ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 3, Summer 1975

p-SUBGROUPS OF CORE-FREE QUASINORMAL SUBGROUPS II

FLETCHER GROSS

1. **Introduction.** The subgroup H is quasinormal in the group G if $HK = KH$ for each subgroup K in G. H is core-free in G if H contains no non-identity normal subgroup of G. Suppose now that H is a core-free quasinormal subgroup of G and that H has exponent p^n where p is a prime. It was shown in [2] that (*i*) H is nilpotent of class at most Max $\{1, p^{n-1} - 1\}$, and (ii) the derived length of H is at most n if p is odd and at most $[(n+1)/2]$ if $p = 2$. Stonehewer [4] constructed examples proving that the upper bound on the derived length is best-possible for $p \neq 2$. One result of the present paper is that the bound in (ii) also is best-possible if $p = 2$. The main purpose of this paper, however, is to obtain a best-possible upper bound on the class of H. Specifically, it is proved that the class of H is at most $\mathrm{Max}\{1, p^{n-2}(p-1)\}$. For each prime p and each positive integer n, there is an example of a core-free quasinormal subgroup H of exponent p^n such that H has nilpotence class equal to $\mathrm{Max}\{1, p^{n-2}(p-1)\}$.

2. **Notation and preliminary results.** If S is a subset of the group G, then $\langle S\rangle$ is the subgroup generated by the elements of S. If G is a p-group and n is a non-negative integer, then $\Omega_n(G) = \langle x \mid x \in G, x^{p^n} = 1\rangle$ and $\mho^n(G) = \langle x^{p^n} \mid x \in G\rangle$. If G is a nilpotent group, then $c(G)$ and $d(G)$ denote the class and derived length of G, respectively. The subgroups $L_n(G)$ are defined inductively by $L_1(G) = G$ and $L_{n+1}(G) = [L_n(G), G]$. The core of H in G is the largest normal subgroup of G contained in H. The group G is said to have exponent n if n is the smallest positive integer such that $x^n = 1$ for all $x \in G$.

The first three of the following lemmas are well known and are stated without proof. These three results will be used implicity throughout the remainder of the paper.

2.1. LEMMA. *If H is a quasinormal subgroup of G and T is a homomorphism of G, then HT is a quasinormal subgroup of GT.*

2.2. LEMMA. *Let H be a subgroup of G and N a normal subgroup of G contained in H. Then H is quasinormal in G if, and only if, H/N is quasinormal in G/N.*

Received by the editors October 30, 1973

2.3. LEMMA. *If H is a quasinormal subgroup of G and K is a subgroup of G, then $H \cap K$ is a quasinormal subgroup of K.*

2.4. LEMMA. *Let A be an abelian group of exponent dividing 4. Let x be an automorphism of A such that $x^4 = 1$ and $[A, x^2, x^2] = 1$. Let $B = [A, x^2]\ [A, x, x]\ \mho^1(A)$. Then $[B, x^2] = [B, x, x] = \mho^1(B) = 1$.*

PROOF. Let V be A written additively and let T be the element in the endomorphism ring of V corresponding to x. Then $V(T^4 - 1) = V(T^2 - 1)^2 = 4V = (0)$ and the lemma is equivalent to showing that $(T - 1)^2(T^2-1) = 2(T^2-1) = (T-1)^4 = 2(T-1)^2 = 0$. Now $(T-1)^4 = (T^2-1)^2 - 4T(T-1)^2 = 0$. $2(T^2-1) = (T^4-1) - (T^2-1)^2 = 0$ and $2(T-1)^2 = 2(T^2-1) - 4(T-1) = 0$. Finally, $(T-1)^2(T^2-1) = (T^2-1)^2 - 2(T^2-1)(T-1) = 0$.

The next two lemmas are used to compute the derived length and class of the examples constructed in §4.

2.5. **Lemma.** *Let G be a finite non-trivial nilpotent group with a normal abelian subgroup M. Assume that there is a basis for M that is the union of conjugacy classes of G. Then* $\mathrm{d}(G) > \mathrm{d}(G/C_G(M))$.

PROOF. Assume that G is a counter-example in which $|M|$ is as small as possible. Since the lemma certainly is true if $|G/C_G(M)| = 1$, we must have $G \neq C_G(M)$. According to the hypothesis, M contains elements $u_1, u_2, \cdots, u_r$ such that u_i and u_j are not conjugate in G for $i \neq j$ but $\{x^{-1}u_ix \mid x \in G, 1 \leqq i \leqq r\}$ is a basis for M. Let M_i be the subgroup generated by all conjugates of u_i. Then M_i is normal in G and $M = M_1 \times M_2 \times \cdots \times M_r$.

First suppose that $r > 1$. Then $|M_i| < |M|$ for $1 \leqq i \leqq r$. The minimality of M implies that $\mathrm{d}(G) > \mathrm{d}(G/C_G(M_i))$ for $1 \leqq i \leqq r$. Since $\bigcap_i C_G(M_i) = C_G(M)$, we obtain $\mathrm{d}(G) > \mathrm{d}(G/C_G(M))$.

Hence $r = 1$. Since $C_G(u_1) \neq G$ and G is nilpotent, we find that $G \neq G'C_G(u_1)$. This implies that $\{x^{-1}u_1x \mid x \in G\}$ is the union of more than one conjugacy class in $G'M$. Let $v_1, \cdots, v_s$ be representatives of all the distinct classes in $G'M$ whose union is $\{x^{-1}u_1x \mid x \in G\}$. Let $N_i = \langle x^{-1}v_ix \mid x \in G'M\rangle$ for $1 \leqq i \leqq s$. Then N_i is normal in $G'M$ and $M = N_1 \times N_2 \times \cdots \times N_s$. Furthermore, $s > 1$ and conjugation by the elements of G transitively permutes the subgroups $N_1, \cdots, N_s$. Since $|N_i| < |M|$, the minimality of M implies that $\mathrm{d}(G'M) > \mathrm{d}(G'M/C_{G'M}(N_i))$ for $1 \leqq i \leqq s$. Since $\bigcap_i C_{G'M}(N_i) = MC_{G'}(M)$, this yields $\mathrm{d}(G'M) > \mathrm{d}(G'/C_{G'}(M)) = \mathrm{d}(G/C_G(M)) - 1$. Now let $T_i = \prod_{j \neq i} N_j$ and $U = [M, G]$. Since conjugation transitively permutes $N_1, \cdots, N_s$ among themselves, $UT_i = M$ for $1 \leqq i \leqq s$.

Hence $G'M = G'T_i$ for $1 \leqq i \leqq s$. $G'M$ is the subdirect product of the isomorphic groups $G'M/T_i$. Thus $\mathrm{d}(G'M) = \mathrm{d}(G'M/T_i) = \mathrm{d}(G'T_i/T_i) = \mathrm{d}(G'/G' \cap T_i)$ for $1 \leqq i \leqq s$. Since $\bigcap_i(G' \cap T_i) = 1$, this implies that $\mathrm{d}(G'M) = \mathrm{d}(G')$. We then obtain $\mathrm{d}(G') > \mathrm{d}(G/C_G(M)) - 1$ which is equivalent to $\mathrm{d}(G) > \mathrm{d}(G/C_G(M))$.

2.6. **Lemma.** *Assume p is an odd prime and V is a vector space of finite dimension n over a field of characteristic p. Assume that x and y are p-elements of* $\mathrm{GL}(V)$ *such that $x^{-1}yx = y^{p+1}$ and $C_V(y)$ has dimension one. Assume further that V contains a subspace U such that $Ux = U$ and V is the direct sum of $C_V(y)$ and U. Then the minimal polynomial of x is $(x-1)^r$ where r is the smallest positive integer such that $rp \geqq n$.*

Proof. Let $G = \langle x,y \rangle$ and $P = \langle x,y^p \rangle$. If g is any nonidentity element of $\langle y \rangle$, then $V(g-1)$ must contain $C_V(y)$. Hence g cannot fix U. Thus $\langle x \rangle \cap \langle y \rangle = 1$. Then $\langle x \rangle$ is quasinormal in G[2, Lemma 4.1(b)]. Let W be the largest subspace of U that is invariant under P. Let z be either of the elements y^p or xy^p. Since $\langle x \rangle$ is quasinormal, $P = \langle x,z \rangle = \langle z \rangle \langle x \rangle$. Now let $W_1/W = C_{U/W}(z)$. Then $W_1P = W_1\langle z \rangle \langle x \rangle = W_1 \langle x \rangle \subset U$. Due to the definition of W, this implies that $W_1 = W$. Hence $C_{V/W}(z)$ has dimension one. Let m be the dimension of V/W. By looking at the Jordan normal form of z acting on V/W, we see that $(V/W)(z-1)^m = (0) \neq (V/W)(z-1)^{m-1}$. Clearly $(V/W)(g-1)^m = 0$ for all $g \in P$ since P is a p-group.

Since P is normal in G, this implies that $V(g-1)^m \subset Wy^i$ for all i. But $\bigcap_i Wy^i = (0)$. Thus $V(g-1)^m = (0)$ for all $g \in P$. Then the minimal polynomial of z must be $(z-1)^m$. Thus $(xy^p-1)^r = 0$ if, and only if, $(y^p-1)^r = 0$. But $xy^p = yxy^{-1}$. Hence $(x-1)^r = 0$ if, and only if, $(y^p-1)^r = 0$. Since $C_V(y)$ has dimension one, the minimal polynomial of y is $(y-1)^n = 0$. This implies that $(y^p-1)^r = 0$ if, and only if, $pr \geqq n$. The lemma now follows.

3. **An upper bound on the class.** Our upper bound on the class is based upon Lemma 3.1 if $p \neq 2$ and upon Lemma 3.3 if $p = 2$.

3.1. Lemma. *Suppose $G = H\langle x \rangle$ is a finite p-group where $|\langle x \rangle| = p^n, n \geqq 3$, H is a core-free quasinormal subgroup, and $p \neq 2$. Let $M = \Omega_{n-1}(G)$ and define $M_0, M_1, \cdots$, inductively by $M_0 = M$ and $M_{i+1} = [M_i, M]\,\mho^1(M_i)$. Let $r = p^{n-3}(p-1)$. Then $M_r = 1$.*

Proof. $H \leqq \Omega_{n-1}(G)$ by [1, Theorem 5.1]. Hence $M = H\langle x^p \rangle$ [2, Lemma 3.1 (b)]. First, suppose $n = 3$. Since H is core-free in G, $H \cap \langle x \rangle = 1$ and $C_G(x) = \langle x \rangle$. Then, since $Z(G)$ must be contained in

$\langle x\rangle$, any nonidentity normal subgroup of G must contain x^{p^2}. M has exponent p^2[**2**, Lemma 3.1(b)] and so if M is abelian, we would have $M_2 = 1$. Since $r = p - 1 \geqq 2$ (this is the only place where the assumption $p \neq 2$ is required), $M_r = 1$ if M is abelian. Suppose M is not abelian. Then $x^{p^2} \in M'$, $M = \langle x^p\rangle H$, and $c(M) \leqq p - 1$ [**2**, Lemma 3.1 (c)]. $M/\Omega_1(G)$ is isomorphic to $\Omega_1(G/\Omega_1(G))$ which is abelian by [**2**, Lemma 3.1]. Hence $M' \leqq \Omega_1(G)$. It follows from all of this that $\mho^1(M) = \langle x^{p^2}\rangle \mho^1(H)$. If K is the core of H in M, then H/K has exponent dividing p[**1**, Theorem 5.1]. Hence $K \geqq \mho^1(H)$. It follows that $[\mho^1(M), M] \leqq K \leqq H$. Since H is core-free in G, $[\mho^1(M), M] = 1$. Since $M_1 \leqq \Omega_1(G)$, we deduce that $M_2 = [M, M, M]$. Then $M_r = M_{p-1} = L_p(M) = 1$.

Thus the lemma is proved if $n = 3$. We now assume that $n > 3$ and proceed by induction on n. Let K be the core of H in M. Then M/K satisfies the hypothesis of the lemma with n replaced by $n-1$. By induction, therefore, M/K satisfies the conclusion of the lemma.

This implies that M contains normal subgroups $N_0, N_1, \cdots, N_s$ such that $N_0 = \langle x^{p^2}\rangle H$, $N_s = K$, $s = p^{n-4}(p-1)$, $N_i \geqq N_{i+1}$, N_i/N_{i+1} is elementary abelian, and $[N_i, \langle x^{p^2}\rangle H] \leqq N_{i+1}$.

Let V_i be N_i/N_{i+1} written additively. Since N_i and N_{i+1} are normal in M, M induces automorphisms of V_i. $\langle x^{p^2}\rangle H$ acts trivially on V_i. Let T_i be the automorphism of V_i induced by x^p. Then $T_i^p - 1 = 0$. Hence $(T_i - 1)^p = 0$. This implies that

$$[N_i, \underbrace{M, \cdots, M}_{p}] \leqq N_{i+1} \quad .$$

$M/N_1 = (\langle x^p\rangle N_1/N_1)(N_0/N_1)$. Hence

$$L_{p+1}(M/N_1) = [N_0, \underbrace{M, \cdots, M}_{p}]\, N_1/N_1 = 1.$$

Therefore $c(M/N_1) \leqq p$. Then $(M/N_1)/Z(M/N_1)$ has class at most $p-1$. If $N_1x^p \notin Z(M/N_1)$, then both $(M/N_1)/Z(M/N_1)$ and $Z(M/N_1)$ have exponent dividing p. It follows from this that $M_p \leqq N_1$ if $N_1x^p \notin Z(M/N_1)$. If, on the other hand, $N_1x^p \in Z(M/N_1)$, then M/N_1 is abelian and $M_p \leqq M_2 \leqq N_1$. Thus it is always true that $M_p \leqq N_1$. Since N_1/N_2 is elementary abelian and

$$[N_1, \underbrace{M, \cdots, M}_{p}] \leqq N_2,$$

we conclude that $M_{2p} \leqq N_2$. Continuing in this way we find that $M_r = M_{sp} \leqq N_s = K \leqq H$. But M_r is a normal subgroup of G and H is core-free in G. Hence $M_r = 1$ and the lemma is proved.

The case $p = 2$ presents more difficulty and we require a preliminary result.

3.2. LEMMA. *Assume $G = CH$ is a finite 2-group where C is cyclic and H is a non-identity core-free quasinormal subgroup of exponent 2^n. Then the following is true:*

(a) $|C| \geqq 2^{n+2}$.
(b) $\Omega_2(C) \leqq Z(G)$.
(c) $c(\Omega_3(G)) \leqq 2$.

PROOF. Let $G_1 = G/\Omega_{n-1}(G)$ and let H_1 and C_1 be the images of H and C, respectively, in G_1. Then H_1 is a core-free quasinormal subgroup of G_1 [2, Lemma 3.1 (b)]. Since $\Omega_{n-1}(G)$ has exponent 2^{n-1} [2, Lemma 3.1 (b)], $H_1 \neq 1$. Hence G_1 is not abelian. But $\Omega_2(G_1)$ is abelian [2, Lemma 3.1 (c)] and $H_1 \leqq \Omega_2(G_1)$. It follows that $|C_1| \geqq 2^3$ which implies (a).

Now let K be the core of H in $H\Omega_2(C)$. (a) applied to $H\Omega_2(C)/K$ yields $|H/K| = 1$. Hence $\Omega_2(C)$ normalizes H. Then $[\Omega_2(C), G] \leqq H$. Since H is core-free in $G = CH$ and G normalizes $[\Omega_2(C), G]$, this implies that $[\Omega_2(C), G] = 1$ and so (b) is proved.

Next let L be the core of $\Omega_3(H)$ in $\Omega_3(G)$. Then $\Omega_3(G)$ is the subdirect product of the isomorphic groups $\{\Omega_3(G)/x^{-1}Lx \mid x \in G\}$. Thus $c(\Omega_3(G)) = c(\Omega_3(G)/L)$. Since $\Omega_3(G) = \Omega_3(H)\Omega_3(C)$ [2, Lemma 3.1 (b)] and $\Omega_3(H)/L$ is a core-free quasinormal subgroup of $\Omega_3(G)/L$, it is sufficient to prove (c) under the assumption that $G = \Omega_3(G)$.

Assume, therefore, that $G = \Omega_3(G)$. Then, by (a), $H \leqq \Omega_1(G)$. Hence $G = \Omega_1(G)C$. Now $|\Omega_1(G)| \leqq 2^2$ by Lemma 3.1 (d) of [2]. This implies that $|G : C| \leqq 2$. Then G is the product of the two normal abelian subgroups C and $\Omega_1(C)$. Thus $c(G) \leqq 2$ and (c) is proved.

3.3. LEMMA. *Suppose $G = H\langle x\rangle$ is a finite 2-group where $|\langle x\rangle| = 2^n$, $n \geqq 4$, and H is a core-free quasinormal subgroup. Let $M = \Omega_{n-2}(G)$ and define $M_0, M_1, \cdots,$ inductively by $M_0 = M$ and $M_{i+1} = [M_i, M][M_i, x^2, x^2]\mho^2(M_i)$. Let $r = 2^{n-4}$. Then $M_r = 1$.*

PROOF. By Lemma 3.2(a), $H \leqq \Omega_{n-2}(G)$. Then $M = H\langle x^4\rangle$. By an induction argument, M_i is a normal subgroup of G for $i = 0, 1, \cdots$. Suppose that $n = 4$. Then M is abelian [2, Lemma 3.1(c)] and $\mho^2(M) = 1$. Therefore $M_1 = [M, x^2, x^2]$. But $M\langle x^2\rangle \leqq \Omega_3(G)$, and, according to Lemma 3.2(c), $\mho_3(G)$ has class at most 2. Hence $M_1 = 1$ and the lemma is proved for $n = 4$.

We now assume that $n > 4$ and proceed by induction on n. Let K be the core of H in $H\langle x^2\rangle$. Then $H\langle x^2\rangle/K$ satisfies the hypothesis of the lemma. By induction, therefore, $H\langle x^2\rangle/K$ satisfies the conclusion.

Thus $H\langle x^2\rangle$ contains normal subgroups $N_0, N_1, \cdots, N_s$ such that $N_0 = H\langle x^8\rangle$, $N_s = K$, $s = 2^{n-5}$, $N_i \geqq N_{i+1}$, N_i/N_{i+1} is abelian of exponent dividing 4, and $[N_i, H\langle x^8\rangle]\ [N_i, x^4, x^4] \leqq N_{i+1}$.

Now $[N_i/N_{i+1}, M] = [N_i/N_{i+1}, x^4]$. Let $P_{2i} = N_i$ and $P_{2i+1} = [N_i, x^2, x^2]\ [N_i, x^4]\, N_{i+1}$. Then x^2 induces an automorphism of order dividing 4 on N_i/N_{i+1} and $[N_i/N_{i+1}, x^4, x^4] = 1$. Lemma 2.4 now implies that $[P_{2i+1}, M]\ [P_{2i+1}, x^2, x^2] \leqq N_{2i}$. Thus $[P_i, M]\ [P_i, x^2, x^2]\ \mho^2(P_i) \leqq P_{i+1}$ for $0 \leqq i \leqq 2s$.

Consider now $M/N_1 = (N_0/N_1)\langle N_1 x^4\rangle$. Since $x^8 \in N_0$ and N_0/N_1 has exponent dividing 4, $\mho^2(M)$ must be contained in $\mho^1(N_0)N_1$. We deduce from this that $M_1 \leqq [N_0, x^4]\ [N_0, x^2, x^2]\ \mho^1(N_0)N_1$. Lemma 2.4 applied to N_0/N_1 being acted upon by x^2 yields $[M_1, x^4]\ [M_1, x^2, x^2]\ \mho^2(M_1) \leqq N_1$. Thus $M_2 \leqq P_2$. Since $[P_2, M]\ [P_2, x^2, x^2]\ \mho^2(P_2) \leqq P_3$, we find that $M_3 \leqq P_3$. Continuing, we conclude that $M_i \leqq P_i$ for $i \geqq 2$. Then $M_r = M_{2s} \leqq P_{2s} = N_s = K \leqq H$. Since M_r is normal in G and H is core-free, this implies that $M_r = 1$.

3.4. Theorem. *Suppose $G = H\langle x\rangle$ is a finite p-group where $|\langle x\rangle| = p^n$ and H is a core-free quasinormal subgroup. Then $c(G) \leqq$* Max$\{1, p^{n-2}(p-1)\}$.

Proof. First suppose $p \neq 2$. If $n \leqq 2$, then the theorem follows from [2, Lemma 3.2]. Therefore we assume $n \geqq 3$. Let $M = H\langle x^p\rangle$ and define $M_0, M_1, \cdots$ inductively by $M_0 = M$ and $M_{i+1} = [M_i, M]\ \mho^1(M_i)$. Lemma 3.1 implies that $M_r = 1$ where $r = p^{n-3}(p-1)$. Let V_i be M_i/M_{i+1} written additively and let T_i be the automorphism of V_i induced by x. M_i/M_{i+1} is elementary abelian and $[M_i/M_{i+1}, x^p] = 1$. Hence $V_i(T_i-1)^p = V_i(T_i^p-1) = 0$. This implies that

$$[M_i, \overbrace{G, \cdots, G}^{p}] \leqq M_{i+1}.$$

Since M/M_1 is a normal abelian subgroup of $G/M_1 = (M/M_1)\langle M_1 x\rangle$, we must have $L_{p+1}(G/M_1) = [M/M_1, G/M_1, \cdots, G/M_1]$. Thus $L_{p+1}(G) \leqq M_1$. Then

$$L_{2p+1}(G) \leqq [M_1, \overbrace{G, \cdots, G}^{p}] \leqq M_2.$$

Continuing, we obtain $L_{ip+1}(G) \leqq M_i$ for $i \geqq 1$. Since $M_r = 1$, $c(G) \leqq rp = p^{n-2}(p-1)$.

Now suppose $p = 2$. If $n \leqq 3$, the result follows from either Lemma 3.2(c) or [2, Lemma 3.2]. Therefore we assume $n \geqq 4$. Let $M = H\langle x^4\rangle$ and define $M_0, M_1, \cdots$ inductively by $M_0 = M$ and $M_{i+1} = [M_i, M]\ [M_i, x^2, x^2]\ \mho^2(M_i)$. Then Lemma 3.3 implies that $M_s = 1$

where $s = 2^{n-4}$. Lemma 2.4 implies that $[M_i/M_{i+1}, x, x, x, x] = 1$. Since $[M_i/M_{i+1}, H] = 1$, we obtain $[M_i, G, G, G, G] \leqq M_{i+1}$. M/M_1 is a normal abelian subgroup in $G/M_1 = (M/M_1)\langle M_1x\rangle$. Thus $L_5(G/M_1) = [M/M_1, G/M_1, G/M_1, G/M_1, G/M_1] = 1$. Hence $L_5(G) \leqq M_1$ and $L_9(G) \leqq [M_1, G, G, G, G] \leqq M_2$. In general, $L_{4i+1}(G) \leqq M_i$ for $i \geqq 1$. Since $M_s = 1$, $c(G) \leqq 4s = 2^{n-2}$.

3.5. THEOREM. *Let H be a core-free quasinormal subgroup of the group G. Suppose K is a subgroup of H such that K has exponent p^n where p is a prime. Then K is nilpotent of class at most* $\mathrm{Max}\{1, p^{n-2}(p-1)\}$.

PROOF. If $x \in G$, let N_x be the core of H in $H\langle x\rangle$. Then K is the subdirect product of the groups $K/(N_x \cap K)$. Thus it suffices to prove the theorem under the assumption that $G = H\langle x\rangle$. If $|G:H|$ is infinite, then x normalizes H [1, Theorem 4.1]. Since H is core-free, this implies that $H = 1$. Thus we may assume that $|G:H|$ is finite. This implies that $|G|$ is finite.

Then H is nilpotent and a Sylow p-subgroup of H is a core-free quasinormal subgroup of a Sylow p-subgroup of G[3]. Thus it suffices to prove the theorem under the assumption that G is a finite p-group and $G = H\langle x\rangle$.

Now let M be the core of $\Omega_n(H)$ in $\Omega_n(G)$. $\Omega_n(G)$ is the subdirect product of the isomorphic groups $\{\Omega_n(G)/y^{-1}My \mid y \in G\}$. Hence $c(K) \leqq c(\Omega_n(G)) = c(\Omega_n(G)/M)$. The theorem now follows from Theorem 3.4.

4. Examples.

4.1. THEOREM. *Let n be a positive integer. Then there is a finite* 2-*group G containing a core-free quasinormal subgroup H such that*

(a) *H has exponent* 2^n.
(b) $c(H) = \mathrm{Max}\{1, 2^{n-2}\}$.
(c) $d(H) = [(n+1)/2]$.

PROOF. Let R_n be the residue classes modulo 2^{n+2}. Let G_n be the permutation group on R_n generated by $\{a_n, b_{n,k} \mid 0 \leqq k \leqq n-1\}$ where $ia_n \equiv i + 1 \pmod{2^{n+2}}$ and

$$ib_{n,k} \equiv \begin{cases} 5i, & \text{if } (2^{n+2}, i) = 2^k, \\ i, & \text{otherwise.} \end{cases}$$

The only difference between this and the definition of the groups constructed by Stonehewer in [4] is that, for an odd prime p, Stonehewer defines $ib_{n,k}$ to be either $(p+1)i$ or i rather than $5i$ or i. As would be expected, many of Stonehewer's arguments carry over to the present case. Therefore, in the proof of the Theorem, some of the details (especially computations) are omitted.

Now let H_n be the stabilzer of 2^{n+2} in G_n. From now on, if there is no danger of confusion, we will write R, G, H, a, and b_k instead of R_n, G_n, H_n, a_n, and $b_{n,k}$, respectively. G_n is transitive and so H_n is core-free in G_n.

First suppose $n = 1$. Then $G = \langle(12345678), (15)(37)\rangle$. It is easily verified that $|G| = 2^4$, that $H = \langle(15)(37)\rangle$, and, by [2, Lemma 4.1], that H is quasinormal in G.

We now suppose $n > 1$. Since $\langle a\rangle$ is regular on R, it follows that $G = H\langle a\rangle$ and $C_H(\langle a\rangle) = 1$. An easy computation shows that a has order 2^{n+2} and that $a^{2^n} \in Z(G)$. Hence G transitively permutes the orbits of $\langle a^{2^{n+1}}\rangle$. These orbits are $\{i, i+2^{n+1}\}$ for $1 \leqq i \leqq 2^{n+1}$. This gives rise to a representation of G_n as a permutation group on R_{n-1}. As in [4], we obtain a homomorphism T of G_n onto G_{n-1} such that $a_nT = a_{n-1}$, $b_{n,k}T = b_{n-1,k}$ for $0 \leqq k \leqq n-2$, $b_{n,n-1}T = 1$, and $H_nT = H_{n-1}$.

Let K be the kernel of T. If $x \in K$, then x fixes the set $\{i, i+2^{n+1}\}$ for all i, $1 \leqq i \leqq 2^{n+1}$. Hence $x^2 = 1$. Thus K is an elementary abelian 2-group. By induction on n, we now conclude that G_n is a 2-group of exponent 2^{n+2} and that H_n has exponent 2^n.

Now if $x \in K$, then either x or $a^{2^{n+1}}x$ fixes 2^{n+2}. This implies that $K = \langle a^{2^{n+1}}\rangle(H \cap K)$. I assert that $K = \Omega_1(G)$. Suppose this is not the the case. Then there exists $x \in G$ such that $x^2 = 1$ but $x \notin K$. Since G is transitive on the orbits of $\langle a^{2^{n+1}}\rangle$, we may assume without loss of generality that x does not fix the set $\{1, 2^{n+1}+1\}$. By induction on n, xT must fix all the orbits of $\langle a_{n-1}^{2^n}\rangle$. This implies that x fixes the set $\{1, 2^n+1, 2^{n+1}+1, 2^{n+1}+2^n+1\}$. It follows from this that x interchanges the two sets $\{1, 2^{n+1}+1\}$ and $\{2^n+1, 2^{n+1}+2^n+1\}$. Since $x^2 = 1$, there are exactly 2 ways that x could operate on $\{1, 2^n+1, 2^{n+1}+1, 2^{n+1}+2^n+1\}$. Both possibilities conflict with the fact that $xa^{2^n} = a^{2^n}x$. Thus $K = \Omega_1(G)$. Then $\Omega_1(H) = H \cap K$.

Since $G_n/\Omega_1(G_n)$ is isomorphic to G_{n-1} and $\Omega_1(G_n) = \Omega_1(\langle a_n\rangle)\Omega_1(H_n)$, it follows that $\Omega_k(G) = \Omega_k(\langle a\rangle)\Omega_k(H)$ for all k.

Since $\Omega_2(\langle a\rangle) \leqq Z(G)$, $\Omega_2(G)'$ must be contained in H. Since H is core-free, this implies that $\Omega_2(G)$ is abelian.

Now let M be a maximal subgroup of G containing H. Since $|G : M| = 2$, we must have $M = H\langle a^2\rangle$. The orbit of 2^{n+2} under M is $\{2i \mid 1 \leqq i \leqq 2^{n+1}\}$. Thus there is a natural representation of M as a permutation group on R_{n-1}. This gives rise to a homomorphism S of M onto G_{n-1} where $a_n{}^2S = a_{n-1}$, $b_{n,k}S = b_{n-1,k-1}$ if $1 \leqq k \leqq n-1$, $b_{n,0}S = 1$, and $H_nS = H_{n-1}$. Let N be the kernel of S. Clearly $N \leqq H$. $M = \langle a^2\rangle H = \Omega_{n+1}(\langle a\rangle)\Omega_{n+1}(H) = \Omega_{n+1}(G)$.

I assert that $\mho^{n+1}(G) = \langle a^{2^{n+1}} \rangle$. By induction, we may assume that $\mho^n(G_{n-1}) = \langle a_{n-1}^{2^n} \rangle$. Hence $\mho^n(G) \leqq \langle a^{2^n} \rangle K \leqq \mho_2(G)$ which is abelian. Thus $\mho^{n+1}(G) \leqq \mho^1(\langle a^{2^n} \rangle)\, \mho^1(K) = \langle a^{2^{n+1}} \rangle \leqq \mho^{n+1}(G)$. Therefore $\mho^{n+1}(G) = \langle a^{2^{n+1}} \rangle$ as claimed.

Now we proceed to prove that H is quasinormal in G. By induction, we may assume that H_{n-1} is quasinormal in G_{n-1}. Hence H/N is quasinormal in M/N and HK/K is quasinormal in G/K. Therefore H is quasinormal in M and HK is quasinormal in G. Let L be any subgroup of G. If $L \leqq M$, then $LH = HL$. Suppose then that $L \nleqq M = \Omega_{n+1}(G)$. This implies that $\mho^{n+1}(L) \neq 1$. It follows that $a^{2^{n+1}} \in L$. Then $HL = H(H \cap K)\langle a^{2^{n+1}} \rangle L = HKL$. But HKL is a subgroup of G since HK is quasinormal in G. Hence $HL = LH$ and H is quasinormal in G. It only remains to calculate $c(H)$ and $d(H)$. Since $HK = H \times \langle a^{2^{n+1}} \rangle$, $c(HK) = c(H)$ and $d(HK) = d(H)$.

Let $c_i = a^{-2^{n-1}-i} b_{n-1} a^{2^{n-1}+i}$ for $1 \leqq i \leqq 2^n$. Then $jc_i \neq j$ if, and only if, $j \equiv i (\mathrm{mod}\, 2^n)$. Thus the points in R not fixed by c_i are $\{i, i+2^n, i+2^{n+1}, i+2^{n+1} + 2^n\}$ which is an orbit of $\langle a^{2^n} \rangle$. Since G transitively permutes the orbits of $\langle a^{2^n} \rangle$ and since each orbit of $\langle a^{2^n} \rangle$ contains exactly one element of the set $\{1, 2, 3, \cdots, 2^n\}$, we see that $c_1, \cdots, c_{2^n}$ are distinct and that $\{c_1, \cdots, c_{2^n}\}$ is a conjugacy class in G. Now $c_i \in K$ and K is elementary abelian. It is immediate that $\{c_1, \cdots, c_{2^n}\}$ is an independent set of elements of K. Hence $|\langle c_i | 1 \leqq i \leqq 2^n \rangle| = 2^{2^n}$. It follows from Lemma 3.1(d) of [2] that $|\Omega_1(G)| \leqq 2^{2^n}$. Hence $\{c_i \mid 1 \leqq i \leqq 2^n\}$ is a basis for K.

I claim that $C_G(K) = \Omega_2(G)$. Clearly $\Omega_2(G) \leqq C_G(K)$ since $K \leqq \mho_2(G)$ and $\Omega_2(G)$ is abelian. Suppose $x \in C_G(K)$. Then $xc_i = c_i x$. It follows that x fixes the set $\{j \mid jc_i \neq j\}$. Hence x fixes each orbit of $\langle a^{2n} \rangle$. Then xT fixes each orbit of $\langle a_{n-1}^{2^n} \rangle$. This implies that $xT \in \Omega_1(G_{n-1})$. Thus $x^2 T = 1$. This shows that $x^2 \in K$. Hence $x^4 = 1$ and so $x \in \Omega_2(G)$.

Now $c(H) \leqq \mathrm{Max}\{1, 2^{n-2}\}$ by Theorem 3.5 and $d(H) \leqq [(n+1)/2]$ by [2, Theorem 3.4(c)]. It only remains to show that $c(H) \geqq \mathrm{Max}\{1, 2^{n-2}\}$ and $d(H) \geqq [(n + 1)/2]$. If $n \leqq 2$, this is trivial. We now assume $n > 2$. Lemma 2.5 implies that $d(H) = d(HK) > d(HK/C_{HK}(K)) = d(H/\Omega_2(H))$. Now $H/\Omega_1(H)$ is isomorphic to H_{n-1} and $\Omega_2(H)/\Omega_1(H) = \Omega_1(H/\Omega_1(H))$. Thus $H/\Omega_2(H)$ is isomorphic to $H_{n-1}/\Omega_1(H_{n-1})$ which is isomorphic to H_{n-2}. Thus $d(H_n) > d(H_{n-2})$. By induction on n, $d(H_{n-2}) = [(n-1)/2]$. Then $d(H_n) \geqq 1 + [(n-1)/2] = [(n+1)/2]$.

Since $H/\Omega_2(H)$ has exponent 2^{n-2}, there is an element $x \in H$ and distinct basis elements $d_1, \cdots, d_{2^{n-2}}$ in K such that $x^{-1} d_i x = d_{i+1}$ if $1 \leqq i \leqq 2^{n-2}$ and $x^{-1} d_{2^{n-2}} x = d_1$. Then

$$[d_1, x, \underbrace{x, \cdots, x}_{2^{n-2}-1}] = d_1 d_2 \cdots d_{2^{n-2}} \neq 1.$$

Thus $c(H) = c(HK) \geqq 2^{n-2}$. This finishes the proof of the theorem.

4.2. **Theorem.** *Let p be a prime and n a positive integer. Then there is a finite p-group G containing a core-free quasinormal subgroup H such that H has exponent p^n and $c(H) = \mathrm{Max}\{1, p^{n-2}(p-1)\}$.*

Proof. If $p = 2$, this follows from the previous theorem, and if $n = 1$, this follows from [2, Lemma 4.1]. Accordingly, we assume that p is odd and $n > 1$. The method of constructing our examples is the same as the method used in [2, page 549]. Let $m = p^{n-1}(p-1)$ and let W be a vector space of dimension m with basis $\{v_1, v_2, \cdots, v_m\}$ over the field of p elements. Let W_1 be the subspace spanned by $\{v_2, v_3, \cdots, v_m\}$. Let Y be the linear transformation of W determined by $v_1 Y = v_1$ and $v_i Y = v_i + v_{i-1}$ for $2 \leqq i \leqq m$. Then the minimal polynomial of Y is $(Y-1)^m$. Hence Y has order p^n. According to [2, Lemma 5.1], there is a p-element X in $\mathrm{GL}(W)$ such that $v_1 X = v_1$, $W_1 X = W_1$, and $X^{-1}YX = Y^{p+1}$. As is shown in [2], X must have order p^{n-1}. By Lemma 2.6, the minimal polynomial of X is $(X-1)^r$ where $r = p^{n-2}(p-1)$.

Now let A be the group generated by two elements a and b subject only to the relations $b^{p^{n+1}} = a^{p^n} = 1$ and $a^{-1}ba = b^{p+1}$. Then $a \to X$, $b \to Y$ determines a homomorphism of A into $\mathrm{GL}(W)$. Let B be the semi-direct product AV relative to the above homomorphism.

Using the same argument as in [2, page 549], it can be shown that $\langle b^{p^n} v_1^{-1}, a^{p^{n-1}} v_2 \rangle$ is a normal elementary abelian subgroup in B.

Let G be the factor group of B modulo this subgroup. Let V, U, x, and y denote the images in G of W, W_1, a, and b, respectively. Since $W_1 X = W_1$, x normalizes U. Let $H = U\langle x \rangle$. Since $(Y^{p^{n-1}} - 1)^{p-1} = (Y-1)^m = 0$, we conclude that

$$[V, \underbrace{y^{p^{n-1}}, y^{p^{n-1}}, \cdots, y^{p^{n-1}}}_{p-1}] = 1.$$

Thus $c(\Omega_2(\langle y \rangle)V) \leqq p-1$. Then Theorem 4.2 of [2] implies that H is quasinormal in G. x has order p^n and $\Omega_1(\langle x \rangle) \leqq U$. Since U is a normal elementary abelian subgroup of H and $H = U\langle x \rangle$, H must have exponent p^n. Since the minimal polynomial of X is $(X-1)^r$ where $r = p^{n-2}(p-1)$, the class of H must be $p^{n-2}(p-1)$.

It only remains to show that H is core-free. If H is not core-free, then H contains an element z of order p such that $z \in Z(G)$. Since $|C_{W_1'}(Y)| = 1$, z cannot belong to U. Since H/U is cyclic, it follows that $U\langle z \rangle =$

$U\langle x^{p^{n-2}}\rangle$. This implies that $[y, x^{p^{n-2}}] \in V$. But $[y, x^{p^{n-2}}] = y^{t-1}$ where $t-1 = (p+1)^{p^{n-2}} - 1 \equiv p^{n-1} (\mathrm{mod}\, p^n)$ [2, Lemma 2.4]. Since $\langle y\rangle \cap V = \langle y^{p^n}\rangle$, this is a contradiction. Hence H is core-free and the theorem is proved.

References

1. R. Bradway, F. Gross, and W. Scott, *The nilpotence class of core-free quasinormal subgroups*, Rocky Mt. J. Math. 1 (1971), 375-382.

2. F. Gross, *p-subgroups of core-free quasinormal subgroups*, Rocky Mt. J. Math. 1 (1971), 541-550.

3. N. Itô and J. Szep, *Über die Quasinormalteiler von endlichen Gruppen*, Acta Sci. Math. (Szeged) 23 (1962), 168-170.

4. S. Stonehewer, *Permutable subgroups of some finite permutation groups*, Proc. London Math. Soc. (to appear).

University of Utah, Salt Lake City, Utah 84112