, Secondary 31B25.
Key words and phrases: potential theory, Dirichlet problem, Poisson formula, Martin boundary.
Received by the editors September 25, 1973.

1. Preliminaries.

DEFINITION 1.1 (MARTIN). A non-negative harmonic function u in a domain D is *minimal* if the only non-negative harmonic functions which are dominated by u in D are multiples of u.

The minimal harmonic functions, suitably normalized, form a subset Δ_1 of the Martin boundary of D over which any non-negative harmonic function can be represented in a unique manner (see [6]). To study the correspondence between these functions and points on the topological boundary the notion of "reduced function" is essential:

> if u is a non-negative harmonic function in D and Γ is a subset of D, then $R_u^\Gamma(x) = \inf\{v(x)$: v is a non-negative superharmonic function in D with $v \geqq u$ on $\Gamma\}$.

If $\partial\Gamma$ contains only regular boundary points of $D\backslash\bar{\Gamma}$ and u is bounded on $\partial\Gamma \cap D$, then R_u^Γ is a continuous superharmonic function in D which is harmonic in $D\backslash\partial\Gamma$, equal to u in $\bar{\Gamma}$, and satisfies $0 \leqq R_u^\Gamma \leqq u$ in D. The definition of the reduced function can be extended to the case when Γ is a closed subset of ∂D by taking $R_u^\Gamma = \inf\{R_u^{D\cap\Omega} : \Omega$ is open and $\Gamma \subset \Omega\}$. Then R_u^Γ is a harmonic function in D and satisfies $0 \leqq R_u^\Gamma \leqq u$. We will use the abbreviation R_u^y in case $\Gamma = \{y\}$.

DEFINITION 1.2. A minimal harmonic function u is *associated to* $y \in \partial D$ if $R_u^y = u$.

If D is a Lipschitz domain, Hunt and Wheeden have shown that a minimal harmonic function is a "kernel function", vanishing at all points of ∂D except one, to which it is associated. Similar results for the heat equation may be found in [3] and [4]. However, an example of Martin shows that, for general domains, the notion of kernel function is not an appropriate characterization of the association between minimal harmonic functions and points on the boundary of D. In fact, the association of Definition 1.2 is more appropriate, as revealed in the following theorem of Brelot [1].

THEOREM 1.3. *Let D be a bounded domain in R^n. If u is a minimal harmonic function in D, then there is at least one point of ∂D to which u is associated.*

2. Influence of Boundary Behavior on the Representing Measure.

Considering the Poisson formula for a bounded harmonic function in a ball as a special case of Martin's representation theorem, there is an obvious relation between the support of the representing measure and the boundary values of the function. Below we examine the analogous relation for a general domain for which the Martin boundary may not coincide with ∂D.

THEOREM 2.1. *Let D be a bounded domain in $\mathbf{R}^n$ and u a non-negative harmonic function in D, with representing measure μ on Δ_1, such that $\lim_{x\to y} u(x) = 0$ for y in a relatively open subset $\mathcal{O}$ of ∂D. Then the set of minimal harmonic functions which are associated to points in $\mathcal{O}$ has zero μ-measure in Δ_1.*

PROOF. For a set $E \subset \partial D$, let $E^* = \{v \in \Delta_1 : R_v^y = v$ for some $y \in E\}$. It suffices to show that $\mu(G^*) = 0$ for any compact subset G of $\mathcal{O}$.

Since μ represents u on Δ_1, we have $u(x) = \int_{\Delta_1} K_z(x)\, d\mu(z)$, where K_z is the minimal harmonic function corresponding to $z \in \Delta_1$. For a compact neighborhood Γ of G in $\mathbf{R}^n$, $R_u^{D\cap\Gamma}(x) = \int_{\Delta_1} R_{K_z}^{D\cap\Gamma}(x)\, d\mu(z)$ so that $0 = R_u^G(x) = \inf_{\Gamma\supset G} \int_{\Delta_1} R_{K_z}^{D\cap\Gamma}(x)\, d\mu(z) = \int_{\Delta_1} R_{K_z}^G(x)\, d\mu(z)$. Since $R_{K_z}^G$ is either K_z or zero, depending on whether $K_z \in G^*$ or not, if the K_z's are normalized by the condition $K_z(x) = 1$ (for some fixed x), we have $\mu(G^*) = 0$, completing the proof.

3. **An application to the Dirichlet Problem.** Martin's example [**6**, p. 165] of a non-minimal Martin boundary point makes use of a domain D, regular for the Dirichlet problem, but having a singular "edge" E such that any positive harmonic function in D which vanishes on $\partial D\backslash E$ must be unbounded near every point of E. Taking a normalized sequence of Green's functions with poles approaching a point of E, such a function can actually be constructed. (A form of the Harnack principle [**5**] and the maximum principle guarantee that the limit function vanishes on $\partial D/E$.) Although this function may not be minimal, there are minimal harmonic functions with the same property.

LEMMA 3.1. *If D is the domain of Martin with singular edge E, then there is a minimal harmonic function associated to a point of E.*

PROOF. Let u be the (not necessarily minimal) harmonic function constructed above which vanishes on $\partial D\backslash E$ and let μ be the representing measure for u on Δ_1. By Theorem 2.1 the μ-measure of the set of minimal harmonic functions associated to points of $\partial D\backslash E$ is zero, so the set of minimal harmonic functions associated to points of E must have positive μ-measure. Any function in that set satisfies the conclusion of the theorem.

It is well known that the solution of a Dirichlet problem for a regular domain with boundary data which is bounded must vanish at a boundary point near which zero data is assigned. Martin [**6**, p. 166, footnote 28] has indicated one method of constructing a Dirichlet problem in a regular domain for which the solution "misbehaves" by not taking on the given boundary data at certain points of continuity. We present here a different approach to the verification of that misbehavior, based on the minimal harmonic function of Lemma 3.1 and the results of the preceeding section.

THEOREM 3.2. *There is a regular domain* Ω *and a resolutive function* f *on* $\partial\Omega$ *such that* f *vanishes in a neighborhood of a point* $y \in \partial\Omega$, *but the solution of the corresponding Dirichlet problem is unbounded at* y.

PROOF. Let u be a minimal harmonic function associated to a point y_1 of E, the singular edge of the domain D in Martin's example. Let B be a sphere, centered at y_1, whose radius is sufficiently small that $E\backslash\bar{B}$ is not empty and take $\Omega = D\backslash\bar{B}$. Since D is regular for the Dirichlet problem, so is Ω.

Define f on $\partial\Omega$ equal to u on $\partial B \cap D$ and equal to zero on $\partial D\backslash B$, which includes a part of E. Since u is an upper function for f on Ω, f is resolutive. In fact, it is easily seen that the Wiener-Perron solution of the Dirichlet problem in Ω with boundary values equal to f is $u(x) = R_u^{D\cap\bar{B}}(x)$, making use of the fact that u is associated to y_1. Since u is known to be unbounded at every point of E, the example is complete.

REFERENCES

1. M. Brelot, *Le Problème de Dirichlet Axiomatique et Frontière de Martin*, J. de Math. **35** (1956), 297-335.

2. R. A. Hunt and R. L. Wheeden, *Positive Harmonic Functions on Lipschitz Domains*, Trans. A.M.S. **147** (1970), 507-527.

3. B. F. Jones, Jr., and C. C. Tu, *On the Existence of Kernel Functions for the Heat Equation*, Math. J., Indiana U. **21** (1972), 857-876.

4. J. T. Kemper, *Temperatures in Several Variables: Kernel Functions Representations, and Parabolic Boundary Values*, Trans. A.M.S. **167** (1972), 243-262.

5. ———, *A Boundary Harnack Principle for Lipschitz Domains and the Principle of Positive Singularities*, Comm. Pure and Applied Math. **25** (1972), 247-255.

6. R. S. Martin, *Minimal Positive Harmonic Functions*, Trans. A.M.S. 49 (1941), 137-172.

THE CITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, NEW YORK, NEW YORK 10031