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TAMELY RAMIFIED EXTENSIONS OF HENSELIAN FIELDS

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Let F be a Henselian field; that is, a valued field whose valuation extends uniquely to any algebraic extension of F [8, p. 175]. Denote the residue class field, value group, and maximal ideal of the valuation ring of F by k_F, Γ_F and M_F, respectively (and similarly for other valued fields). Recall that a finite algebraic extension K/F is *tamely ramified* [1, pp. 67–68] if k_K/k_F is separable, $(\Gamma_K : \Gamma_F)$ is not divisible by the characteristic of k_F, and

$$(1) \qquad [K:F] = [k_K : k_F](\Gamma_K : \Gamma_F).$$

Now let k be a finite separable extension of k_F and Γ be an ordered group containing Γ_F with $(\Gamma : \Gamma_F)$ finite and not divisible by the characteristic of k_F. A field extension K/F is a *(k, Γ)-extension* when $\Gamma_K = \Gamma$, k_K is k_F-isomorphic to k, and (1) holds. We construct a bijection from the set of F-isomorphism classes of (k, Γ)-extensions to the set of orbits of $k^\times \otimes \Gamma/\Gamma_F$ (tensor product as abelian groups; $k^\times$ denotes the multiplicative group of nonzero elements of k) under the action of the group g of k_F-automorphisms of k (§ 1). We next show (§ 2) how the orbit of a (k, Γ)-extension determines the automorphism group of the extension, considered as a group extension of $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ by the stabilizer in g of an element of the orbit. A four term exact sequence describes which such group extensions are the Galois groups of (k, Γ)-extensions, and which Galois groups of (k, Γ)-extensions split as group extensions. This section has a considerable overlap with [10, pp. 70–73]. Under some restrictions on F (which hold for certain "ultracomplete fields" [3, 5]) we characterize the abelian tame extensions and count, for example, the number of normal (k, Γ)-extensions (§ 3). Under additional restrictions on F (which hold for nonArchimedean local fields), we show how the orbit of a normal (k, Γ)-extension determines the structure of its norm factor group, as a group extension of $k_F{}^\times/k_F{}^{\times e}$ by $\Gamma_F/f\Gamma_F$ (e and f denote the ramification index and residue class degree, respectively) (§ 4). For abelian extensions, this allows us to show that the norm factor group and Galois group are isomorphic as group extensions; the isomorphisms we construct in an elementary way can also be obtained for local fields from the prop-

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erties of the norm residue symbol of local class field theory. Finally (§ 5), we compute the structure of the "multiplicative congruence group" $K^\times/1 + M_K$ of a (k, Γ)-extension K/F (again, as a certain group extension). This leads to a new proof of the basic bijection of § 1 using the Δ-functor of [4]. The close connection of orbits with the "signatures" of [2] is also observed.

The cardinality of a set A is denoted by $|A|$. Z denotes the set of rational integers.

1. **Orbits.** Let v denote the valuation on F; we also denote by v its unique extension to the algebraic closure of F. Let g denote the automorphism group of the extension k/k_F, i.e., the group of automorphisms of k which leave k_F fixed. g acts on $k^\times \otimes \Gamma/\Gamma_F$ in the obvious way; the *orbit* of an element of $k^\times \otimes \Gamma/\Gamma_F$ is the set of its images under the elements of g.

Let us fix once and for all a set $A \subset F$ which v maps bijectively to Γ_F (i.e., a system of representatives for Γ_F). Fix a subset $\gamma_1, \cdots, \gamma_t$ of Γ mapping bijectively to a basis for Γ/Γ_F. Let e_i denote the order of $\gamma_i + \Gamma_F$ $(i \leqq t)$. Let a_i denote the element of A with $v(a_i) = -e_i\gamma_i$ $(i \leqq t)$.

Now let K/F be a (k, Γ)-extension. Pick $b_i \in K$ with $v(b_i) = \gamma_i$ $(i \leqq t)$. Set

$$\alpha_K = \sum_{i \leqq t} (a_i b_i^{e_i} + M_K) \otimes \gamma_i + \Gamma_F \; (\in k_K^\times \otimes \Gamma/\Gamma_F).$$

1.1 Note. α_K is independent of the choice of b_i.

Proof. If $v(b_i') = \gamma_i$, then because $e_i(\gamma_i + \Gamma_F) = 0$,

$$\begin{aligned}&(a_i b_i^{e_i} + M_K \otimes \gamma_i + \Gamma_F)(a_i b_i'^{e_i} + M_K \otimes \gamma_i + \Gamma_F)^{-1}\\ &= (b_i/b_i' + M_K)^{e_i} \otimes \gamma_i + \Gamma_F\\ &= 1 \otimes 0.\end{aligned}$$

Any k_F-isomorphism of k_K into k carries α_K into an element of $k^\times \otimes \Gamma/\Gamma_F$. Any two such isomorphisms differ by an automorphism in g, so the orbit of this element is uniquely determined. We call it the *orbit of the extension K/F*.

1.2 Theorem. *Assigning to each (k, Γ)-extension its orbit induces a bijection from the set of F-isomorphism classes of (k, Γ)-extensions to the set of orbits of elements of $k^\times \otimes \Gamma/\Gamma_F$.*

We will prove 1.2 in this section using standard techniques on Henselian fields. A different insight is provided by the proof of 1.2 that we sketch in Remark 5.2 A. For some related results and special cases of

1.2, see [**12**, pp. 240–241] and [**13**].

Isomorphic (k, Γ)-extensions of F clearly have the same orbit. That each orbit (of an element) of $k^\times \otimes \Gamma/\Gamma_F$ is the orbit of one and only one F-isomorphism class of (k, Γ)-extensions follows from Lemmas 1.4 and 1.5 below (respectively). The notation of Lemma 1.4 will be used in later sections.

1.3 Note. Since the γ_i map to a basis of Γ/Γ_F, every element of $k^\times \otimes \Gamma/\Gamma_F$ can be written in the form $\sum \alpha_i \otimes \gamma_i + \Gamma_F$ $(\alpha_i \in k^\times)$. We have a g-module isomorphism

$$\sum_{i \leqq t} \alpha_i \otimes \gamma_i + \Gamma_F \to \prod_{i \leqq t} \alpha_i k^{\times e_i}$$

from $k^\times \otimes \Gamma/\Gamma_F$ onto $\prod_{i \leqq t} k^\times/k^{\times e_i}$. The reader who wishes to generalize the results of this section to tamely ramified algebraic (k, Γ)-extensions of infinite degree (but with Γ/Γ_F admitting an infinite basis $\{\gamma_i + \Gamma_F : i \in I\}$) should replace $k^\times \otimes \Gamma/\Gamma_F$ by $\prod_{i \in I} k^\times/k^{\times e_i}$.

1.4 Lemma. *Let $\alpha = \sum \alpha_i \otimes \gamma_i + \Gamma_F$ $(\alpha_i \in k^\times)$. Let E be an unramified extension of F admitting a k_F-isomorphism μ from k_E onto k. Let c_i be a unit of E with $\mu(c_i + M_E) = \alpha_i$ $(i \leqq t)$. Let b_i be an e_ith root of c_i/a_i (in an algebraic closure of E). Then $K = E[b_1, \cdots, b_t]$ is a (k, Γ)-extension of F with $\mu(\alpha_K) = \alpha$. (Indeed, $k_E = k_K$ and $\mu(a_i b_i^{e_i} + M_K) = \alpha_i$ for all $i \leqq t$.)*

The proof of 1.4 is immediate.

1.5 Lemma. *Let K and K' be (k, Γ)-extensions of F. Suppose there is a k_F-isomorphism $\sigma : k_K \to k_{K'}$ with $\sigma(\alpha_K) = \alpha_{K'}$. Then σ is induced by an F-isomorphism from K to K'.*

Proof. We may suppose K is the field of Lemma 1.4. σ is induced by an F-isomorphism σ_0 of E into K'. (A standard argument. $E = F[\beta]$ where β is a unit whose residue class generates the extension k_E/k_F. By Hensel's lemma [**8**, Theorem 4, p. 185], the irreducible polynomial of β over F has a zero in K', call it β', whose residue class is $\sigma(\beta + M_E)$. It now suffices to map β to β'.) Note that for all $i \leqq t$, $a_i x^{e_i} - c_i$ is irreducible over $E[b_1, b_2, \cdots, b_{i-1}]$ (an extension by a zero would have ramification index at least e_i). It suffices to show that $\sigma_0(c_i/a_i)$ has an e_ith root in K' for all $i \leqq t$ (for then σ_0 extends to an isomorphism $K \to K'$). So fix $i \leqq t$. Let $v(b_i') = \gamma_i$ where $b_i' \in K'$. Since $\sigma(\alpha_K) = \alpha_{K'}$,

$$\sigma(c_i + M_K)/(a_i b_i'^{e_i} + M_{K'}) \in k_{K'}^{\times e_i}$$

(cf. Note 1.3). By Hensel's lemma [8, p. 185], $1 + M_{K'} \subset (K')^{\times e_i}$. Hence

$$\sigma_0(c_i/a_i) \in b_i'^{e_i}\ U_{K'}^{e_i}\ (1 + M_{K'}) \subset (K')^{\times e_i}.$$

(Here, $U_{K'}$ denotes the set of units of K'.)

2. **Automorphisms of (k, Γ)-extensions.** Let $\alpha \in k^\times \otimes \Gamma/\Gamma_F$. Let $\mathcal{S}(\alpha) = \{\sigma \in g : \sigma(\alpha) = \alpha\}$; $\mathcal{S}(\alpha)$ is the *stabilizer* of α in g. $\mathcal{S}(\alpha)$ acts on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ in a canonical way (namely, composition of functions). Let K/F be a (k, Γ)-extension with orbit generated by α. We now compute the automorphism group $G = G_K$ of the extension K/F as a group extension of $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ by $\mathcal{S}(\alpha)$ (with the above canonical action of $\mathcal{S}(\alpha)$ on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$). (For factor sets and group extensions see [1, pp. 108–117] or [7, pp. 108–112].)

We adopt the notation of Lemma 1.4; we can do this without loss of generality by Theorem 1.2. The isomorphism μ of Lemma 1.4 will be treated as an identification of k and k_K.

Let ϵ_i denote the number of e_ith roots of unity in k $(i \leqq t)$. Thus k (and hence K) has a primitive ϵ_ith root of unity.

2.1 PROPOSITION. *We have an exact sequence*

$$1 \to \mathrm{Hom}(\Gamma/\Gamma_F, k^\times) \xrightarrow{\delta} G_K \xrightarrow{\delta'} \mathcal{S}(\alpha) \to 1. \tag{2}$$

Here, δ' assigns to each element of G_K the induced automorphism of k. δ assigns to each $f \in \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ the unique $\sigma \in \ker \delta'$ with $f(v(a) + \Gamma_F) = \sigma(a)/a + M_K$ $(a \in K^\times)$. Moreover, the action of $\mathcal{S}(\alpha)$ on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ induced by the group extension (2) is the natural one.

PROOF. By Note 1.1, δ' maps into $\mathcal{S}(\alpha)$; Lemma 1.5 says that δ' is surjective. When K/F is normal then k/k_F is normal and the sequence

$$1 \to \mathrm{Hom}(\Gamma/\Gamma_F, k^\times) \to G_K \to g \to 1 \tag{3}$$

is well known to be exact (cf. [11, pp. 67–78] or argue as in [1, p. 76]). Set $L = E[b_1^{\epsilon_1}, b_2^{\epsilon_2}, \cdots, b_t^{\epsilon_t}]$ (notation as in Lemma 1.4). K is the splitting field of $\Pi_{i \leqq t} X^{\epsilon_i} - b_i^{\epsilon_i}$ over L, so K/L is normal. Hence $L \supset K^G$, so $\Gamma_L = \Gamma_F + \sum_{i \leqq t} Z\epsilon_i \gamma_i \supset \Gamma_{K^G} \supset \Gamma_F$. But for each $f \in \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$, f maps $\gamma_i + \Gamma_F$ into an ϵ_ith root of unity $(i \leqq t)$. Hence f kills Γ_{K^G}/Γ_F. Consequently we can identify $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ with $\mathrm{Hom}(\Gamma/\Gamma_{K^G}, k^\times)$. We then deduce the exactness of (2) from the exactness of (3) in the case $F = K^G$.

For each $\sigma \in \mathcal{S}(\alpha)$ pick a "coset representative" $u_\sigma \in \delta'^{-1}(\sigma)$. For any $\tau \in \ker \delta'$, let $f_\tau = \delta^{-1}(\tau)$. For any $\gamma = v(a) + \Gamma_F \in \Gamma/\Gamma_F$ we have

$$\begin{aligned} f_{u_\sigma \tau u_\sigma^{-1}}(\gamma) &= u_\sigma \tau u_\sigma^{-1}(a)/a + M_K \\ &= \sigma(\tau(u_\sigma^{-1}(a))/u_\sigma^{-1}(a) + M_K) \\ &= \sigma(f_\tau(\gamma)) \end{aligned}$$

so $\mathcal{S}(\alpha)$ acts on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ by composition. The proposition is proved.

We can now prove a known condition for normality [**10**, p. 71, Theorem 7] by an easy counting argument.

Let e denote the characteristic exponent of Γ/Γ_F; so e is the least common multiple of the e_i. Note that e is not necessarily the ramification index.

2.2 Corollary. *K/F is normal if and only if k/k_F is normal, α is invariant under g, and k has a primitive eth root of unity.*

Proof. Clearly, $|\mathcal{S}(\alpha)| \leqq |g| \leqq [k : k_F]$ and $|\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)| \leqq (\Gamma : \Gamma_F)$. Equality holds everywhere above if and only if $|G_K| = [K : F]$ (Proposition 2.1 and display (1)), i.e., K/F is normal.

2.3 Remark. We can read off from (2) the degree, residue class degree, and ramification index of K/K^G. Also it is easy to see that $\Gamma_{K^G} = \Gamma_F + \sum_{i \leqq t} Ze_i\gamma_i$.

We now show how α determines a factor set for the group extension (2). Write $\alpha = \sum_{i \leqq t} \alpha_i \otimes \gamma_i + \Gamma_F$ $(\alpha_i \in k^\times)$. For each $\sigma \in \mathcal{S}(\alpha)$ and $i \leqq t$, $\sigma(\alpha_i)\alpha_i^{-1}$ has an e_ith root in $k^\times$ (since $\sigma(\alpha_i)\alpha_i^{-1} \otimes \gamma_i + \Gamma_F = 1 \otimes 0$); pick one and denote it by $c_{\sigma,i}$. Define

$$c = c_{\sigma,\tau} : \mathcal{S}(\alpha) \times \mathcal{S}(\alpha) \to \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$$

by

$$c_{\sigma,\tau}(\gamma_i + \Gamma_F) = c_{\sigma,i}\sigma(c_{\tau,i})c_{\sigma\tau,i}^{-1} \quad (i \leqq t).$$

2.4 Proposition. *c is a factor set for the group extension* (2).

Proof. For each $\sigma \in \mathcal{S}(\alpha)$ pick a representative $u_\sigma \in \delta'^{-1}(\sigma)$; let $g_{\sigma,\tau}$ be the associated factor set for the group extension (2). Since $v(b_i) = \gamma_i$ $(i \leqq t)$ (notation of 1.4), we have

$$\begin{aligned} g_{\sigma,\tau}(\gamma_i + \Gamma_F) &= b_i^{-1} u_\sigma u_\tau u_{\sigma\tau}^{-1}(b_i) + M_K \\ &= \sigma\tau(b_i^{-1} u_{\sigma\tau}^{-1}(b_i) + M_K / b_i^{-1}(u_\sigma u_\tau)^{-1}(b_i) + M_K) \end{aligned}$$

$$= (b_i/u_{\sigma\tau}(b_i) + M_K)/(b_i/u_\sigma u_\tau(b_i^{-1}) + M_K)$$
$$= (u_\sigma(b_i)/b_i)u_\sigma(u_\tau(b_i)/b_i)(b_i/u_{\sigma\tau}(b_i)) + M_K$$
$$= d_{\sigma,i}\sigma(d_{\tau,i})d_{\sigma\tau,i}^{-1}$$

where, for each $\sigma \in \mathcal{S}(\alpha)$ and $i \leqq t$, we set $d_{\sigma,i} = u_\sigma(b_i)/b_i + M_K$. Since

$$d_{\sigma,i}^{e_i} = u_\sigma(a_i b_i^{e_i})/a_i b_i^{e_i} + M_K = \sigma(\alpha_i)/\alpha_i = c_{\sigma,i}^{e_i},$$

there exist e_ith roots of unity $\zeta_{\sigma,i} \in k^\times$ with $c_{\sigma,i} = \zeta_{\sigma,i}\, d_{\sigma,i}$ $(i \leqq t, \sigma \in \mathcal{S}(\alpha))$. Now define $\zeta_\sigma \in \operatorname{Hom}(\Gamma/\Gamma_F, k^\times)$ $(\sigma \in \mathcal{S}(\alpha))$ by setting $\zeta_\sigma(\gamma_i + \Gamma_F) = \zeta_{\sigma,i}$ for all $i \leqq t$. Then $c_{\sigma,\tau} = g_{\sigma,\tau}\zeta_\sigma\sigma(\zeta_\tau)\zeta_{\sigma\tau}^{-1}$. Thus c is a factor set cohomologous to $g_{\sigma,\tau}$. Proposition 2.4 is proved.

The differences between the above calculation of G_K and the calculations of [**10**, pp. 71–72] (at least in the normal case) arise from our desire to compute G_K entirely in terms of α, k, k_F, Γ and Γ_F.

We now suppose k/k_F is normal and k has a primitive eth root of unity. By Corollary 2.2, the normal (k, Γ)-extensions are those whose orbit consists of a single element of $(k^\times \otimes \Gamma/\Gamma_F)^g$ (the set of elements invariant under g). In the next proposition we calculate when the sequence (2) splits and identify those group extensions of $\operatorname{Hom}(\Gamma/\Gamma_F, k^\times)$ by g which are the Galois groups of normal (k, Γ)-extensions (compare with [**10**, Theorem 8, p. 73]).

2.5 Proposition. *With the above hypotheses we have an exact sequence*

$$k_F^\times \otimes \Gamma/\Gamma_F \xrightarrow{\varphi} (k^\times \otimes \Gamma/\Gamma_F)^g \xrightarrow{\varphi'} H^2(g, \operatorname{Hom}(\Gamma/\Gamma_F, k^\times))$$
$$\xrightarrow{\varphi''} H^2(g, \oplus_{i \leq t} k^\times).$$

Here, g acts on $\operatorname{Hom}(\Gamma/\Gamma_F, k^\times)$ *and* $\oplus_{i\leq t}k^\times$ *in the obvious way. φ is induced by the inclusion $k_F \to k$. φ' assigns to each element of $k^\times \otimes \Gamma/\Gamma_F$ the Galois group (considered as a group extension) of the corresponding (k, Γ)-extension (cf. 2.1). φ'' is induced by the map taking each $f \in$* $\operatorname{Hom}(\Gamma/\Gamma_F, k^\times)$ *to* $\oplus_{i \leq t} f(\gamma_i + \Gamma_F) \in \oplus_{i\leq t} k^\times$.

Proof. One checks directly that φ' is a group homomorphism; suppose α is in its kernel. Let K/F be the (k, Γ)-extension with orbit $\{\alpha\}$. Then G_K has a subgroup H mapping bijectively to g. K/K^H is unramified of degree $[k : k_F]$, so K^H/F is totally ramified with value group Γ. Hence there exist $b_i \in K^H$ with $v(b_i) = \gamma_i$ $(i \leqq t)$. But then $a_i b_i^{e_i} + M_K \in k_F$ $(i \leqq t)$ so α is in the image of $k_F^\times \otimes \Gamma/\Gamma_F$ (cf. Note 1.1). Conversely, suppose $\alpha = \sum \alpha_i \otimes \gamma_i + \Gamma_F \in k^\times \otimes \Gamma/\Gamma_F$ where each $\alpha_i \in k_F$. We use the notation of Lemma 1.4; we may suppose

$c_i \in F$ for all $i \leqq t$. Thus K is an unramified extension of $F[b_1, \cdots, b_t]$ with residue class field extension k/k_F. The Galois group of $K/F[b_1, \cdots, b_t]$ maps bijectively to g, so the exact sequence (2) splits. This proves exactness at $(k^\times \otimes \Gamma/\Gamma_F)^g$.

One easily checks that $\varphi''\varphi' = 0$. (In the notation of 2.4, c maps to the coboundary of the 1-cochain taking each $\sigma \in g$ to $\oplus_{i \leq t} c_{\sigma,i}$.) Now suppose $h_{\sigma,\tau} : g \times g \to \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ is a factor set whose cohomology class is killed by φ''. Then there exist $d_{\sigma,i} \in k^\times$ with

$$h_{\sigma,\tau}(\gamma_i + \Gamma_F) = d_{\sigma,i}\, \sigma(d_{\tau,i}) d_{\sigma\tau,i}^{-1}$$

($\sigma \in g, i \leqq t$). (Take $\oplus_{i \leq t} d_{\sigma,i}$ to be any 1-cochain in $\oplus_{i \leq t} k^\times$ whose coboundary is $\varphi''(h_{\sigma,\tau})$.) Since $h_{\sigma,\tau}(\gamma_i + \Gamma_F)^{e_i} = 1$ for each $i \leqq t$, $\{d_{\sigma,i}^{e_i} : \sigma \in g\}$ is a solution to Noether's equations. Hence there exist [**1**, p. 118, Lemma] $\alpha_i \in k^\times$ with $d_{\sigma,i}^{\epsilon_i} = \sigma(\alpha_i)\alpha_i^{-1}$ for all $\sigma \in g$, $i \leqq t$. But then $\sum \alpha_i \otimes \gamma_i + \Gamma_F$ is in $(k^\times \otimes \Gamma/\Gamma_F)^g$ and maps to the cohomology class of $h_{\sigma,\tau}$.

3. **Applications to ultracomplete fields.** We assume in this section that F is a Henselian field whose residue class field k_F is either locally finite, real closed, or algebraically closed. Among such valued fields are those fields ultracomplete at a Harrison prime [**5**] or an extended prime spot [**3**].

The next proposition generalizes [**12**, p. 242].

3.1 PROPOSITION. *A finite dimensional tamely ramified extension K/F is abelian if and only if k_F has a primitive eth root of unity, where e is the characteristic exponent of Γ_K/Γ_F.*

The reader will find it easy to generalize 3.1 to extensions of infinite dimension.

PROOF. We may suppose that K/F is a (k, Γ)-extension. First suppose K/F is abelian. Then k has a primitive eth root of unity (Corollary 2.2) which must indeed lie in k_F since g acts trivially on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ (cf. Proposition 2.4). Now suppose k_F has a primitive eth root of unity. Then g acts trivially on $\mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$. Also g is cyclic, and hence the extension (2) admits a symmetric factor set. Thus G_K is abelian. To show K/F is normal, it suffices to show that g acts trivially on $k^\times \otimes \Gamma/\Gamma_F$ (Corollary 2.2). This is obvious if k is of characteristic zero (for then $k^\times \otimes \Gamma/\Gamma_F$ or g is trivial). So suppose k is locally finite. Let $\beta \in k^\times, \gamma \in \Gamma/\Gamma_F$, and $\sigma \in g$. Let $P = |k_0[\zeta]|$ where k_0 is the prime subfield of k and ζ is a primitive eth root of unity. For some integer i, $\sigma(\beta) = \beta^{p^i}$ (every conjugate of β over k_F is a conjugate of β over $k_0[\zeta]$, and hence has the above form). But $e \mid p^i - 1$.

Hence $\sigma(\beta \otimes \gamma)(\beta \otimes \gamma)^{-1} = \beta^{p^i-1} \otimes \gamma = 1 \otimes 0$. Thus every $\beta \otimes \gamma$ is invariant under g. The proposition is proved.

For the remainder of this section we assume k has a primitive eth root of unity, e the characteristic exponent of Γ/Γ_F. We now count the number of normal (k, Γ)-extensions. Since k/k_F is normal, this amounts to computing $|(k^\times \otimes \Gamma/\Gamma_F)^g|$ (Theorem 1.2 and Corollary 2.2).

3.2 Proposition. *Suppose k is locally finite. Then $(k^\times \otimes \Gamma/\Gamma_F)^g$ and $k_F{}^\times \otimes \Gamma/\Gamma_F$ are isomorphic as groups.*

The above proposition can be extended (trivially) to include the case that k_F is real or algebraically closed by replacing $k_F{}^\times \otimes \Gamma/\Gamma_F$ above by $N_g k^\times \otimes \Gamma/\Gamma_F$.

Lemma. *Let $a, b \in k^\times$ (hypotheses as in* 3.2*). There exists a finite subfield k_1 of k and a generator σ of g such that* (i) *k_1 contains a, b, and all eth roots of unity,* (ii) *$\sigma(c) = c^{p+1}$ for all $c \in k_1$, where $p = |k_1{}^\times \cap k_F|$, and* (iii) *$N_g(c) = c^{P/p}$ for all $c \in k_1$, where $P = |k_1{}^\times|$.*

Proof. Write $k = k_F[d]$. Let k_1 be any finite subfield of k containing a, b, d and the eth roots of unity. Each element of g is determined by its action on d; hence the restriction of any generator of g to k_1 has fixed fields $k_1 \cap k_F$ and order $|g|$. Hence the restriction map from g to the Galois group of $k_1/k_1 \cap k_F$ is bijective. The lemma can now be checked.

Proof of 3.2. Let s be the number of eth roots of unity in k_F; thus $k_F{}^{\times e} = k_F{}^{\times s}$ (this follows from the special case when $k_F{}^\times$ is finite and hence cyclic). We first show $(k^\times/k^{\times e})^g$ and $k_F{}^\times/k_F{}^{\times e}$ are both isomorphic to $k^\times/k^{\times s}$. The norm map $N_g : k \to k_F$ is surjective, so $N_g(k^{\times s}) = k_F{}^{\times s} = k_F{}^{\times e}$. On the other hand, if $N_g(a) \in k_F{}^{\times e}$ $(a \in k^\times)$, then $N_g(a) = N_g(b^e)$ for some $b \in k^\times$. Hence (notation as in the Lemma) $ab^{-e} = c^p$ for some $c \in k$. But then $a = b^e c^p \in k^{\times s}$ (for, $s = (e, p)$). Thus N_g induces an isomorphism $k^\times/k^{\times s} \to k_F{}^\times/k_F{}^{\times e}$. Now consider the map $\varphi : K^\times \to k^\times/k^{\times e}$ with $\varphi(a) = a^{e/s}k^{\times e}(a \in k^\times)$. Clearly, $\varphi(k^\times s) = 1$. On the other hand if $a^{e/s} = b^e$ for some $a, b \in k^\times$, then (with notation as in the Lemma) $a^{P/s} = b^p = 1$, so $a \in k^{\times s}$. (After all, $k_1{}^\times$ is cyclic of order P.) Hence $k^{\times s}$ is the kernel of φ. We compute the image. For any $a \in k^\times$ and σ as in the Lemma,

$$\sigma(\varphi(a)) = (a^{e/s})^{p+1}k^{\times e} = a^{e/s}k^{\times e} = \varphi(a)$$

(since $s \mid p$). Hence φ maps into $(k^\times/k^{\times e})^g$. Now suppose $ak^{\times e} \in (k^\times/k^{\times e})^g$. Then $\sigma(a)a^{-1} = a^p = c^e$ for some $c \in k^\times$. $N_g(c) = c^{P/p}$ is an eth root of unity in k_F, and hence an sth root of unity. But then

$a^{P/(e/s)} = a^{p(Ps/pe)} = (c^{P/p})^s = 1$ (note that $pe \mid Ps$). Hence $a \in k^{\times e/s}$. Thus $ak^{\times e}$ is in the image of φ. We conclude that $(k^\times/k^{\times e})^g$ is isomorphic to $k_F{}^\times/k_F{}^{\times e}$. The proposition now follows from the isomorphic of Note 1.3, noting that the above argument works when e is replaced by any e_i, $i \leqq t$.

3.3 Corollary. *Suppose k_F is finite and has a primitive eth root of unity. Then there are exactly $(\Gamma : \Gamma_F)$ isomorphism classes of (k, Γ)-extensions of F.*

3.4 Corollary. *Suppose k is finite and has a primitive eth root of unity. Then there are exactly $\Pi_{i \leqq t}\, s_i$ normal (k, Γ)-extensions of F (up to F-isomorphism), where s_i is the number of e_ith roots of unity in k_F.*

We give an alternate proof of the second corollary. If k_F is finite, the map $k^\times \otimes \Gamma/\Gamma_F \to \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ taking $\sum \beta_i \otimes \gamma_i + \Gamma_F$ to $f \in \mathrm{Hom}(\Gamma/\Gamma_F, k^\times)$ where $f(\gamma_i + \Gamma_F) = \beta_i{}^{P/e_i}$ $(i \leqq t, P = |k^\times|)$ is a g-module isomorphism. Hence it maps $(k^\times \otimes \Gamma/\Gamma_F)^g$ to $\mathrm{Hom}(\Gamma/\Gamma_F, k_F{}^\times)$, which clearly has $\Pi_i s_i$ elements.

3.5 Remarks. Assume k_F is finite. For simplicity we will also assume that Γ/Γ_F is cylcic of order e (so we take $t = 1$). Let $P = |k^\times|$ and $p = |k_F{}^\times|$. We list the orders of the groups in the exact sequence of Proposition 2.5. First note that $H^2(g, \oplus_{i \leqq t} k^\times)$ is trivial (it will always be trivial when $N_g : k \to k_F$ is surjective and g is cyclic). The number of normal (k, Γ)-extensions is $|(k^\times \otimes \Gamma/\Gamma_F)^g| = (e, p)$. The number of these with split Galois groups is $|\varphi(k_F{}^\times \otimes \Gamma/\Gamma_F)| = e(e, P/p)^{-1}$. The number of "Galois groups" of (k, Γ)-extensions is

$$|H^2(g, \mathrm{Hom}(\Gamma/\Gamma_F, k^\times))| = (e, p)(e, P/p)/e.$$

We thank Joel Schneider for pointing out to us that Burnside's lemma [**6**, p. 136] implies that the number of F-isomorphism classes of (k, Γ)-extensions is

$$(1/f) \sum_{0 \leqq i < f} (e, P, (p+1)^i - 1)$$

where $f = [k : k_F]$.

4. **Local fields.** We now assume that F is a Henselian field with k_F finite and Γ_F infinite cyclic (e.g. a nonArchimedian local field) and that K/F is a normal (k, Γ)-extension. Let $e = (\Gamma : \Gamma_F)$ and $f = [k : k_F]$. Let π_K and π_F be prime elements of K and F, respectively; we can (and do) assume that $\pi_K{}^e$ lies in the maximal unramified subextension E of K/F (cf. Lemma 1.4). We will identify the orbit of K/F with $\alpha k^{\times e}$ where $\alpha = \pi_K{}^e \pi_F{}^{-1} + M_K$. (Recall that the orbit is a

singleton (Corollary 2.2) and that $k^\times \otimes \Gamma/\Gamma_F$ may be identified with $k^\times/k^{\times e}$ (Note 1.3).)

Let $N = N_{K/F} : K \to F$ and $N_g : k \to k_F$ be the norm maps. We may also regard $N_{K/F}$ as a map $k \to k_F$.

4.1 Lemma. *We have an exact sequence*

$$(4) \qquad 1 \to k_F{}^\times/k_F{}^{\times e} \xrightarrow{\eta} F^\times/NK^\times \xrightarrow{\eta'} \Gamma_F/f\Gamma_F \to 0$$

where η is induced by the inclusion $U_F \to F^\times$ and η' by the valuation. (U_F denotes the group of units of F.)

Proof. We first show $1 + M_F \subset NK^\times$. Let $a \in M_F$. By Hensel's lemma there exists $b \in M_F$ with $1 + a = (1 + b)^e$. There exists $\beta \in U_E$ with $k_K = k_F[\beta + M_E]$. The irreducible polynomial h of β over F has degree f, so $N_{E/F}(\beta) = (-1)^f h(0)$. The polynomial $h^* = h + b\,h(0)$ is congruent to h modulo M_F and hence by Hensel's lemma has a zero β^* in E. Then

$$1 + a = N_{K/E}(1 + b) = N_{K/E}N_{E/F}(\beta^*/\beta) \in NK^\times.$$

The lemma now follows from the exactness of

$$U_F \to F^\times/NK^\times \to \Gamma/[K:F]\Gamma_F \to 0$$

and the fact that $Nk^\times = k_F{}^{\times e}$ (since N_g is surjective and $|G_K| = e|g|$).

4.2 Note. If no restriction is put on k_F and Γ_F (but K is still a normal (k, Γ)-extension of a Henselian field F), then the above argument gives an exact sequence

$$1 \to k_F{}^\times/(N_g k^\times)^e \to F^\times/NK^\times \to \Gamma_F/[K:F]\Gamma \to 0.$$

We now show how α determines a factor set for the group extension (4). Set $\rho = v(\pi_F) + f\Gamma_F$. Let $h(i\rho, j\rho)$ be 1 if $i + j < f$, and $N_g(\alpha)^{-1}k_F{}^{\times e}$ otherwise. Here $i, j \in \{0, 1, \cdots, f-1\}$, so h maps $\Gamma_F/f\Gamma_F \times \Gamma_F/f\Gamma_F$ into $k_F{}^\times/k_F{}^{\times e}$.

4.3. Proposition. *h is the factor set for the group extension* (4) *associated with the system of representatives $(-\pi_F NK^\times)^i$ $(0 \leqq i < f)$ in $F^\times/NK^\times$ for $\Gamma_F/f\Gamma_F$.*

Proof. Let h' be the factor set associated with the above system of representatives. $h'(i\rho, j\rho)$ is 1 if $i + j < f$ and $\eta^{-1}((-\pi_K)^f NK^\times)$ otherwise. Since $\pi_K{}^e \in E$, $N_{K/E}(\pi_K) = (-1)^{e+1}\pi_K{}^e$. For each $\tau \in g$ pick a representative $u_\tau \in G_K$. Then

$$\begin{aligned}N_g(\alpha)^{-1} &= \Pi_{\tau\in g}u_\tau(\pi_F\pi_K{}^{-e}) + M_K\\ &= \Pi_{\tau\in g}u_\tau(-\pi_F N_{K/E}(-\pi_K{}^{-1})) + M_K\\ &= (-\pi_F)^f N_{K/F}(-\pi_K{}^{-1}) \in \eta^{-1}((-\pi_F)^f NK^\times).\end{aligned}$$

Hence $h = h'$.

For the remainder of this section we suppose that K/F is abelian. We will construct (e,f) isomorphisms of the group extensions (2) and (4); when F is a local field one of these is (induced by) the reciprocity map of local class field theory (cf. Remark 4.6).

Let $p = |k_F|$ (this is not the use of "p" in § 3). Let $\sigma \in g$ be the Frobenius automorphism. Thus $|k| = p^f$, $e \mid p-1$ (Proposition 3.1), and for $\beta \in k^\times$, $\sigma(\beta) = \beta^p$ and $N_g(\beta) = \beta^{(p^f-1)/(p-1)}$.

4.4 Note. There are exactly (e,f) elements u in G_K with u mapping to $\sigma \in g$ and $u^f(\pi_K)/\pi_K + M_K = \alpha^{(p^f-1)/e}$. Note 4.4 will be proved along with,

4.5 Theorem. *Let $u \in G_K$ be as in Note 4.4. We have an isomorphism of group extensions (cf. (2) and (4) above)*

$$(5)\qquad \begin{array}{ccccccc} 1 \to & k_F{}^\times/k_F{}^{\times e} & \xrightarrow{\eta} & F^\times/NK^\times & \xrightarrow{\eta'} & \Gamma_F/f\Gamma_F & \to 0\\ & \downarrow\Theta & & \downarrow\Theta_u & & \downarrow\Theta'' & \\ 1 \to & \mathrm{Hom}(\Gamma/\Gamma_F, k^\times) & \xrightarrow{\delta} & G_K & \xrightarrow{\delta'} & g & \to 1\end{array}$$

where

(i) $\Theta(\beta k_F{}^{\times e})(v(\pi_K) + \Gamma_F) = \beta^{(1-p)/e}\quad (\beta \in k_F{}^\times)$,

(ii) $\Theta''(v(\pi_F) + f\Gamma_F) = \sigma$,

(iii) $\Theta_u(-\pi_F NK^\times) = u$.

Note that Θ and Θ'' are independent of u and are uniquely determined by (i) and (ii) above since $v(\pi_K) + \Gamma_F$ and $v(\pi_F) + f\Gamma_F$ generate Γ_K/Γ_F and $\Gamma_F/f\Gamma_F$, respectively). Θ_u is uniquely determined by the conditions (i) and (iii) (every element of $F^\times$ is the product of a unit in F with a power of π_F).

We now prove 4.4 and 4.5.

First note that we have (unique) isomorphisms Θ and Θ'' satisfying (i) and (ii). For Θ'', it suffices to note that both $\Gamma_F/f\Gamma_F$ and g are cyclic of order f. That (i) defines an isomorphism is immediate from the fact that the map $\beta k_F{}^{\times e} \to \beta^{(p-1)/e}$ is a bijection from $k_F{}^\times/k_F{}^{\times e}$ to the eth roots of unity. (After all, $k_F{}^\times$ is cyclic of order $p-1$ and $e \mid p-1$.)

Now let c be the factor set of Proposition 2.4 for the group extension (2), where we take $t = 1$ and $c_{\sigma^i,1} = \alpha^{(p^i-1)/e}$ for $0 \leqq i < f$. We claim that Θ and Θ'' carry h into c (cf. Proposition 4.3); that is,

$$\Theta(h(\Theta''^{-1}(\sigma^i), \Theta''^{-1}(\sigma^j))) = c(\sigma^i, \sigma^j) \tag{6}$$

for $i, j \in \{0, 1, \cdots, f-1\}$. If $i + j < f$, one checks that c is trivial and hence equal to h. Suppose $i + j \geqq f$. Then at $v(\pi_K) + \Gamma_F$ the value of $c(\sigma^i, \sigma^j)$ is

$$\begin{aligned} &\alpha^{(p^i-1)/e}(\alpha^{(p^j-1)/e})^{p^i}(\alpha^{(p^{i+j-f}-1)/e})^{-1} \\ &= ((\alpha^{(p-1)/e})^{(p^f-1)/(p-1)})^{p^{i+j-f}} \\ &= N_g(\alpha^{(p-1)/e}) = h(\Theta''^{-1}(\sigma^i), \Theta''^{-1}(\sigma^j))^{(1-p)/e} \end{aligned}$$

which is the value of the left hand side of (6) at $v(\pi_K) + \Gamma_F$. The claim is proved. Consequently, there must exist an isomorphism Θ': $F^\times/NK^\times \to G_K$ inducing Θ and Θ''. Such a map Θ' is determined by its value u on $-\pi_F NK^\times$. Such an element u must map to σ and have $\delta^{-1}(u^f)(v(\pi_K) + \Gamma_F) = N(\alpha)^{(p-1)/e}$ (since its powers give a set of coset representatives in G_K for g giving rise to the factor set c), whence $u^f(\pi_K)\pi_K^{-1} + M_K = \alpha^{(p^f-1)/e}$. Conversely, any such $u \in G_K$ determines an isomorphism Θ_u making (5) commute. We leave to the interested reader the task of checking that there are exactly (e, f) such $u \in G_K$.

4.6 Remark. The isomorphisms Θ_u of the above proposition are precisely those isomorphisms from $F^\times/NK^\times$ to G_K which, with Θ and Θ'', give an isomorphism of the group extensions (2) and (4). We now assume F is a local field and show that the reciprocity map $(\,, K/F)$ is such an isomorphism [**1**, 9]. First, $(\,, K/F)$ does induce an isomorphism of the group extensions (2) and (4) and also induces Θ'' [**9**, p. 205]. Now let d be a unit of F. There exists a unit d' of E with $N_{E/F}(d') = d$. (As usual, E is the inertia field of K/F.) Then [**9**, p. 205]

$$(d, K/F)(\pi_K)\pi_K^{-1} + M_K = (d', K/E)(\pi_K)\pi_K^{-1} + M_K$$

which by [**9**; Proposition 6, p. 215, and Corollary, p. 217] equals

$$d'^{(1-p^f)/e} + M_K = (d + M_K)^{(1-p)/e}.$$

Hence $(\,, K/F)$ also induces Θ.

It would be interesting to have an elementary description of the automorphism $u \in G_K$ that has $\Theta_u = (\,, K/F)$. Of course if $(e, f) = 1$ then u is uniquely determined and $\Theta_u = (\,, K/F)$.

5. **Multiplicative congruence.** We return in this section to the notation and situation of § 1; we assume nothing beyond Note 1.1. Note that the map $\Upsilon_F : \Gamma_F \times \Gamma_F \to k^\times$ given by $\Upsilon_F(\gamma, \gamma') = a_\gamma a_{\gamma'} a_{\gamma+\gamma'}^{-1} + M_F$ is a factor set for the group extension

$$(7) \qquad 1 \to k_F{}^\times \to F^\times/1 + M_F \to \Gamma_F \to 0$$

(here, a_γ denotes the unique element of A of value γ). We now show how the orbit of a (k, Γ)-extension K/F determines the structure of $K^\times/1 + M_K$ as an extension of $k^\times$ by Γ.

Let $\alpha_1, \cdots, \alpha_t \in k^\times$. We define a map $\Upsilon : \Gamma \times \Gamma \to k^\times$ depending on the α_i. For each $1 \leqq i \leqq t$, let $\Gamma_i = \Gamma_F + \sum_{j \leqq i} Z\gamma_j$. Thus $\Gamma_0 = \Gamma_F$ and $\Gamma_t = \Gamma$. For $\gamma, \gamma' \in \Gamma_F$, set $\Upsilon(\gamma, \gamma') = \Upsilon_F(\gamma, \gamma')$. Suppose inductively that Υ has been defined on $\Gamma_{i-1} \times \Gamma_{i-1}$. Then for any $\gamma, \gamma' \in \Gamma_{i-1}$ and $r, r' \in \{0, 1, \cdots, e_i - 1\}$, we set $\Upsilon(\gamma + r\gamma_i, \gamma' + r'\gamma_i)$ equal to $\Upsilon(\gamma, \gamma')$ if $r + r' < e_i$ and equal to

$$\alpha_i \Upsilon(\gamma, \gamma')/\Upsilon(-e_i\gamma_i, \gamma + \gamma' + e_i\gamma_i)$$

if $r + r' \geqq e_i$.

In the next proposition K/F is a (k, Γ)-extension with orbit generated by $\sum \alpha_i \otimes \gamma_i + \Gamma_F$. Thus there is a k_F-isomorphism $\tau : k_K \to k$ and elements $b_i \in K$ with $(v(b_i) = \gamma_i$ and) $\tau(a_i b_i^{e_i} + M_K) = \alpha_i$, $1 \leqq i \leqq t$.

5.1 Proposition. *Υ is the factor set for*

$$1 \to k^\times \xrightarrow{\tau^{-1}} K^\times/1 + M_K \xrightarrow{v} \Gamma \to 0$$

associated with the system of representatives

$$(8) \qquad a b_1{}^{i_1} b_2{}^{i_2} \cdots b_t{}^{i_t} (1 + M_K) \quad (a \in A, 0 \leqq i_j < e_j, 1 \leqq j \leqq t)$$

in $K^\times/1 + M_K$ for Γ.

Proof. Let Υ' be the factor set associated with (8). Υ and Υ' agree on $\Gamma_F \times \Gamma_F$; suppose they agree on $\Gamma_{i-1} \times \Gamma_{i-1}$ where $i \leqq t$. For $\gamma \in \Gamma$, let a_γ be the corresponding element of the form (8). Then if $\gamma, \gamma' \in \Gamma_{i-1}$ and $r, r' \in \{0, 1, \cdots, e_i - 1\}$ and $r + r' \geqq e_i$, we have

$$\begin{aligned} &\Upsilon'(\gamma + r\gamma_i, \gamma' + r'\gamma_i) \\ &\quad = a_\gamma b_i^r a_{\gamma'} b_i^{r'} (a_{\gamma+\gamma'+e_i\gamma_i} b^{r+r'-e_i})^{-1}(1 + M_K) \\ &\quad = (a_\gamma a_{\gamma'} a_{\gamma+\gamma'}^{-1} a_{-e_i\gamma_i} b_i^{e_i} / a_{\gamma+\gamma'+e_i\gamma_i} a_{\gamma+\gamma'}^{-1} a_{-e_i\gamma_i})(1 + M_K) \\ &\quad = \Upsilon(\gamma + r\gamma_i, \ \gamma' + r'\gamma_i). \end{aligned}$$

The case when $r + r' < e_i$ is similar, but easier.

5.2 REMARKS. A. We sketch a new proof of Theorem 1.2 using [4]. Recall that $\Delta F = B_F/b_F$ where B_F is the ring of formal series $\sum_{\gamma\in\Gamma_F} c_\gamma t^\gamma \in \mathcal{S}(F, \Gamma_F, 1)$ (cf. [4] or [10, pp. 22–24]) with $v(c_\gamma) \geqq \gamma$ for all $\gamma \in \Gamma_F$, and where b_F is the ideal of all $\sum c_\gamma t^\gamma \in B_F$ with $v(c_\gamma) > \gamma$ for all $\gamma \in \Gamma_F$. ΔF is a valued field with valuation $\sum c_\gamma t^\gamma + b_F \to \inf\{\gamma : v(c_\gamma) = \gamma\}$. Δ is a functor inducing a bijection between the isomorphism classes of (k, Γ)-extensions of F and ΔF [4, proof of Theorem 1].

The set A induces an isomorphism of valued fields σ_A from ΔF to $\mathcal{S}(k_F, \Gamma_F, \Upsilon_F)$; $\sum_{\gamma\in\Gamma_F} c_\gamma t^\gamma + b_F$ is carried to $\sum_{\gamma\in\Gamma_F}(a_\gamma{}^{-1}c_\gamma + M_F)t^\gamma$ [4, Proposition]. A also induces a system of representatives in ΔF for Γ_F, namely $A' = \{at^{v(a)} + b_F : a \in A\}$. One easily verifies that the orbit of a (k, Γ)-extension K/F is also the orbit of $\Delta K/\Delta F$, provided that orbits of (k, Γ)-extensions of ΔF are defined with respect to A'.

Now suppose $\alpha_1, \cdots, \alpha_t \in k^\times$; we do not assume $\sum \alpha_i \otimes \gamma_i + \Gamma_F$ is the orbit of a (k, Γ)-extension of F. One can check directly that the map Υ (defined before Proposition 5.1) is a factor set. Υ restricts on $\Gamma_F \times \Gamma_F$ to Υ_F, so we may regard $\mathcal{S} = \mathcal{S}(k, \Gamma, \Upsilon)$ as a (k, Γ)-extension of ΔF (use the embedding σ_A). Hence there exists a (k, Γ)-extension K'/F with $\Delta K'/\Delta F$ isomorphic to $\mathcal{S}/\Delta F$. But the orbit of $\mathcal{S}/\Delta F$ is checked to be generated by $\sum \alpha_i \otimes \gamma_i + \Gamma_F$. Hence $\sum \alpha_i \otimes \gamma_i + \Gamma_F$ generates the orbit of K'/F. This proves surjectivity in Theorem 1.2. Now suppose K/F is any (k, Γ)-extension with orbit generated by $\sum \alpha_i \otimes \gamma_i + \Gamma_F$. Then by Proposition 5.1 (and [4, Proposition], applied to K), $\Delta K/\Delta F$ is isomorphic to $\mathcal{S}/\Delta F$, so K/F is isomorphic to K'/F. This proves the injectivity in Theorem 1.2.

B. We can choose the set $\gamma_1, \cdots, \gamma_t$ so that for any $\alpha_i \in k^\times$, the sequence $\langle(\alpha_i, \gamma_i)\rangle_{i\leqq t}$ generates a signature in the sense of [2, Definition (7.3)]. The factor set Υ constructed above is precisely the factor set associated with this signature [2, pp. 480–481]. The (k, Γ)-extension of F with orbit generated by $\sum \alpha_i \otimes \gamma_i + \Gamma$ is generated as an extension of F by a zero of the generator of this signature [2, Lemma (3.5) and §7]. Notice that we have, trivially, a condition for the birational equivalence of the generators of signatures of the above form: namely, the corresponding elements of $k^\times \otimes \Gamma/\Gamma_F$ must have the same orbit [2, pp. 469–470, 478].

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