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A TRANSFORMATION FORMULA FOR PRODUCTS ARISING IN PARTITION THEORY[1]

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ABSTRACT. We obtain a transformation formula involving Euler products. The formula can be utilized to obtain a large variety of partition-theoretic identities.

1. **A transformation formula.** Let $f(a, x)$ be the product given by

$$f(a,x) = \prod_{n=1}^{\infty} (1 - a^{\alpha(n)}x^n)^{g(n)/n}, \tag{1.1}$$

where $\alpha(n)$, $g(n)$ are totally multiplicative functions of n (that is, $\alpha(mn) = \alpha(m)\alpha(n)$, $g(mn) = g(m)g(n)$ for all positive integers m and n). Then we shall prove in this note that

$$\prod_{r=0}^{k-1} f(a, \omega^r x) = \prod_{d|k} \prod_{\delta|(k/d)} f(a^{(k/d)\alpha(d\delta)}, x^{k\delta})^{(g(d\delta)/\delta)\mu(\delta)}, \tag{1.2}$$

ω being a primitive k-th root of unity.

This result is a generalization of the identity proved earlier in [3]:

$$\prod_{r=0}^{k-1} \phi(\omega^r x) = \prod_{d|k} \{\phi(x^{kd})\}^{\sigma(k/d)\mu(d)}, \tag{1.3}$$

where

$$\phi(x) = \prod_{n=1}^{\infty} (1 - x^n), \tag{1.4}$$

and $\sigma(n)$ denotes the sum of the positive divisors of n. This is an important tool in deriving partition-theoretic identities such as the celebrated Ramanujan identity

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$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\{\phi(x^5)\}^5/\{\phi(x)\}^6,$$

$p(n)$ denoting as usual the number of unrestricted partitions of n.

The result (1.3) is easy when k is a prime and was noted by Kolberg [1], while the proof of (1.3) for general values or k was given by Subrahmanyasastri [3] by using multiplicative induction of k.

Many partition functions have generating functions of the form (1.1). For example,

(1) When $g(n) = n^2$, $a = 1$, $f(a,x)^{-1}$ generates the plane partitions, for which an asymptotic formula was obtained by Wright [5].

(2) When $g(n) = n \min(k,n)$, $a = 1$, $f(a,x)^{-1}$ generates $p^{(k)}(n)$, the number of k-rowed partitions of n. In this case $g(n)$ is not a totally multiplicative function. However, the $f(a,x)$ in this case can be related to the function for which $g(n) = n$, and n is a totally multiplicative function. Whenever the generating function is related to an $f(a,x)$ with a totally multiplicative $g(n)$, the formula (1.2) will be useful.

(3) When

$$g(n) = \begin{cases} n^2, & \text{if } n = 2^\alpha, \alpha \geqq 0 \\ 0, & \text{otherwise,} \end{cases}$$

and $a = 1$, we have a simple and interesting case. Here $g(n)$ is totally multiplicative and $f(a,x)^{-1}$ generates $P(n)$, the number of partitions of n into powers of 2 (including 1), with each summand occurring at most in as many different colors as the magnitude of the summand, with repetitions allowed. That is, n has representations of the form

$$n = \sum_{\alpha=0}^{\infty} \sum_{j=1}^{2^\alpha} a_{\alpha j}(2^\alpha)_j, \quad (a_{\alpha j} \geqq 0),$$

$a_{\alpha j}$ denoting the multiplicity of the summand 2^α in the color j. The notion of partitions with summands occurring in different colors goes back to MacMahon [2]. We can also interpret $P(n)$ as the number of weighted partitions into summands 2^α ($\alpha \geqq 0$), where the weight of the summand 2^α (of mulitplicity a_α) in a partition of

$$n = \sum_{\alpha=i_1}^{i_t} a_\alpha 2^\alpha$$

is to be taken as (the binomial coefficient)

$$\binom{2^{\alpha}+a_{\alpha}-1}{a_{\alpha}}, \quad (\alpha = i_1, i_2, \cdots, i_t).$$

In other words,

$$P(n) = \sum \binom{2^{i_1}+a_{i_1}-1}{a_{i_1}}\binom{2^{i_2}+a_{i_2}-1}{a_{i_2}}\cdots\binom{2^{i_t}+a_{i_t}-1}{a_{i_t}},$$

the summation being over all those non-negative integers i_r and a_{i_r} for which $n = a_{i_1}2^{i_1} + a_{i_2}2^{i_2} + \cdots + a_{i_t}2^{i_t} + \cdots$.

To illustrate the applications of (1.2) we shall derive the following simple partition-theoretic identities for $p(n)$, $p^{(3)}(n)$ and $P(n)$.

(A) In the case $g(n) = n, a = 1, k = 4$, we derive

$$\sum_0^\infty p(4n)x^{2n} = \frac{1}{2}\frac{\phi(x^2)}{\phi^3(x)}\phi(x^{24})A_1(x) + \frac{1}{2}\frac{\phi^3(x)\phi^3(x^4)}{\phi^8(x^2)}\phi(x^{24})A_2(x), \tag{1.5}$$

$$\sum_0^\infty p(4n+1)x^{2n} = \frac{1}{2}\frac{\phi(x^2)\phi(x^{24})}{\phi^3(x)}A_3(x) + \frac{1}{2}\frac{\phi^3(x)\phi^3(x^4)\phi(x^{24})}{\phi^8(x^2)}A_4(x), \tag{1.6}$$

$$\sum_0^\infty p(4n+2)x^{2n} = \frac{1}{2}\frac{\phi(x^2)\phi(x^{24})}{\phi^3(x)}A_1(x) - \frac{1}{2}\frac{\phi^3(x)\phi^3(x^4)\phi(x^{24})}{\phi^8(x^2)}A_2(x), \tag{1.7}$$

and

$$\sum_0^\infty p(4n+3)x^{2n+1} = \frac{1}{2}\frac{\phi(x^2)\phi(x^{24})}{\phi^3(x)}A_3(x) - \frac{1}{2}\frac{\phi^3(x)\phi^3(x^4)\phi(x^{24})}{\phi^8(x^2)}A_4(x), \tag{1.8}$$

where

$$A_1(x) = \prod_{m=1}^{\infty}(1+x^{24m-13})(1+x^{24m-11})$$

$$- x \prod_{m=1}^{\infty} (1 + x^{24m-19})(1 + x^{24m-5}),$$

$$A_2(x) = \prod_{m=1}^{\infty} (1 - x^{24m-13})(1 - x^{24m-11})$$

$$+ x \prod_{m=1}^{\infty} (1 - x^{24m-19})(1 - x^{24m-5}),$$

$$A_3(x) = \prod_{m=1}^{\infty} (1 + x^{24m-17})(1 + x^{24m-7})$$

$$- x^2 \prod_{m=1}^{\infty} (1 + x^{24m-23})(1 + x^{24m-1}),$$

and

$$A_4(x) = \prod_{m=1}^{\infty} (1 - x^{24m-17})(1 - x^{24m-7})$$

$$- x^2 \prod_{m=1}^{\infty} (1 - x^{24m-23})(1 - x^{24m-1}).$$

(B) In the case $g(n) = n \min(3, n)$, $a = 1$, we derive

$$\begin{aligned}\sum_{n=0}^{\infty} p^{(3)}(3n)x^{3n} = {} & \frac{\phi^3(x^9)\phi^6(x)}{\phi^{12}(x^3)}(1 + 2x^3) \\ & + \frac{6x\phi^6(x^9)\phi^3(x)}{\phi^{12}(x^3)}(1 + 2x^3) \\ & + \frac{9x^2\phi^9(x^9)}{\phi^{12}(x^3)}(1 - 2x + 2x^3 - x^4),\end{aligned} \tag{1.9}$$

$$\begin{aligned}\sum_{n=0}^{\infty} p^{(3)}(3n+1)x^{3n+1} = {} & \frac{-x\phi^3(x^9)\phi^6(x)}{\phi^{12}(x^3)}(2 + x^3) \\ & + \frac{3x\phi^6(x^9)\phi^3(x)}{\phi^{12}(x^3)}(1 - 4x + 2x^3 - 2x^4) \\ & + \frac{9x^2\phi^9(x^9)}{\phi^{12}(x^3)}(1 - 2x + 2x^3 - x^4),\end{aligned} \tag{1.10}$$

$$
\begin{aligned}
(1.11)\quad \sum_{n=0}^{\infty} p^{(3)}(3n+2)x^{3n+2} &= \frac{-3x^2\phi^6(x^9)\phi^3(x)}{\phi^{12}(x^3)}(2+x^3) \\
&\quad + \frac{9x^2\phi^9(x^9)}{\phi^{12}(x^3)}(1-2x+2x^3-x^4).
\end{aligned}
$$

Incidentally, we note from (1.11) that

$$
(1.12)\qquad p^{(3)}(3n+2) \equiv 0 \pmod 3.
$$

(C) In the case

$$
g(n) = \begin{cases} n^2, & \text{if } n = 2^\alpha, \alpha \geqq 0 \\ 0, & \text{otherwise} \end{cases}, \quad a = 1,
$$

we derive

$$
\begin{aligned}
(1.13)\quad \left(\sum_{0}^{\infty} P(4n)x^{4n}\right)x &= \left(\sum_{0}^{\infty} P(4n+1)x^{4n+1}\right) \\
&= \frac{x(1+3x^4)}{f(x)(1+x^2)^3(1+x)}
\end{aligned}
$$

and

$$
\begin{aligned}
(1.14)\quad \left(\sum_{0}^{\infty} P(4n+2)x^{4n+2}\right)x &= \left(\sum_{0}^{\infty} P(4n+3)x^{4n+3}\right) \\
&= \frac{x^3(3+x^4)}{f(x)(1+x^2)^3(1+x)}.
\end{aligned}
$$

2. **Proof of the formula** (1.2). We require the following

LEMMA 2.1. *Let A be any set of positive integers and $F(k,n)$ any arithmetic function with values in the complex number field. Then for every positive integer k*

$$
(2.1)\qquad \prod_{\substack{n\in A \\ (n,k)=1}} F(k,n) = \prod_{d\mid k} \prod_{\substack{m \\ md\in A}} \{F(k,md)\}^{\mu(d)},
$$

where $\mu(d)$ is the Möbius function.

This is easily proved using the Möbius inversion formula by setting $L(k,n) = \log F(k,n)$ (the principal value), $\sum_{n\in A \text{ and } (k,n)=d} L(k,n) = G(k/d)$, $\sum_{nd\in A} L(k,nd) = H(k/d)$ and noting that $\sum_{d\mid k} G(k/d) = H(k)$.

PROOF OF (1.2). Left side of (1.2) =

$$\prod_{d|k} \prod_{\substack{n=1 \\ (n,k)=d}}^{\infty} \prod_{r=0}^{k-1} (1 - a^{\alpha(n)} \omega^{rn} x^n)^{g(n)/n} \tag{2.2}$$
$$= \prod_{d|k} \prod_{\substack{n_1=1 \\ (n_1,k_1)=1}}^{\infty} F(k_1, n_1)$$

with $k = k_1 d$, $n = n_1 d$ and

$$F(k_1, n_1) = \prod_{r=0}^{k_1 d - 1} (1 - a^{\alpha(n_1 d)} \omega^{r n_1 d} x^{n_1 d})^{g(n_1 d)/n_1 d}$$
$$= \prod_{r=0}^{k_1 d - 1} (1 - a^{\alpha(n_1 d)} \omega_1^{\,n_1 r} x^{n_1 d})^{g(n_1 d)/n_1 d},$$

where $\omega_1 = \omega^d$, a primitive k_1-th root of unity. $\omega_2 = \omega_1^{\,n_1}$ is also a primitive k_1-th root of unity, and as r runs through a complete residue system mod k once, it runs through a complete residue system (mod k_1) d times. Hence

$$F(k_1, n_1) = \prod_{r=0}^{k_1 - 1} (1 - a^{\alpha(n_1 d)} \omega_2^{\,r} x^{n_1 d})^{d g(n,d)/n_1 d}$$
$$= (1 - a^{k_1 \alpha(n_1 d)} x^{k_1 n_1 d})^{g(n_1 d)/n_1},$$

so that by Lemma 2.1

$$\prod_{\substack{n_1=1 \\ (n_1,k_1)=1}}^{\infty} F(k_1, n_1) = \prod_{\delta|k_1} \prod_{m=1}^{\infty} (1 - a^{k_1 \alpha(m\delta d)} x^{k_1 m\delta d})^{g(md\delta)\mu(\delta)/m\delta}.$$

Substituting this in (2.2) and using the fact that $\alpha(n)$ and $g(n)$ are totally multiplicative, (1.2) follows.

COROLLARY. *In the case $a = 1$, (1.2) takes the form*

$$\prod_{r=0}^{k-1} f(\omega^r x) = \prod_{\delta|k} \{f(x^{k\delta})\}^{h(k/\delta)\mu(\delta)g(\delta)/\delta}, \tag{2.3}$$

where

$$f(x) = \prod_{n=1}^{\infty} (1 - x^n)^{g(n)/n} \tag{2.4}$$

and $h(m) = \sum_{d|m} g(d)$.

We shall give a simple alternate proof in this case. It is well known ([4], theorem 5, special case) that, if $h(n) = \sum_{d|n} g(d)$, then

$$h(kM) = \sum_{d|k,d|M} h(k/d)h(M/d)g(d)\,\mu(d). \tag{2.5}$$

We also recall that

$$\eta(k, m) \equiv \sum_{r=0}^{k-1} \omega^{rm} = \begin{cases} 0, & k \nmid m \\ k, & k \mid m \end{cases} \tag{2.6}$$

and that

$$\sum_{m=1}^{\infty} \frac{g(m)x^m}{1 - x^m} = \sum_{\ell=1}^{\infty} h(\ell)x^{\ell}. \tag{2.7}$$

From (2.5) and (2.6), we have

$$\begin{aligned}\sum_{m=1}^{\infty} h(m)\eta(k, m)x^m &= k \sum_{M=1}^{\infty} h(kM)x^{kM} \\ &= \sum_{M=1}^{\infty} \sum_{\substack{d|k \\ nd=M}} kh(k/d)\mu(d)g(d)h(n)x^{kM} \\ &= \sum_{d|k} kh(k/d)\mu(d)g(d) \sum_{n=1}^{\infty} h(n)x^{kdn},\end{aligned}$$

which on using (2.6) and (2.7) can be written as

$$\sum_{r=0}^{k-1} \sum_{m=1}^{\infty} \frac{g(m)\omega^{rm}x^m}{1 - x^m\omega^{rm}} = \sum_{d|k} h(k/d)\mu(d)kg(d) \sum_{n=1}^{\infty} \frac{g(n)x^{kdn}}{1 - x^{kdn}}.$$

We now restrict x to be such that $0 < x < 1$ (we can at the end extend the result to $|x| < 1$ by analytic continuation). Dividing both sides by x and integrating with respect to x, we obtain

$$\begin{aligned}&\sum_{r=0}^{k-1} \sum_{m=1}^{\infty} \frac{g(m)}{m} \log(1 - \omega^{rm}x^m) \\ &= \sum_{d|k} h(k/d)\mu(d) \frac{g(d)}{d} \sum_{n=1}^{\infty} \frac{g(n)}{n} \log(1 - x^{kdn}),\end{aligned}$$

the constant of integration being zero as can be seen by setting $x = 0$. Thus we have

$$\sum_{r=0}^{k-1} \log f(\omega^r x) = \sum_{d|k} h(k/d)\,\mu(d)\,\frac{g(d)}{d}\log f(x^{kd}),$$

which is the same as relation (2.3).

3. **Proof of the identities** (1.5) **to** (1.8). Choosing $g(n) = n$, $a = 1$, $k = 4$, (1.2) yields

$$\phi(x)\phi(ix)\phi(-x)\phi(-ix) = \frac{\phi^7(x^4)}{\phi^3(x^8)}, \tag{3.1}$$

i being an imaginary square root of -1. Also

$$4\sum_{0}^{\infty} p(4n+\ell)x^{4n+\ell} = \frac{1}{\phi(x)} + \frac{i^{-\ell}}{\phi(ix)} + \frac{i^{-2\ell}}{\phi(-x)} + \frac{i^{-3\ell}}{\phi(-ix)} \quad (\ell = 0, 1, 2, 3). \tag{3.2}$$

We shall also need the well-known identity of Jacobi:

$$\sum_{k=-\infty}^{\infty} y^k z^{k^2} = \phi(z^2)\prod_{m=1}^{\infty}(1 + yz^{2m-1})(1 + y^{-1}z^{2m-1}). \tag{3.3}$$

Using Euler's identity

$$\phi(x) = \sum_{-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}, \tag{3.4}$$

we can write

$$\phi(x) = g_0(x) + g_1(x) + g_2(x) + g_3(x), \tag{3.5}$$

where

$$g_\ell(x) = \sum_{-\infty}^{\infty} (-1)^n x^{n(3n+1)/2}, \ \ell = 0, 1, 2, 3, \qquad n(3n+1)/2 \equiv \ell \pmod 4). \tag{3.6}$$

Then

$$\begin{aligned}\phi(-x) &= g_0(-x) + g_1(-x) + g_2(-x) + g_3(-x)\\ &= g_0(x) - g_1(x) + g_2(x) - g_3(x),\end{aligned}$$

in view of (3.6), so that on using (3.6) and (3.4),

$$\begin{aligned}\phi(x)+\phi(-x) &= 2\{g_0(x)+g_2(x)\}\\ &= 2\sum_{-\infty}^{\infty} x^{2k(12k+1)} - 2\sum_{-\infty}^{\infty} x^{(4k+1)(6k+2)}\\ &= 2\phi(x^{48})\left\{\prod_1^{\infty}(1+x^{48m-26})(1+x^{48m-22})\right.\\ &\qquad \left. - x^2\prod_1^{\infty}(1+x^{48m-38})(1+x^{48m-10})\right\}\\ &= 2\phi(x^{48})A_1(x^2).\end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned}\phi(ix)+\phi(-ix) &= 2\phi(x^{48})A_1((ix)^2)\\ &= 2\phi(x^{48})A_1(-x^2)\\ &= 2\phi(x^{48})A_2(x^2)\end{aligned} \tag{3.8}$$

in terms of $A_1(x)$ and $A_2(x)$ given in (A) of §1.

With $g(n) = n$, $k = 2$, $a = 1$ (1.2) yields

$$\phi(x)\phi(-x) = \frac{\phi^3(x^2)}{\phi(x^4)}, \tag{3.9}$$

and so, from (3.1),

$$\phi(ix)\phi(-ix) = \frac{\phi^8(x^4)}{\phi^3(x^8)\phi^3(x^2)}. \tag{3.10}$$

Hence, from (3.2) and (3.7) to (3.10), we obtain

$$\begin{aligned}\sum_0^{\infty} p(4n)x^{4n} &= \frac{1}{4}\left\{\frac{\phi(x)+\phi(-x)}{\phi(x)\phi(-x)} + \frac{\phi(ix)+\phi(-ix)}{\phi(ix)\phi(-ix)}\right\}\\ &= \frac{1}{2}\frac{\phi(x^4)}{\phi^3(x^2)}\phi(x^{48})A_1(x^2) + \frac{1}{2}\frac{\phi^3(x^2)\phi^3(x^8)}{\phi^8(x^4)}\phi(x^{48})A_2(x^2),\end{aligned}$$

which is the same as (1.5). (1.6) to (1.8) follow on similar lines.

4. **Proof of the identities** (1.9) **to** (1.11). The generating function $\psi(x)^{-1}$ of $p^{(3)}(n)$ is given by

$$\psi(x) = \prod_{n=1}^{\infty}(1-x^n)^{\min(3,n)} \tag{4.1}$$

$$= \frac{\phi^3(x)}{(1-x)^2(1-x^2)}.$$

If ω is a primitive cube root of unity, then

$$\begin{aligned}(4.2)\quad 3\sum_0^\infty p^{(3)}(3n+\ell)x^{3n+\ell} &= \frac{1}{\psi(x)} + \frac{\omega^{2\ell}}{\psi(\omega x)} + \frac{\omega^\ell}{\psi(\omega^2 x)} \\ &= \frac{\psi(\omega x)\psi(\omega^2 x) + \omega^{2\ell}\psi(x)\psi(\omega^2 x) + \omega^\ell \psi(x)\psi(\omega x)}{\psi(x)\psi(\omega x)\psi(\omega^2 x)} (\ell = 0, 1, 2).\end{aligned}$$

Also,

$$(4.3)\qquad \phi^3(x) = h_0(x) + h_1(x) + h_2(x)$$

with

$$h_\ell(x) = \sum_{n=0}^\infty (-1)^n(2n+1)x^{n(n+1)/2}.$$

$$n(n+1)/2 \equiv \ell (\mathrm{mod}\ 3),$$

so that

$$(4.4)\qquad \phi^3(\omega^\ell x) = h_0(x) + \omega^\ell h_1(x) + \omega^{2\ell} h_2(x).$$

In fact,

$$h_0(x) = \phi^3(x) + 3x\phi^3(x^9),$$

$$h_1(x) = -3x\phi^3(x^9),$$

$$h_2(x) = 0 \qquad \text{(See Kolberg [1] p. 82).}$$

From (4.1) and (1.3), we obtain

$$(4.5)\qquad \psi(x)\psi(\omega x)\psi(\omega^2 x) = \frac{\phi^{12}(x^3)}{\phi^3(x^9)(1-x^3)^2(1-x^6)}.$$

Further, from (4.1) and (4.4) we obtain

$$\psi(\omega x)\psi(\omega^2 x) = \frac{(h_0^2(x) + h_1^2(x) - h_0(x)h_1(x))(1 - 2x + 2x^3 - x^4)}{(1-x^3)^2(1-x^6)}$$

and similar expressions for $\psi(\omega^2 x)\psi(x)$ and $\psi(x)\psi(\omega x)$.

Taking $\ell = 0$ in (4.2) and using (4.5) and the above expressions for $\psi(\omega x)\psi(\omega^2 x)$ etc., we obtain

$$3\sum_0^\infty p^{(3)}(3n)x^{3n} = 3\{h_0^2(x)(1+2x^3) - h_1^2(x)(2x+x^4)\}\frac{\phi^3(x^9)}{\phi^{12}(x^3)}.$$

which yields (1.9) on substituting the above Kolberg's expressions for $h_0(x)$ and $h_1(x)$. (1.10) and (1.11) follow on the same lines.

5. **Proof of** (1.13) **and** (1.14). Choosing

$$g(n) = \begin{cases} n^2, & \text{if } n = 2^\alpha\ (\alpha \geqq 0) \\ 0, & \text{otherwise} \end{cases}, \text{ and } a = 1, k = 4,$$

we have from (1.2)

$$f(x)f(ix)f(-x)f(-ix) = \frac{f^{21}(x^4)}{f^{10}(x^8)}. \tag{5.1}$$

We can also verify that in this case $f(x)$ satisfies

$$(1 - x)f^2(x^2) = f(x) \tag{5.2}$$

or

$$f^2(x^2) = f(x)(1 + x + x^2 + \cdots),$$

so that if we put $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $(a_0 = 1)$, it is easily seen that the coefficients a_n are given by the recursion formulae

$$a_n = -\sum_{r=0}^{n-1} a_r, \qquad \text{if } n \text{ is odd,} \tag{5.3}$$

and

$$a_n = \sum_{j=0}^{n/2} a_j a_{(n/2)-j}, \quad \text{if } n \text{ is even.} \tag{5.4}$$

These equations (5.3) and (5.4) determine $f(x)$. However, these are not required for the proof of (1.13) and (1.14).

We shall indicate the proof of the first half of (1.13). First we note that

$$4\sum_0^\infty P(4n + \ell)x^{4n+\ell} = \frac{1}{f(x)} + \frac{i^{-\ell}}{f(ix)} + \frac{i^{-2\ell}}{f(-x)} + \frac{i^{-3\ell}}{f(-ix)} \qquad (\ell = 0, 1, 2, 3). \tag{5.5}$$

From (5.2) we have $f(-x) = (1 + x)f^2(x^2)$, so that

$$f(x) + f(-x) = 2f^2(x^2), \tag{5.6}$$

$$f(x)f(-x) = f^4(x^2)(1 - x^2), \tag{5.7}$$

and

$$\frac{f(-x^2)}{f(x^2)} = \frac{(1 + x^2)}{(1 - x^2)}. \tag{5.8}$$

Taking $\ell = 0$ in (5.5), and using (5.6), (5.7), (5.1) and similarly (5.8) we obtain

$$\sum P(4n)x^{4n} = \frac{1}{4}\left\{\frac{f(x)+f(-x)}{f(x)f(-x)} + \frac{f(ix)+f(-ix)}{f(ix)f(-ix)}\right\}$$

$$= \frac{1}{4}\frac{(f(x)+f(-x))f(ix)f(-ix)+(f(ix)+f(-ix))f(x)f(-x)}{f(x)f(ix)f(-x)f(-ix)}$$

$$= \frac{f^{10}(x^8)}{2f^{21}(x^4)}\{f^2(x^2)f^4((ix)^2)(1-i^2x^2)$$

$$+ f^2((ix)^2 f^4(x^2)(1-x^2)\} \tag{5.9}$$

$$= \frac{f^{10}(x^8)}{2f^{21}(x^4)} f^6(x^2)\frac{(1+x^2)^2}{(1-x^2)^4}\{(1+x^2)^3+(1-x^2)^3\}.$$

But by repeated use of (5.2) raised to the suitable exponents, we obtain

$$\frac{f^{10}(x^8)f^6(x^2)}{f^{21}(x^4)} = \frac{f^{10}(x^8)}{f^5(x^4)}\frac{f^8(x^2)}{f^{16}(x^4)}\frac{1}{f^2(x^2)}$$

$$= (1-x^2)^8\frac{1}{(1-x^4)^5}\frac{(1-x)}{f(x)} = \frac{(1-x^2)^3(1-x)}{(1+x^2)^5 f(x)}.$$

The first half of the identity (1.13) follows on substituting this in (5.9). The other half of (1.13) and (1.14) follow on using similar arguments.

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