ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 5, Number 4, Fall 1975

A SURVEY OF PROPERTIES OF THE CONVEX COMBINATION OF UNIVALENT FUNCTIONS

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This paper surveys the known results concerning various combinations of univalent functions and then generalizes and extends many of these results to various families of locally univalent functions.

The first problems concerning properties of real linear combinations of various normalized univalent functions appeared in a general list of function theory problems in 1962 [20]. However many of the results which have since appeared were first stated, albeit without proof in several cases, by Rahmanov from 1952 to 1953 [21], [22]. His results are virtually unknown outside of the USSR. The problems which have been investigated have been one of the following five types: (1) For some suitably normalized class of univalent functions $\mathcal{M}$ (convex, starlike, close-to-convex) consider the set $\mathcal{N}$ of all functions of the form $\lambda f(z) + (1-\lambda)g(z)$, f, g in $\mathcal{M}$, λ in $[0, 1]$. What is the radius of convexity, starlikeness or univalence of elements of $\mathcal{N}$. What are the valence properties of functions in $\mathcal{N}$? This type of question has been examined by Goodman, MacGregor, and Labelle and Rahman. (2) If in question (1) we allow λ and $1-\lambda$ to be complex numbers, then how do the various radii change as a function of the argument of λ? This has been examined by Stump. (3) If in question (1) (and then in question (2)) we take convex combinations of n functions, how do the radii vary with n? Both MacGregor and Rahmanov have examined this question. (4) Since $F(z) = \lambda f(z) + (1-\lambda)g(z)$ is not in general univalent, when is $F(z)$ univalent if $f(z)$ is a fixed function but $g(z)$ is allowed to range through some restricted family? This has been examined by Rahmanov, Trimble, and Chichra and Singh. (5) Finally, if $f(z)$ and $g(z)$ are intimately related as in $F(z) = \lambda f(z) + (1-\lambda)e^{i\theta}f(e^{-i\theta}z)$, θ in $[0, 2\pi]$, λ in $[0, 1]$, $f(z)$ in some family, then what univalence properties must such functions have? This was investigated by Robertson who showed that this naturally leads to posing problems about $F(z) = \int_0^{2\pi} e^{i\theta}f(e^{-i\theta}z)\, d\mu(\theta)$ where $\int_0^{2\pi} d\mu(\theta) = 1$.

Most of the results that have been proved do not depend on the functions being restricted to a univalent family nor do they depend on being posed for real convex combinations $\lambda f + (1-\lambda)g$. Ques-

Received by the Editors March 25, 1974, and in revised form July 25, 1974.

tions 1, 2, and 3 are best posed for the closed convex hull of linearly invariant families of finite order. Question (5) really concerns the behavior of the argument of derivative of functions in a linearly invariant family of finite order.

1. **Preliminaries.** Let D be the open unit disc $\{z: |z| < 1\}$ and $\mathcal{LS}$ be the set of all locally univalent $(f'(z) \neq 0)$ analytic functions in D with normalization $f(z) = z + \cdots$. Let $K(\alpha)$ be the class of functions in $\mathcal{LS}$ which are convex of order α, $\alpha \leqq 1$; that is, $\operatorname{Re}(1 + zf''(z)/f'(z)) \geqq \alpha$, $|z| < 1$. Let $St(\beta)$ be the class of functions in $\mathcal{LS}$ which are starlike of order β, $\beta \leqq 1$; that is $\operatorname{Re} zf'(z)/f(z) \geqq \beta$, $|z| < 1$. Let V_k be the class of functions in $\mathcal{LS}$ which are of boundary rotation $\leqq k\pi$, $k \geqq 2$; that is

$$\sup_{r<1} \int_0^{2\pi} |\operatorname{Re}[1 + re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})]|\,d\theta \leqq k\pi.$$

Let $C(\gamma)$ be the class of functions in $\mathcal{LS}$ which are close-to-convex of order γ, $\gamma \geqq 0$; that is, for some θ and some $g(z)$ in $K(0)$,

$$|\arg e^{i\theta}f'(z)/g'(z)| \leqq \gamma\pi/2, \quad |z| < 1.$$

Finally, let $\mathcal{S}$ be the class of univalent functions in $\mathcal{LS}$. The families $K(0)$, $St(0)$, and $C(1)$ will be denoted by K, St, and C respectively. A family $\mathcal{M}$ of functions in $\mathcal{LS}$ is linearly invariant if for each f in $\mathcal{M}$ and for each Moebius transformation $\phi(z)$ of D onto D, the function

$$\Lambda_\phi[f(z)] = \frac{f(\phi(z)) - f(\phi(0))}{f'(\phi(0))\phi'(0)} = z + \cdots$$

is again in $\mathcal{M}$. If $\mathcal{M}$ is a linearly invariant family, the order of $\mathcal{M}$ is defined to be $\alpha = \sup\{|f''(0)|/2 : f(z) \in \mathcal{M}\}$, or equivalently [19],

$$\alpha = \sup_{f \in \mathcal{M}} \sup_{|z|<1} \left| -\bar{z} + \frac{1 - |z|^2}{2} \cdot \frac{f''(z)}{f'(z)} \right|.$$

A family $\mathcal{M}$ is said to be closed under rotations and conjugation if whenever $f(z)$ in in $\mathcal{M}$ then $e^{i\theta}f(e^{-i\theta}z)$ and $\overline{f(\bar{z})}$ are in $\mathcal{M}$. It is elementary to verify that $K(\alpha)$, $St(\beta)$, V_k, $C(\gamma)$, $\mathcal{S}$, and the universal linearly invariant family of order α are all rotation and conjugation invariant.

2. **Closed Convex Hulls of Locally Univalent Families.** In [20] and [11] it is asked whether the sum of normalized functions of finite valence are of finite valence and more specifically does the arithmetic average of two functions in K belong to St. The necessity

for normalization is obvious since otherwise letting $g(z) \equiv -f(z)$ or $f(z) = z + z^n/n$, $g(z) = -z + z^n/n$, would make the problem of little interest.

In 1968 Goodman [8] proved that $(f + g)/2$ could be of infinite valence for f, g in $\mathcal{S}$ and in [10] he proved that $\lambda f + (1 - \lambda)g$ could be of infinite valence for $.042 < \lambda < .958$. It is still open whether the result holds for all λ in $[0, 1]$. He conjectured in [10] that if f, g were in St, then $F = \lambda f + (1 - \lambda)g$ would be at most of finite valence. Styer and Wright [28] showed that this was false by proving the existence of functions f, g in St for which $f + g$ has an infinite number of zeros on $(0, 1)$. Although MacGregor had shown f, g in K did not imply $f + g$ in St, it was conjectured [9] that $f + g$ would be at most 2-valent in this case. Styer and Wright's work showed that this was false and $f + g$ could be 3-valent. They conjectured that $f + g$ could be of infinite valence for f, g in K. Thus interest in linear combinations of various function classes quickly passed from valence results to questions concerning the radius of convexity, starlikeness or univalence of linear combinations.

There is an interesting alternate way of showing for each λ in $(0, 1)$ that $\lambda f + (1 - \lambda)g$ need not be starlike for f, g in K. Let $f(z) = z/(1 + z)$ and $g(z) = z/(1 - z)$. Then

$$
\begin{aligned}
F(z) = \lambda f(z) + (1 - \lambda)g(z) &= [z + z^2(1 - 2\lambda)]/(1 - z^2) \\
&= \sum_{n=1}^{\infty} [1 - \lambda + (-1)^{n-1}\lambda]\, z^n \\
&= \sum_{n=1}^{\infty} c_n z^n.
\end{aligned}
$$

First we note that since $F(z)$ is not a rotation of $z/(1 - z)$, its coefficients would have to go to zero if $F(z)$ were convex [5, Theorems A and 3]. Thus $F(z)$ cannot be convex for any λ in $(0, 1)$. Furthermore, because $F(z)$ is not of the form $z/(1 - e^{i\theta_1}z)(1 - e^{i\theta_2}z)$, then for $F(z)$ to be starlike we would have to have $|c_{n+1}| - |c_n| = O(n^{-\delta})$ for some positive δ [18, Theorem 4]. It is simple to check that $\big|\, |c_{n+1}| - |c_n| \,\big|$ is 2λ, if $\lambda \leqq 1/2$, and $2(1 - \lambda)$, if $\lambda \geqq 1/2$. Thus $F(z)$ can not be starlike. We remark that in order for a family of functions $\mathcal{M}$ closed under rotations to have $(f + g)/2$ in $\mathcal{S}$ for all f, g in $\mathcal{M}$, it is necessary (but clearly not sufficient) that the coefficients of each function in $\mathcal{M}$ be bounded. For let $f(z)$ be in $\mathcal{M}$ and consider $g(z) = -f(-z)$. Then $F(z) = (f(z) + g(z))/2$ is an odd schlicht function whose coefficients are those of $f(z)$. Since odd schlicht functions have bounded coefficients, we see that $f(z)$ has bounded odd coefficients. By a result of Goluzin [7, p. 190], the even coefficients will have to be bounded also.

The first radius results for convex combinations of univalent functions are due to MacGregor [14] who determined the sharp radius of univalence for convex combinations of functions in K, St, and $\mathcal{S}$. Stump treated the more general case in which λ is complex. The bound he found for the radius of univalence of $\lambda f + (1-\lambda)g$, f, g in $\mathcal{S}$, was sharp only for real λ. The sharp result will be obtained in a more general setting [Theorem 1].

Let M be a family in $\mathcal{LS}$. By the closed α convex hull of M (written $H_\alpha(M)$) we mean the closure in the topology of uniform convergence on compacta of the set of all functions of the form

$$F(z) = \sum_{k=1}^{n} \lambda_k f_k(z), \qquad \sum_{k=1}^{n} \lambda_k = 1, \quad |\arg \lambda_k| \leqq \alpha, \quad f_k \in M.$$

The closed 0-convex hull of M is simply the ordinary closed convex hull and is denoted by $H(M)$.

THEOREM 1. *Let $\mathcal{M}$ be a compact rotation and conjugate invariant family in $\mathcal{LS}$ with*

$$G(r) = G(r, \mathcal{M}) = \sup\{\arg f'(z) : |z| = r,\ f \in \mathcal{M}\}.$$

If α is less than $\pi/2$, then the sharp radius of close-to-convexity and the sharp radius of univalence of $H_\alpha(\mathcal{M})$ is the smallest root of $\alpha + G(r) = \pi/2$.

PROOF. The compactness of $\mathcal{M}$ and elementary considerations guarantee that $G(r)$ is a real valued nondecreasing continuous function on $(0, 1)$ with $G(0) = 0$. Thus it makes sense to speak about the smallest root r_0 of $\alpha + G(r) = \pi/2$ (unless $G(r)$ is strictly less than $\pi/2 - \alpha$ for all r in $(0, 1)$ in which case we shall see that each function in $H_\alpha(\mathcal{M})$ is close-to-convex in D). In order to show that each function in $H_\alpha(\mathcal{M})$ is close-to-convex in $|z| < r_0$, it suffices to show that any function $F(z) = \sum_{j=1}^{n} \lambda_j f_j(z)$, $\sum \lambda_j = 1$, $|\arg \lambda_j| \leqq \alpha$, satisfies $\operatorname{Re} F'(z) > 0$ in $|z| < r_0$. This will be true if $\operatorname{Re} \lambda_j f_j'(z) > 0$ for $|z| < r_0$. This is exactly what r_0 guarantees. Not only is the result sharp in $H_\alpha(\mathcal{M})$, but for any $n \geqq 2$ we can find f_j in $\mathcal{M}$, and complex numbers λ_j such that $|\arg \lambda_j| \leqq \alpha$, $\sum \lambda_j = 1$, for which $F(z) = \sum_{j=1}^{n} \lambda_j f_j(z)$ satisfies $F'(z_0) = 0$ for some z_0, $|z_0| = r_0$. We first prove the result for $n = 2$ assuming $r_0 < 1$. The compactness of $\mathcal{M}$ guarantees the existence of a function $f_1(z)$ in $\mathcal{M}$ with $\arg f_1'(z_0) = \pi/2 - \alpha$ for some z_0, $|z_0| = r_0$. By rotation and conjugation invariance, the function

$f_2(z) = \epsilon \bar{f}(\epsilon \bar{z})$, $\epsilon = \exp(2i \arg z_0)$, is in $\mathcal{M}$ and satisfies $\arg f_2'(z_0) = -\pi/2 + \alpha$. Thus

$$F(z) = \frac{e^{i\alpha}}{2 \cos \alpha} f_1(z) + \frac{e^{-i\alpha}}{2 \cos \alpha} f_2(z)$$

satisfies $F'(z_0) = 0$. For $n > 2$ we set $\lambda_1 = e^{i\alpha}/2 \cos \alpha$, $\lambda_k = e^{-i\alpha}/2(n-1) \cos \alpha$ and $f_k = f_2$ for $k = 2, \cdots, n$ and then choose f_1 and f_2 as above. If $\alpha = 0$, then we let $\lambda_1 = 1/2$ and $\lambda_k = 1/2(n-1)$ for $k = 2, \cdots, n$. This concludes the proof.

We note the following applications. If $\mathcal{M} = K$, then $G(r) = 2 \arcsin r$; consequently theorem 1 asserts the radius of close-to-convexity and univalence, r_u, for $H_\alpha(K)$ is $\sin(\pi/4 - \alpha/2)$. If $\mathcal{M} = \mathcal{S}$, then $G(r) = 4 \arcsin r$ for $r < 1/\sqrt{2}$; consequently, $r_u = \sin(\pi/8 - \alpha/4)$. When $\alpha = 0$ we obtain MacGregor's results [14]. Since [32, p. 112], [30, Theorem 5], and [31, Theorem 5], respectively,

$$|\arg f'(z)| \leqq k \arcsin(z), \qquad f \text{ in } V_k,$$
$$|\arg f'(z)| \leqq 2(\gamma + 1) \arcsin |z|, \quad f \text{ in } C(\gamma),$$
$$|\arg f'(z)| \leqq 2(1 - \beta) \arcsin |z|, \quad f \text{ in } K(\beta),$$

and the bounds are sharp, we can obtain new results for these classes. We state only one such result which generalizes MacGregor's result for convex functions. The radius of close-to-convexity and univalence of $H_\alpha(V_k)$ is $\sin[(\pi - 2\alpha)/2k]$; in particular, $H(K) = 1/\sqrt{2}$.

The solution $r_u = \sin(\pi/8 - \alpha/4)$ for $H(\mathcal{S})$ sharpens and extends the results of Stump [26, Theorem 3] in the following sense. Stump had posed the problem of determining the radius of univalence of $\lambda_1 f_1 + \lambda_2 f_2$, $\lambda_1 + \lambda_2 = 1$, in terms of the joint parameter $\alpha = \arg \lambda_1/\lambda_2$. Such a formulation discourages a generalization of the problem to arbitrary finite combinations $\sum \lambda_j f_j$, $\sum \lambda_j = 1$ and suggests Stump's problem be reformulated. If we reformulate the problem as "determine the radius of univalence of $\lambda_1 f_1 + \lambda_2 f_2$, $\lambda_1 + \lambda_2 = 1$ in terms of a uniform bound on $\arg \lambda_1$ and $\arg \lambda_2$," then $r_u = \sin(\pi/8 - \alpha/4)$ is the complete sharp solution for not only finite combinations of functions in $\mathcal{S}$ but for the entire closed linear hull of the univalent functions.

The theorem can also be used to generalize some results of Silverman [24], and Styer and Wright [28]. Silverman noted that $\lambda f + (1 - \lambda) g$, λ in $(0, 1)$, is close-to-convex univalent if f and g are convex of order $1/2$ since $\arg f'(z)$ is bounded in this case by $\arcsin |z|$. Theorem 1 shows that the result is actually true for the larger class of close-to-convex functions with $|\arg f'(z)| \leqq \arcsin |z|$ of which $K(1/2)$ is but a proper subset. The result will also be true for the closed α-convex hull of the family for a suitable restriction on $|\arg f'(z)|$. Styer and Wright observed that $\lambda f + (1-\lambda)g$, λ in $(0, 1)$, is close-to-convex if f and g are odd convex functions. Again a stronger statement is possible. An easy modification of Goluzin's proof [**6**, pp. 184–5] shows that: if $f(z) = z + a_{n+1}z^{n+1} + \cdots$ is convex of order β, $-\infty < \beta \leqq 1$, then

$$(1) \quad |\arg f'(z)| \leqq \frac{2}{n}(1-\beta) \arcsin r^n.$$

In particular if we let $\mathcal{M}$ be the set of convex functions whose second coefficient is zero, when $H(\mathcal{M}) \subset C$ which extends Styer and Wright's Theorem 1.

Since K, $\mathcal{S}$ and V_k are linearly invariant families one might ask if there are any linearly invariant families whose convex hull consists of univalent functions. The answer, under a simple restriction, is no. The procedure sheds some light on the constant $1/\sqrt{2}$ for $H(K)$.

THEOREM 2. *Let $\mathcal{M}$ be any linearly invariant family closed under conjugation. Then the radius of univalence and close-to-convexity of $H(\mathcal{M})$ is $\leqq 1/\sqrt{2}$.*

PROOF. We need not assume $\mathcal{M}$ is of finite order since a result of Campbell and Ziegler [3] shows that for any linearly invariant family $\mathcal{M}$ $\sup\{\arg f'(z) : |z| = r,\ f \in \mathcal{M}\} \geqq 2 \arcsin r$. Thus for any $r_0 > 1/\sqrt{2}$ we can find an f in $\mathcal{M}$ such that for some z of modulus r_0 we have $\arg f'(z) > \pi/2$. By continuity of $f'(z)$ and by a rotation, if necessary, we can assume that $f'(z)$ has the property $\arg f'(r_0) = \pi/2$. Since $\mathcal{M}$ is conjugate invariant, $\mathcal{M}$ contains the function $g(z) = \overline{f(\bar{z})}$. Thus $F(z) = (f(z) + g(z))/2$ has the property $F'(r) = 0$. This shows the radius of close-to-convexity is no greater than $1/\sqrt{2}$.

This completes the generalization of known results for the radius of univalence for (complex) convex combinations of functions from various well known classes and explicitly presents the machinery to determine the sharp radius for other function classes.

3. **Bounds for the Radius of Convexity and Starlikeness.** The radius of convexity of $(f+g)/2$, f, g in K, was first examined by Labelle and Rahman [13] who showed $.39 < R_0 < .40$. An extension was made in [2] to $\lambda f + (1-\lambda)g$, λ in $(0, 1)$, f, g in a linearly invariant family of order β (the results erroneously state λ is in $(-\infty, \infty)$ which

is manifestly false). Rahmanov claimed in [22] that the radius of convexity of $\sum_{j=1}^{n} f_j(z)/n$, f_j in K, is at least .39 and if the f_j are in $\mathcal{S}$, then $r_0 > .188$. However, as with many of his results in this area, no proofs were given. Stump examined the radius of convexity for $\lambda f + (1-\lambda)g$, f, g in K, λ complex. The machinery to estimate the radius of convexity for the closed α-convex hull of any linearly invariant family of order β is now presented.

Lemma 3. *If* $|w-a| \leqq d$, $a \geqq 0$, *and* w_0 *is an arbitrary complex number, then* $\operatorname{Re} ww_0 \geqq |w_0|\,\{a\cos(\arg w_0) - d\}$.

Proof. This is immediate upon noting

$$|ww_0 - a|\,w_0|e^{i \arg w_0}| \leqq d|w_0|.$$

Lemma 4. *Let* $w_j, j = 1, \cdots, n$, *lie in the infinite sector* $|\arg w| \leqq \gamma$, $0 \leqq \gamma < \pi/2$. *Then for all* n $|\sum_{j=1}^{n} w_j| \geqq \cos\gamma \sum_{j=1}^{n} |w_j|$.

Proof. Let $w_j = |w_j| e^{i\gamma_j}$. Then

$$\left|\sum_{j=1}^{n} w_j\right| \geqq \left|\operatorname{Re}\left(\sum_{j=1}^{n} w_j\right)\right| \geqq \sum_{j=1}^{n} |w_j|\cos\gamma_j$$

$$\geqq \cos\gamma \sum_{j=1}^{n} |w_j|.$$

Lemma 5. *Let* $w_j, j = 1, \cdots, n$ *satisfy* $|w_j - a| \leqq d$, $a \geqq 0$. *Let* $v_j, j = 1, \cdots, n$, *be complex numbers satisfying* $|\arg v_j| \leqq \gamma$, $0 \leqq \gamma < \pi/2$. *Then*

$$\operatorname{Re}\left[\sum_{j=1}^{n} w_j v_j \Big/ \sum_{k=1}^{n} v_k\right] \geqq a - d\sec\gamma.$$

Proof. If we let $x_j = v_j/\sum_{k=1}^{n} v_k$, $j = 1, \cdots, n$, then by Lemma 3

$$\operatorname{Re}\left[\sum_{j=1}^{n} w_j v_j \Big/ \sum_{k=1}^{n} v_k\right] = \sum_{j=1}^{n} \operatorname{Re}(w_j x_j)$$

$$\geqq \sum_{j=1}^{n} |x_j|\,\{a\cos(\arg x_j) - d\}$$

$$= a\sum_{j=1}^{n} \operatorname{Re} x_j - d\sum_{j=1}^{n} |x_j|$$

$$= a - d\sec\gamma,$$

where the last line uses Lemma 4 and the fact that $\sum_{j=1}^n \operatorname{Re} x_j = \operatorname{Re} \sum_{j=1}^n x_j = \operatorname{Re} 1 = 1$.

Lemma 5 is the key to the following series of bounds on the radius of convexity for closed α-convex hulls of linearly invariant families.

THEOREM 6. *Let $\mathcal{M}$ be a linearly invariant family of order β with $G(r) = \sup\{\arg f'(z) : |z| = r,\ f \in \mathcal{M}\}$. Then r_k, the radius of convexity of any element in $H_\alpha(\mathcal{M})$, $0 \leqq \alpha < \pi/2$, satisfies $r_k \geqq r_0$ where r_0 is the positive root of* $1 + r^2 = 2r\beta \sec(G(r) + \alpha)$.

PROOF. It suffices to prove the claim for any $F(z)$ of the form $\sum_{j=1}^n \lambda_j f_j(z)$, f_j in $\mathcal{M}$, $|\arg \lambda_j| \leqq \alpha$, $\sum_{j=1}^n \lambda_j = 1$, since functions of this form are dense in $H_\alpha(\mathcal{M})$. A direct computation yields,

$$1 + zF''(z)/F'(z) = 1 + z \frac{\sum_{j=1}^n \lambda_j f_j''}{\sum_{k=1}^n \lambda_k f_k'}$$

$$= 1 + \sum_{j=1}^n \frac{z f_j''}{f_j'} \left[\lambda_j f_j' \Big/ \sum_{k=1}^n \lambda_k f_k' \right]$$

$$= 1 + \sum_{j=1}^n w_j v_j \Big/ \sum_{k=1}^n v_j,$$

where $w_j = z f_j''/f_j'$ and $v_j = \lambda_j f_j'$. Since $\mathcal{M}$ is a linearly invariant family of order β we have for each f in $\mathcal{M}$ [19]

$$|-\bar{z} + \tfrac{1}{2}(1 - |z|^2) f''(z)/f'(z)| \leqq \beta, \quad |z| < 1,$$

that is $|w_j - 2r^2/(1 - r^2)| \leqq 2\beta r/(1 - r^2)$ while $|\arg v_j| \leqq G(r) + \alpha$. Thus by Lemma 5,

$$\operatorname{Re}(1 + zF''(z)/F'(z)) \geqq \frac{1 + r^2}{1 - r^2} - \frac{2\beta r}{1 - r^2} \sec(G(r) + \alpha),$$

so long as $G(r) + \alpha$ is less than $\pi/2$. Since the order of a linearly invariant family is always $\geqq 1$ [19], it is clear that the root of $1 + r^2 - 2\beta r \sec(G(r) + \alpha) = 0$ will always occur while $G(r) + \alpha$ is less than $\pi/2$. Thus $\operatorname{Re}(1 + zF''(z)/F'(z)) \geqq 0$ for $|z|$ less than the positive root of $1 + r^2 - 2\beta r \sec(G(r) + \alpha) = 0$.

COROLLARY 7. *The radius of convexity for $H(V_k)$, $H(K)$ and $H(\mathcal{S})$ is bounded below by the positive root of $(1 + r^2)\cos(k \arcsin r) - kr = 0$, $1 - 3r + 2r^2 - 2r^3 = 0$, $1 - 4r - 7r^2 + 8r^6 = 0$ respectively; that is $r_k \geqq .395$ and $r_k \geqq .188$ for $H(K)$ and $H(\mathcal{S})$.*

PROOF. We obtain the result for $H(V_k)$ upon noting that the order of V_k is $k/2$ and $|\arg f'(z)| \leqq k \arcsin |z|$. Since $H(K)$ is $H(V_2)$ we need the positive root of $(1 + r^2)\cos(2 \arcsin r) - 2r = 0$. This equation yields $1 - 2r - r^2 - 2r^4 = 0$. If we remove the trivial root $r = -1$ we obtain $1 - 3r + 2r^2 - 2r^3 = 0$. In fact for k an integer, the elementary formula for $\cos k\theta$ in terms of $\cos\theta$ and $\sin\theta$ yields that $H(V_k)$ has a radius of convexity at least as large as the root of the expression

$$(1 + r^2) \sum_{j=0}^{[k/2]} \binom{k}{2j} (-1)^j (1 - r^2)^{(k-2j)/2} r^{2j} - kr = 0,$$

which is equivalent to a polynomial expression in r.

For the class $\mathcal{S}$ we have $G(r) = 4 \arcsin r$ when $r < 1/\sqrt{2}$. Furthermore, we need only consider $G(r)$ when $G(r) < \pi/2$. Thus it suffices to solve $1 + r^2 = 4r \sec(4 \arcsin r)$. This yields $1 - 4r - 7r^2 + 8r^6 = 0$.

The bound for $\lambda f + (1 - \lambda)g$, λ in $(0, 1)$ was done for K by Labelle and Rahman [13], and for $\mathcal{S}$, V_k and other classes by Campbell [2]. Theorem 6 also yields the results claimed without proof by Rahmanov [22] for the average $(1/n)\sum f_j(z)$, f_j in K, f_j in $\mathcal{S}$, respectively. It also yields Stump's results for $\lambda f + (1 - \lambda)g$, f, g in K, λ complex if we reformulate Stump's problem in a form more appropriate for finite combinations of functions as was discussed in the remarks following Theorem 1. One can express the bound for the radius of convexity of $H_\alpha(K)$ as a polynomial whose coefficients depend on α.

A careful examination of the proof of Theorem 6 shows how to apply the results to classes which are not linearly invariant such as $St\,\beta$ or $K(\beta)$. All that is really needed is an explicit bound for $\arg f'(z)$, f in $\mathcal{M}$, and that $\mathcal{M}$ lie in a linearly invariant family of finite order β. For example, $K(\beta)$ is not a linear invariant family but does lie in K, which is a linear invariant family of order 1. Since $|\arg f'(z)| \leqq 2(1 - \beta) \arcsin r$ [31, Theorem 5] then $r_k \geqq r_0$ where r_0 is the positive root of

$$1 + r^2 = 2r \sec[2(1 - \beta) \arcsin r + \alpha].$$

We now apply Lemma 5 to bound the radius of starlikeness of the closed α-convex hull of various families of starlike functions.

THEOREM 8. *Let $\mathcal{M}$ be a class of starlike functions satisfying $|zf'(z)/f(z) - a(r)| \leqq d(r)$, $|z| = r < 1$. If $a(r)$ and $d(r)$ are continuous* *functions of r with* $\lim d(r)/a(r) > 0$ *as* $r \to 1$, *then* r_{st}, *the radius of starlikeness of* $H_\alpha(\mathcal{M})$, *satisfies* $r_{st} \geqq r_0$, *where* r_0 *is the first positive root of* $a(r) - d(r)\sec(\alpha + b(r)) = 0$ and $b(r) = \max\{\arg f(z)/z : |z| = r, f \in \mathcal{M}\}$.

PROOF. As in Theorem 6 it suffices to prove this for

$$F(z) = \sum_{j=1}^{n} \lambda_j f_j(z), \ \sum_{j=1}^{n} \lambda_j = 1, \ |\arg \lambda_j| \leqq \alpha, \ f_j \text{ in } \mathcal{M}.$$

We note that

$$z\frac{F'(z)}{F(z)} = z\frac{\sum_{j=1}^{n} \lambda_j f_j'(z)}{\sum_{k=1}^{n} \lambda_j f_j(z)}$$

$$= \sum_{j=1}^{n} \frac{z f_j'(z)}{f_j(z)} \cdot \frac{\lambda_j f_j(z)/z}{\sum_{k=1}^{n} \lambda_k f_k(z)/z},$$

so that we may apply Lemma 5 and the hypothesis of the theorem to obtain

$$\text{(2)} \qquad \operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right] \geqq a(r) - d(r)\sec(\alpha + b(r)), \ |z| = r,$$

under the condition that $\alpha + b(r)$ is less than $\pi/2$. Since the limit of $a(r)/d(r)$ as $r \to 1$ is assumed to be greater than zero, we see that the root of (2) occurs while $\alpha + b(r)$ is less than $\pi/2$. Thus $F(z)$ is starlike for $|z|$ less than the first positive root of (2).

COROLLARY 9. *The radius of starlikeness of* $H_\alpha(St\,1/2)$ *is precisely*

$$\text{(3)} \qquad \left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)/\sqrt{2}.$$

PROOF. If $f(z)$ is in $St(1/2)$, then $|zf'(z)/f(z) - 1/(1 - r^2)| \leqq r/(1 - r^2)$ and $|\arg f(z)/z| \leqq \arcsin r$ [**16**], [**27**]. Thus we determine the root of $\cos(\alpha + \arcsin r) = r$ which turns out to be $(\cos \alpha/2 - \sin \alpha/2)\sqrt{2}$. Since $St(1/2) \supset K$, and from Theorem 1 the sharp radius of univalence of $H_\alpha(K)$ is (3), we see that the radius of starlikeness is sharp.

Stump obtained the same sharp result but only claimed it for $\lambda f + (1 - \lambda)g$, f, g convex. He stated the result in terms of the parameter $\alpha^* = \arg \lambda/(1-\lambda)$. He also gave a direct computation of the sharpness. The above corollary applies to starlike functions whose second coefficient is zero (in particular odd starlike functions) since these are starlike of order 1/2.

COROLLARY 10. *Let $\mathcal{M}$ be the subclass of St consisting of functions satisfying $|zf'(z)/f(z) - 1| \leqq 1$. Then $H_\alpha(\mathcal{M})$ has a radius of starlikeness at least as large as the positive root of* $\cos(\alpha + r) = r$.

PROOF. This class was considered by R. Singh [25]. It is elementary to verify that if $f(z)$ is in $\mathcal{M}$, then $|\arg f(z)/z| \leqq |z|$ and $|zf'(z)/f(z) - 1| \leqq |z|$. The assertion of the corollary is now immediate from the theorem.

COROLLARY 11. *The radius of starlikeness of $H_\alpha(St)$ is at least as great as the positive root of* $\alpha = \arccos[2r/(1+r^2)] - 2 \arcsin r$.

PROOF. This is immediate from the theorem upon noting

$$\left| zf'(z)/f(z) - \frac{1+r^2}{1-r^2} \right| \leqq \frac{2r}{1-r^2}$$

and $|\arg f(z)/z| \leqq 2 \arcsin r$.

4. **Other Convex Combinations which are Univalent.** Since $F(z) = \lambda f(z) + (1-\lambda)g(z)$ is not generally univalent when both f and g are allowed to range through the same function class, one can restrict $f(z)$ and $g(z)$ to different function classes and hope for univalence. For example, Trimble [29] let $f(z)$ be identically z and let $g(z)$ be in $St(1/2)$ and showed that $F(z) = \lambda z + (1-\lambda)g(z)$, $0 \leqq \lambda \leqq 1$, is always close-to-convex univalent. The result was actually first stated, again, by Rahmanov [21].

THEOREM 12. *Let β, γ in $[0, 1]$ satisfy $\beta + \gamma \geqq 3/2$. Let $f_j(z)$ be in $K(\beta)$, $g(z)$ be in $St(\gamma)$ and λ_j in $[0, 1]$ satisfy $\sum_{j=1}^{n} \lambda_j = 1$. Then*

$$F(z) = \sum_{j=1}^{n-1} \lambda_j f_j(z) + \lambda_n g(z) \tag{4}$$

is close-to-convex.

PROOF. Let $G(z) = \int_0^z g(t)/t\, dt$, $|z| < 1$. Then $G(z)$ is convex in D, and

$$\mathrm{Re} \frac{F'(z)}{G'(z)} = \sum_{j=1}^{n=1} \lambda_j \, \mathrm{Re} \frac{zf_j'(z)}{g(z)} + \lambda_n \, \mathrm{Re} \frac{zg'(z)}{g(z)}.$$

The conclusion will therefore follow if $|\arg zf_j'(z)/g(z)| \leqq \pi/2$ in D. But functions in $K(\beta)$ and $St(\gamma)$ satisfy [6], [15], $|\arg f'(z)| \leqq \pi(1-\beta)$, $f \in K(\beta)$, $|\arg g(z)/z| \leqq \pi(1-\gamma)$, $g \in St(\gamma)$. Hence $|\arg zf_j'(z)/g(z)| \leqq \pi(2-(\gamma+\beta)) \leqq \pi/2$ since $\beta+\gamma \geqq 3/2$. This concludes the proof of the theorem.

COROLLARY 13. *The function* $F(z) = \lambda z + (1-\lambda)g(z)$ *is close-to-convex for all* g *in* $St(1/2)$ *and all* λ *in* $[0, 1]$.

The corollary was proved by Trimble [29] but is found in a weakened form in Rahmanov [21] who also remarked that $F(z) = (z+f(z))/2$ is univalent if $f(z)$ is starlike and has second coefficient zero.

Similar problems have been investigated by Chichra and Singh [4] who showed (A) f in K implies $\lambda z + 2(1-\lambda)z^{-1} \int_0^z f(t)\,dt$ is starlike, (B) f in St and $\operatorname{Re} f' > 0$ implies $\lambda z + (1-\lambda)f(z)$ is starlike, (C) f odd convex implies $\lambda z + (1-\lambda)f(z)$ is starlike.

There is a simple geometric condition for convex functions so that $\lambda f + (1-\lambda)g$ will be close-to-convex univalent. Let $\mathcal{M}$ be the set of all f in K such that there exists points z_n' converging to $z = 1$ and z_n'' converging to $z = -1$ for which $\lim_{n\to\infty} \operatorname{Re} f(z_n') = \sup_{|z|<1} \operatorname{Re} f(z)$, $\lim_{n\to\infty} \operatorname{Re} f(z_n'') = \inf_{|z|<1} \operatorname{Re} f(z)$. Then $\mathcal{H}(\mathcal{M})$ is a subset of the close-to-convex univalent functions. For by a result of Hengartner and Schober [12, Theorem 1], every element in $\mathcal{M}$ satisfies $\operatorname{Re}(1-z^2)f'(z) \geqq 0$ since convex functions are automatically univalent and convex in the direction of the imaginary axis. Consequently, every function in $\mathcal{H}(\mathcal{M})$ satisfies $\operatorname{Re}(1-z^2)f'(z) \geqq 0$, i.e., is close-to-convex with respect to $(1+z)/(1-z)$.

There is a simple sufficient condition for $\lambda f + (1-\lambda)g$ to be at least *locally* univalent. Let $h(z)$ be a fixed function in C, θ be fixed in $[-\pi/2, \pi/2]$ and $\mathcal{M} = \{g \in K;\ \operatorname{Re}(h'(z)/e^{i\theta}g'(z)) \geqq 0\}$. Then $\mathcal{H}(\mathcal{M}) \subset \mathcal{LS}$ since the positivity of $\operatorname{Re}(e^{-i\theta} \sum \lambda_j g_j'(z)/h'(z))$, g_j in $\mathcal{M}$, implies that $F'(z) = \sum \lambda_j g_j'(z)$ cannot vanish. It is an open problem whether $\mathcal{H}(\mathcal{M})$ is a subset of K, St or $\mathcal{S}$.

5. **Convex Combinations and Integrals.** We now consider convex combinations of functions which are closely related, in particular, $F(z) = \lambda f(z) + (1-\lambda)\, e^{-i\theta} f(z\, e^{i\theta})$. This is not an artificial condition since it immediately leads to information on

$$F(z) = \int_0^{2\pi} e^{-i\theta} f(z\, e^{i\theta})\, d\mu(\theta)$$

where $\mu(\theta)$ is a nondecreasing function of θ on $[0, 2\pi]$ with $\int_0^{2\pi} d\mu(\theta) = 1$.

In this direction it is interesting to recall MacGregor's observation of the striking difference between convex combinations of meromorphic convex functions and analytic convex functions. In [17, p. 277] C. Pommerenke proved: if the meromorphic functions $f_k(z) = z + \sum_{m=0}^{\infty} a_m^{(k)} z^{-m}$ map $|z| > 1$ univalently onto the complement of a convex set, then the convex combination $F(z) = \sum_{k=1}^{n} \lambda_k f_k(z)$ is a meromorphic close-to-convex univalent function in $|z| > 1$. In the context of the theme of this paper, his result can be stated more strongly and given an interpretation in accordance with the direction of this section. The closed convex hull of the normalized convex meromorphic functions is a subset of the normalized meromorphic close-to-convex univalent functions. In particular, for any nondecreasing function $\mu(\theta)$ on $[0, 2\pi]$ with weight $\int_0^{2\pi} d\mu(\theta) = 1$, the function

$$F(z) = \int_0^{2\pi} e^{-i\theta} f(z\, e^{i\theta})\, d\mu(\theta)$$

is a normalized meromorphic close-to-convex function for any normalized meromorphic convex function $f(z)$ in $|z| > 1$. In order to prove this we approximate $\mu(\theta)$ be a sequence of step functions $\{\mu_n(\theta)\}$ where $\mu_n(\theta)$ has $2n$ jumps t_k, $k = 1, \cdots, 2n$, $0 < t_k < 1$, $\sum_{k=1}^{2n} t_k = 1$. Thus $F_n(z) = \sum_{k=1}^{2n} t_k\, e^{-i\theta_k} f(z\, e^{i\theta_k})$ clearly converges to $F(z)$ and each $F_n(z)$ is meromorphic close-to-convex by Pommerenke's result.

According to an argument given by Robertson [23], it is sufficient to consider expressions of the form $F(z) = tf(z) + (1 - t)e^{-i\theta} f(ze^{i\theta})$ in order to study $\int_0^{2\pi} e^{-i\theta} f(ze^{i\theta})\, d\mu(\theta)$. If $f(z)$ is analytic in D the resulting functions are not necessarily univalent. Sharpness results for the radius of univalence are now much more difficult as Robertson [23] observed since we are deprived of the device of letting $g(z) = \overline{f(\bar{z})}$ as in Section 2. Nevertheless we obtain sharp results for most of the well known function classes.

THEOREM 14. *Let $k \geqq 2$ and $\mathcal{M}$ be a linearly invariant family whose elements satisfy $|\arg f'(z)| \leqq k \arcsin |z|$ in D. Let $\mu(\theta)$ be a nondecreasing function on $[0, 2\pi]$ for which $\int_0^{2\pi} d\mu(\theta) = 1$. Then the function*

$$F(z) = \int_0^{2\pi} e^{-i\theta} f(z\, e^{i\theta})\, d\mu(\theta) \tag{5}$$

is analytic in D and close-to-convex univalent for $|z| < r_0$ where r_0 is the first positive root in $(0, 1)$ of the equation

$$\arctan\left[\frac{2r^2xy}{1-r^2+2r^2y^2}\right]+\frac{k}{2}\arctan\left[\frac{2ry}{1-r^2}\right]=\frac{\pi}{2} \tag{6}$$

and

$$x=\left[\left(\frac{k^2}{4}(1-r^2)^2+8r^2(1+r^2)\right)^{1/2}-\frac{k}{2}(1-r^2)\right]\Big/4r,$$

$$y=(1-x^2)^{1/2}.$$

If the linearly invariant family $\mathcal{M}$ contains the function

$$K(z)\equiv\frac{1}{k}\left(\left(\frac{1+z}{1-z}\right)^{k/2}-1\right), \tag{7}$$

then there exists a $\mu(\theta)$ and a point z_0 on the circle $|z|=r_0$ for which $F'(z_0)=0$; that is, the result is sharp.

PROOF. Let $f(z)$ be an arbitrary element in $\mathcal{M}$ and z and ζ be arbitrary in D. Then by the linear invariance of $\mathcal{M}$ the function

$$\phi(\zeta)=\frac{f((\zeta+z)/(1+\bar{z}\zeta))-f(z)}{(1-|z|^2)f'(z)}$$

is again in $\mathcal{M}$ and satisfies

$$|\arg\phi'(\zeta)|\leqq k\arcsin|\zeta| \tag{8}$$

by hypothesis. Letting $\zeta=z(e^{i\theta}-1)/(1-e^{i\theta}|z|^2)$ yields

$$\frac{f'(ze^{i\theta})}{f'(z)}=\left(\frac{1-|z|^2}{1-e^{i\theta}|z|^2}\right)^2\phi'\left[\frac{z(e^{i\theta}-1)}{1-e^{i\theta}|z|^2}\right].$$

But

$$\left|\frac{z(e^{i\theta}-1)}{1-e^{i\theta}|z|^2}\right|=\frac{2r\sin\theta/2}{[(1-r^2)^2+4r^2\sin^2(\theta/2)]^{1/2}}, \tag{9}$$

hence

$$\arcsin\left|\frac{z(e^{i\theta}-1)}{1-e^{i\theta}|z|^2}\right|=\arctan\left(\frac{2r\sin\theta/2}{1-r^2}\right), \tag{10}$$

while

$$\begin{aligned}\arg(1-r^2e^{i\theta})^{-2}&=2\arctan\frac{r^2\sin\theta}{1-r^2\cos\theta}\\&=2\arctan\left[\frac{2r^2\sin(\theta/2)\cos(\theta|2)}{1-r^2+2r^2\sin^2(\theta/2)}\right].\end{aligned} \tag{11}$$

Thus,

$$U(\theta) \leqq \frac{1}{2}\arg\frac{f'(ze^{i\theta})}{f'(z)} \leqq V(\theta) \tag{12}$$

where

$$V(\theta) = V = \arctan\left[\frac{2r^2 \sin\theta/2 \cos\theta/2}{1 - r^2 + 2r^2 \sin^2\theta/2}\right] + \frac{k}{2}\arctan\left(\frac{2r\sin\theta/2}{1-r^2}\right),$$

$$U(\theta) = U = \arctan\left[\frac{2r^2 \sin\theta/2 \cos\theta/2}{1 - r^2 + 2r^2 \sin^2\theta/2}\right] - \frac{k}{2}\arctan\left(\frac{2r\sin\theta/2}{1-r^2}\right).$$

We first determine when $tf(z) + (1-t)e^{-i\theta}f(ze^{i\theta})$ is close-to-convex for any t in $(0, 1)$, θ in $[0, 2\pi]$, and f in $\mathcal{M}$. A simple modification of Robertson's Lemma [23, p. 414] shows that if $\mathcal{N}$ is any subset of $\mathcal{L.S.}$, then all of the functions $G(z) = tf(z) + (1-t)e^{-i\theta}f(ze^{i\theta}), 0 < t < 1$, $0 \leqq \theta < 2\pi$, f in $\mathcal{N}$, will be close-to-convex univalent with respect to some $e^{-i\gamma}z$ in $|z| < \rho_0$ if and only if

$$|\arg f'(ze^{i\theta})/f'(z)| < \pi/2$$

for all f in $\mathcal{N}$, all $|z| < \rho_0$, and all θ in $[0, 2\pi]$. Since $V(2\pi - \theta) = -U(\theta)$, then by (12) and the above remark we need only determine when $V(\theta) < \pi/2$ for all θ in $[0, 2\pi]$ and all r less than some ρ_0.

A computation shows

$$\frac{dV}{d\theta} = \frac{2r^2\cos^2(\theta/2) + (k/2)\,r(1-r^2)\cos(\theta/2) - r^2(1+r^2)}{1 - 2r^2\cos\theta + r^4}. \tag{13}$$

The numerator of (13) is a quadratic in $\cos\theta/2$. We shall see that the root

$$4r\cos\theta/2 = \left(\frac{k^2}{4}(1-r^2)^2 + 8r^2(1+r^2)\right)^{1/2} - \frac{k}{2}(1-r^2) \tag{14}$$

yields the maximum for V. If $\cos\theta/2$ is given by (14) then it cannot be ± 1 for any r in $(0, 1)$. Otherwise (14) would reduce to $2r = \pm k$ which is absurd for $0 < r < 1$, $k \geqq 2$. A simple computation shows that $d^2V/d\theta^2$ evaluated at the point θ given by (14) is

$$\frac{d^2V}{d\theta^2} = \frac{-\frac{1}{2}r\sin\theta/2[(k^2/4)(1-r^2)^2 + 8r^2(1+r^2)]^{1/2}}{(1 - 2r^2\cos\theta + r^4)}.$$

Thus $d^2V/d\theta^2$ is clearly negative at this point. Hence for fixed r and variable θ, V attains its maximum when $\cos\theta/2$ is given by (14).

Solving for $\sin \theta/2$ and substituting into $V(\theta)$ we obtain

$$V(r) = \arctan \left[\frac{2r^2xy}{1 - r^2 + 2r^2y^2} \right] + \frac{k}{2} \arctan \left[\frac{2ry}{1 - r^2} \right],$$

where $x = \cos \theta/2$ is given by (14) and $y = (1 - x^2)^{1/2}$. The function $V(r)$ is a continuous function of r, $V(r) \to 0$ as $r \to 0$, and $V(r) \to \pi/2(1 + k/2)$ as $r \to 1$. Thus the first positive root r_0 of $V(r) = \pi/2$ is in $(0, 1)$.

The estimate for the radius of univalence is sharp for any linearly invariant family containing the appropriate generalized Koebe function (7). For r_0 determined by (6), let θ_0 be the solution of (14) and set

$$G(z) = \quad [K(z) + e^{i\theta_0} K(e^{-i\theta_0}z)].$$

Then for the generalized Koebe function

$$G'(r_0e^{i\theta_0/2}) = 2Re \left[\left(\frac{1 + r_0e^{i\theta_0/2}}{1 - r_0e^{i\theta_0/2}} \right)^{k/2} \frac{1}{1 - r_0{}^2e^{i\theta_0}} \right].$$

This latter expression will be zero if its argument is zero. But the argument of this last term is precisely $V(r_0)$ which is zero by construction of r_0. The proof that $G(z) = \int_0^{2\pi} e^{i\theta} f(ze^{-i\theta})\, d\mu(\theta)$ has the properties claimed in the theorem is identical to Robertson [23, p. 417] and is omitted.

Corollary 15. *Let r_0 be determined as in Theorem 14. If $\mathcal{M}$ is a linearly invariant family whose elements satsify*

$$|\arg f'(z)| \leqq k \arcsin |z|, \; |z| \leqq 2r_0/(1 + r_0{}^2),$$

then the claims of the theorem hold and are still sharp.

Proof. It suffices to note the proof only requires (8) to hold for those $\zeta = z(e^{i\theta} - 1)/(1 - e^{i\theta}|z|^2)$ with θ in $[0, 2\pi)$ and z in $|z| < r_0$. But by (9) such ζ are bounded by

$$|\zeta| = \left| \frac{z(e^{i\theta} - 1)}{1 - e^{i\theta}|z|^2} \right| = \frac{2|z| \sin \theta/2}{[(1 - |z|^2)^2 + 4|z|^2 \sin^2 \theta/2]^{1/2}}$$

$$\leqq \frac{2|z|}{1 + |z|^2} \leqq \frac{2r_0}{1 + r_0{}^2}.$$

Corollary 16. *The sharp radius of close-to-convexity for $F(z) = \int_0^{2\pi} e^{-i\theta} f(ze^{i\theta})\, d\mu(\theta)$, $f(z)$ in $\mathcal{S}$ is the root of $r^6 + 5r^4 + 79r^2 - 13 = 0$.*

Proof. With k equal to 4, Robertson showed (6) is equivalent to $r^6 + 5r^4 + 79r^2 - 13 = 0$ whose root is approximately .4035. In view of corollary 15 we need only have $|\arg f'(z)| \leqq 4$ arcsin $|z|$ in $|z| < .694$. This is the case since $|\arg f'(z)| \leqq 4$ arcsin $|z|$ for $|z| < 1/\sqrt{2} \approx .707107$.

Corollary 17. *For the linearly invariant family K, the function $F(z)$ defined in Theorem 14 is close-to-convex univalent for $|z| < 1/\sqrt{2}$. The result is sharp.*

Proof. Since K is a linearly invariant family satisfying $|\arg f'(z)| \leqq 2 \arcsin |z|$ and since $f(z) = ((1+z)/(1-z) - 1)/2 = z/(1-z)$ is in K, the sharp radius of close-to-convexity is given by r_0, the first positive root in $(0, 1)$ of the equation $V(r) = \pi/2$ where

$$V(r) = \arctan \frac{2r^2xy}{1 - r^2 + 2r^2y^2} + \arctan \frac{2ry}{1 - r^2}$$

and $x = r$, $y = (1 - x^2)^{1/2}$. Since

$$\tan V(r) = \frac{2r(1 + 3r^2)(1 - r^2)^{1/2}}{1 + r^2 - 6r^4}$$

we need only find the positive root of $1 + r^2 - 6r^4$ which is $1/\sqrt{2}$.

Robertson [23] showed by an alternate method that the functions are actually starlike in $|z| < 1/\sqrt{2}$.

Theorem 14 yields obvious corollaries for the families V_k, $k \geqq 2$, and $C(\gamma)$, $\gamma \geqq 0$.

Corollary 15 is quite important for applications. It allows us to obtain the results of Theorem 14 for linearly invariant families, which obey $|\arg f'(z)| \leqq k \arcsin |z|$ for small $|z|$ but may have $|\arg f'(z)|$ growing rapidly for $|z|$ close to 1. This is precisely the case for the family $\mathcal{S}$ (corollary 16).

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