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A NEW FUNCTIONAL EQUATION WITH SOME SOLUTIONS†

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1. **Introduction.** Functional equations arising naturally in applied mathematics are not too common (see for example Aczel's treatise [1] on the general topic), and here we introduce a system springing from elementary concepts in queueing theory. In a previous paper [2] the authors have developed a system of generalized discrete probability distributions by considering equations of the form $x = y\,g(x)$, where $g(x)$ is the generating function of a random variable defined on the non-negative integers, and x, as a function of y, in general generates a Lagrangian probability generating function.

It now turns out that there is a tie up with the behavior of the random variable which describes the length of a busy period in a single server queueing system where the 'arrivals' follow some defined probability distribution. From general considerations of this situation, we were led to the functional equation

$$(1) \quad H(x, y) = 1 + xy(1 - xy)^{-1}\{1 - \psi(\lambda - \lambda xy)\}\, H(\psi(\lambda - \lambda xy), y),$$

where x and y are defined over the real domain, λ is a real number, and H, ψ are real, continuous and infinitely differentiable functions. The coefficients of the successive powers of x and y represent the probabilities of different lengths of busy periods for queues initiated by 1, 2, 3, $\cdots$ customers when $\psi(\lambda - \lambda xy)$ is a Laplace transform associated with the inter-arrival and inter-service density functions. The queues initiated by one customer have been considered in several studies; however it is a matter of common experience at medical clinics, automobile service stations and factory supply offices etc., that the First Busy Period is usually initiated by a larger number of customers than one. Thus the problem becomes a little more complex.

Our equation (1) is a specal case of a general functional equation

$$(2) \qquad H(x, y) = 1 + \phi(xy)\, H(\psi(xy), y),$$

where $\phi(xy)$ and $\psi(xy)$ are real and continuous functions of xy having power series expansions and such that $\phi(0) = 0$. The observation that

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the functional equation (2) represents an expansion can be verified in the following manner.

If the iterates $\psi(yx)$, $\psi(y\psi(yx))$, $\psi(y\psi(y\psi(yx)))$, $\cdots$ are denoted by ψ_1, ψ_2, ψ_3, $\cdots$ respectively then equation (2) can be written in the form given below:

$$
\begin{aligned}
H(x, y) &= 1 + \phi(xy)\,[1 + \phi(y\psi_1)\,H(\psi_2, y)] \\
&= 1 + \phi(xy) + \phi(xy)\phi(y\psi_1) + \phi(xy)\,\phi(y\psi_1)\,\phi(y\psi_2) + \cdots \qquad (3) \\
&= 1 + \sum_{s=0}^{\infty} \left\{ \prod_{i=0}^{s} \phi(y\psi_i) \right\},
\end{aligned}
$$

where $\psi_0 = x$, and some restrictions may have to be imposed on x and y for its convergence. Thus each term of the infinite series consists of a product of a number of functions.

In this paper we provide a number of simple solutions of equations (1) and (2) in §§ 2 and 3, respectively, and derive two alternative forms for the general solution of equation (2) in §§ 4 and 5. It is presumed for the existence of the solutions that $\psi(xy)$ and $\phi(xy)$ are given in the form of absolutely convergent power series of xy.

2. **Some Simple Solutions of Equation (1).** We may observe that the equation does not possess a solution when (i) $xy = 1$ or (ii) $\psi(\lambda - \lambda xy) = (xy)^{-1}$, and that it satisfies two initial conditions:

$$H(0, y) = H(x, 0) = 1. \qquad (4)$$

The equation possesses two trivial solutions given below:

$$
\left\{
\begin{aligned}
&H(x, y) = 1, && \text{for} \quad \psi(\lambda - \lambda xy) = 1 \\
&H(x, y) = (1 - xy)^{-1}, && \text{for} \quad \psi(\lambda - \lambda xy) = 0.
\end{aligned}
\right. \qquad (5)
$$

The following three cases will reasonably convince anyone that the solutions of the equation are quite intricate even for simple values of $\psi(\lambda - \lambda xy)$ and that the general solution is non-trivial.

Case I. When $\psi(\lambda - \lambda xy) = xy$,

$$H(x, y) = 1 + xy\, H(xy, y),$$

which , recursively, gives the series

$$
\begin{aligned}
H(x, y) &= 1 + xy + xy(xy^2) + xy(xy^2)(xy^3) + \cdots \qquad (6) \\
&= \sum_{r=0}^{\infty} \{xy^{(r+1)/2}\}^r,
\end{aligned}
$$

where the series is convergent for all values of x in the real domain if $|y| < 1$.

Case II. When $\psi(\lambda - \lambda xy) = x^2y^2$,

$$H(x, y) = 1 + xy(1 + xy)\, H(x^2y^2, y),$$

whose solution can be written down recursively in the form

$$H(x, y) = 1 + \sum_{s=1}^{\infty} \left[x^{2^s - 1} y^{2^{s+1} - s - 2} \prod_{i=1}^{s} (1 + x^{2^{i-1}} y^{2^{i}-1}) \right]. \tag{7}$$

The series can be shown to be convergent for all x in the real domain if $|y| < 1$.

Case III. When $\psi(\lambda - \lambda xy) = (xy)^{-2}$,

$$H(x, y) = 1 - (1 + xy)\,(xy)^{-1} H((xy)^{-2}, y),$$

whose solution, according to (3), will be an alternating series. However, each negative term can be easily cancelled with the first part of the next positive term. Thus

$$\begin{aligned} H(x, y) &= 1 + x(1 + xy) + x^3y(1 + xy)(1 + x^2y)(1 + x^4y^3) \\ &\quad + x^{11}y^6(1 + xy)(1 + x^2y)(1 + x^4y^3)(1 + x^8y^5)\cdot \\ &\qquad (1 + x^{16}y^{11}) + \cdots \\ &= 1 + x(1 \pm xy) \left[1 + \sum_{s=1}^{\infty} x^{(2^{2s} - 1)2/3} y^{(2^{2s+2} - 4 - 3s)/9} \right. \\ &\quad \left. \left\{ \prod_{i=1}^{s} (1 + x^{2^{2i-1}} y^{(2^{2i}-1)/3}) (1 + x^{2^{2i}} y^{(2^{2i-1}+1)/3}) \right\} \right]. \end{aligned} \tag{8}$$

The above series is absolutely convergent if $|x| < 1$, $|y| < 1$.

Evidently, the soltuions of (1) will become much more involved if $\psi(\lambda - \lambda xy)$ is a sum of two or more power terms. In fact, the number of terms begins to explode very fast as soon as $\psi(\lambda - \lambda xy)$ is taken as a sum of 4 simple terms.

Though the above remarks clearly indicate that the solutions of equation (1) are generally in the form of a power series in x and y, the next case will show that some of these intricate solutions can be reduced to closed forms as well.

Case IV. When $\psi(\lambda - \lambda xy) = k_1(k_2 - \lambda xy)^{-1}$, let us suppose, if possible, that the function $H(x, y) = [1 - xg(y)]^{-1}$ for some specific values of k_1, k_2 and $g(y)$. It may be noted that $[1 - xg(y)]^{-1}$ represents an infinite series in x and y which satisfies (4) and has to satisfy the above conditions as well as (1). The values of $H(x, y)$ and $\psi(\lambda - \lambda xy)$, on substitution in (1), should give us the conditions for the validity of the supposed solution. Thus

$$\frac{1}{1 - xg(y)} = 1 + \frac{xy(k_2 - k_1 - \lambda xy)}{(1 - xy)(k_2 - \lambda xy)} \cdot \frac{1}{1 - k_1 g(y)/(k_2 - \lambda xy)}$$

or

$$\frac{xg(y)}{1 - xg(y)} = \frac{xy(k_2 - k_1 - \lambda xy)}{(1 - xy)[k_2 - \lambda xy - k_1 g(y)]},$$

which gives the quadratic equation

$$(9) \qquad k_1(g(y))^2 - g(y)\frac{[k_2 - (k_1 + \lambda)xy]}{1 - xy} + y\frac{k_2 - k_1 - \lambda xy}{1 - xy} = 0.$$

If $k_2 = k_1 + \lambda$, the coefficients of y and $g(y)$ become λ and $-(k_1 + \lambda)$, respectively, and equation (9) becomes independent of x. The value of $g(y)$ will then be given by (9) in the form

$$(10) \qquad g(y) = \frac{k_1 + \lambda - \sqrt{\{(k_1 + \lambda)^2 - 4k_1\lambda y\}}}{2k_1}$$

which exists for all values of k_1. Hence the functional equation (1) has a closed form solution $H(x, y) = [1 - xg(y)]^{-1}$ for all k_1, and $g(y)$ is given by (10).

3. **Second Functional Equation.** Since $\phi(0) = 0$, the function $\phi(xy)$ must be of the form $\phi(xy) = xyf(xy)$, and $H(0, y) = H(x, 0) = 1$.

If $x = x_0$, $\psi(xy) = a + bxy = x_1$, $a + bx_1y = x_2$, $a + bx_2y = x_3$, $\cdots$, then the solution of the functional equation (2) is given by (3) as

$$(11) \qquad H(x, y) = 1 + \sum_{k=0}^{\infty} \left\{ \prod_{s=0}^{k} \phi(yx_s) \right\},$$

where

$$(12) \qquad x_s = a\{1 - (by)^s\}(1 - by)^{-1} + x(by)^s.$$

We shall now adopt a procedure similar to Case IV of the last section to determine some closed form solutions of (2).

Case I. Let us suppose (if possible) that

$$(13) \qquad H(x, y) = [1 - x^m g(y)]^{-1}$$

for some values of $\phi(xy)$, $\psi(xy)$, $g(y)$ and m. The above value must satisfy equation (2). Therefore

$$\frac{1}{1 - x^m g(y)} = 1 + \frac{\phi(xy)}{1 - \{\psi(xy)\}^m g(y)}$$

which, on simplification, gives the quadratic equation in $g(y)$ as

$$\{g(y)\}^2 - \frac{\phi(xy) + 1}{\{\psi(xy)\}^m} g(y) + y^m \frac{\phi(xy)}{\{xy\,\psi(xy)\}^m} = 0.$$

The above equation may give realistic values of $g(y)$ if the coefficients of y and $g(y)$ are independent of xy. If we suppose

$$\frac{\phi(xy)}{[xy\,\psi(xy)]^m} = k_1, \quad \frac{\phi(xy) + 1}{\{\psi(xy)\}^m} = k_2,$$

then

(14) $$\psi(xy) = [k_2 - k_1(xy)^m]^{-1/m},$$

(15) $$\phi(xy) = k_1(xy)^m [k_2 - k_1(xy)^m]^{-1},$$

and

(16) $$g(y) = \tfrac{1}{2}[k_2 - \sqrt{(k_2{}^2 - 4k_1 y^m)}].$$

Thus for any set of compatible values of k_1, k_2 and m, one can determine $\psi(xy)$ and $\phi(xy)$, by (14) and (15), for which (13) is a solution of the functional (2) provided (16) gives real values of $g(y)$. Evidently, the functional equation will have an unlimited number of such solutions.

Case II. When $\psi(xy) = \alpha(1 + bxy)^{-1}$, the successive values of ψ_i, $i = 1, 2, 3, \cdots$ in solution (3) can be written down from the theory of continued fractions and are given by

(17) $$\psi_i = \frac{\alpha(\Delta_i + \Delta_{i-1} bxy)}{\Delta_{i+1} + \Delta_i\, bxy},$$

where

(18) $$\Delta_i = \lambda_1{}^i - \lambda_2{}^i, \quad \text{and}$$
$$\lambda_1 = \{1 + \sqrt{(1 + 4aby)}\}/2, \ \lambda_2 = \{1 - \sqrt{(1 + 4aby)}\}/2.$$

The values of ψ_i can be alternatively written in the form of successive convergents $\psi_i = aq_{i-1}/q_i$, where

$$q_0 = 1,\ q_1 = 1 + bxy,\ q_2 = (1 + bxy) + aby \tag{19}$$

$$q_i = (1 + bxy)\left[\sum_{s=0}^{k} \binom{i-s-1}{s} (aby)^s\right] + \frac{aby[1 - (aby)^{i-2}]}{1 - aby},$$

for $i \geqq 3$ and $k = (1/2)\, i$ or $(1/2)\,(i - 1)$ according as i is even or odd, respectively.

Now, if $\phi(xy) = (xy)^m$, the solution (3) of the functional equation (2) becomes

$$\begin{aligned} H(x, y) = 1 &+ \\ (xy)^m &\left[1 + \left(\frac{(ay)}{q_1}\right)^m + \left(\frac{(ay)^2}{q_2}\right)^m + \left(\frac{(ay)^3}{q_3}\right)^m + \cdots\right]. \end{aligned} \tag{20}$$

Thus, it is clear that the functional equation has many different forms of solution, but they are not obvious and need further study.

4. **General Solution of Second Functional Equation as a Power Series in xy.** Let the power series expansions in xy, in some common domain of $|x| < 1$, and $|y| < 1$, of the known functions $\psi(xy)$ and $\phi(xy)$ be given by

$$\psi(xy) = \sum_{r=0}^{\infty} a_r (xy)^r / r! \tag{21}$$

$$\phi(xy) = \sum_{s=1}^{\infty} b_s (xy)^s,\ b_0 = 0, \tag{22}$$

so that $a_0 = \psi(0)$, $a_r = \psi^{(r)}(0)$, where $\psi^r(0)$ denotes the value at $xy = 0$ of the r-th derivative of $\psi(xy)$ with respect to xy.

The form of the functional equation (2) is such that one can reasonably assume that $H(x, y)$ has a power series expansion in terms of xy and that the coefficient of every term is a function of y. Therefore, let

$$H(x, y) = \sum_{k=0}^{\infty} \frac{(xy)^k}{k!} F(k, y), \tag{23}$$

where $F(0, y) = 1$.

The substitution of the values of $\phi(xy)$ and $H(\psi(xy), y)$ by (22) and (23), respectively, in the functional equation (2) gives the right hand side as

$$1 + \left\{ \sum_{s=1}^{\infty} b_s(xy)^s \right\} \left\{ \sum_{k=0}^{\infty} \frac{y^k \{\psi(xy)\}^k}{k!} F(k, y) \right\}. \tag{24}$$

Since the product of the two series in (24) must be identical with the series in (23), the coefficients of the successive powers of x can be equated with each other. Equating the coefficients of $(xy)^m$ on both sides,

$$\begin{aligned} F(m, y) &= m! \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} b_{m-s} \frac{y^k}{k!} F(k, y) \times \text{coef of } z^s \text{ in } [\psi(z)]^k \\ &= m! \sum_{s=0}^{m-1} \sum_{k=0}^{\infty} \Bigg[F(k, y) b_{m-s} \frac{y^k}{k!} \\ &\quad \left\{ \sum_k \frac{k!}{p_0! p_1! \cdots p_s!} (a_0)^{p_0} \left(\frac{a_1}{1!}\right)^{p_1} \cdots \left(\frac{a_s}{s!}\right)^{p_s} \right\} \Bigg], \end{aligned} \tag{25}$$

where the third summation is over all the partitions of k such that

$$\begin{cases} p_0 + p_1 + p_2 + \cdots + p_s = k, \\ p_1 + 2p_2 + \cdots + sp_s = s. \end{cases} \tag{26}$$

The equations (25), subjected to the conditions in (26), provide us with an infinite set of relations as $m = 1, 2, 3, \cdots$ for the determination of the specific expressions for $F(m, y)$. The values of $F(m, y)$, for each integral value of m, are dependent upon all $F(m, y)$. For example:

$$F(1, y) = b_1 \sum_{k=0}^{\infty} (a_0 y)^k F(k, y)/k!$$

$$F(2, y) = 2! \sum_{k=0}^{\infty} [b_2 + b_1 k(a_1/a_0)] \frac{(a_0 y)^k}{k!} F(k, y)$$

$$\begin{aligned} F(3, y) = 3! \sum_{k=0}^{\infty} \Bigg[b_3 + \frac{b_2 a_1 k}{a_0} + b_1 \Bigg\{ &\frac{k!(a_2/a_0)}{(k-1)!1!} \\ &+ \frac{k!(a_1/a_0)^2}{(k-2)!2!} \Bigg\} \Bigg] \frac{(a_0 y)^k}{k!} F(k, y) \end{aligned}$$

$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$

We observe that $[F(m, y)]_{y=0} = m!\, b_m$, and the successive derivatives of $F(m, y)$ at $y = 0$ can also be written down systematically, for each m, from our result (25). Thus the functions $F(m, y)$, for each integral value of m, can be obtained as a power series in y by writing the coefficient of y^n in $F(m, y)$ equal to $(n!)^{-1} \times$ n-th derivative of $F(m, y)$ at $y = 0$. Denoting the coefficient of y^n in $F(m, y)$ by some symbol or else the coefficient of $x^m y^{m+n}$ in $H(x, y)$ by $F_{m+n}(m)$, we may use the result (25) to obtain its value in the form

$$(27)\qquad \begin{aligned} F_{m+n}(n) = \sum_{k=0}^{n} \Bigg[\binom{n}{k} \frac{1}{n!} \left\{ \frac{d^{n-k}}{dy^{n-k}} F(k, y) \right\}_{y=0} \sum_{s=0}^{m-1} b_{m-s} \\ \left\{ \sum_{k} \frac{k!}{p_0! p_1! \cdots p_s!} (a_0)^{p_0} \left(\frac{a_1}{1!}\right)^{p_1} \cdots \left(\frac{a_s}{s!}\right)^{p_s} \right\} \Bigg] . \end{aligned}$$

The above expression for the coefficient of $x^m y^{m+n}$ in $H(x, y)$ is rather complex, but it is not surprising. Though the expression (27) contains derivatives of $F(k, y)$ at $y = 0$ on the right hand side, it can still be evaluated for specific values of m and n by using the infinite set of relations, mentioned earlier, and their successive derivatives at $y = 0$. We quote below some expressions and their values for $m = 1$ and $m = 2$ as an illustration:

For $m = 1$

$$(28)\qquad F_{n+1}(1) = \sum_{k=1}^{n} \frac{b_1 a_0^k}{k!(n-k)!} \left\{ \frac{d^{n-k}}{dy^{n-k}} F(k, y) \right\}_{y=0}$$

and

$$(29)\qquad \begin{aligned} &F_1(1) = b_1,\ F_2(1) = b_1^2 a_0, \\ &F_3(1) = b_1 a_0^2 (b_2 + b_1^2), \\ &F_4(1) = b_1 a_0^2 (b_3 + 2b_2 b_1 a_0 + b_1^2 a_1 + b_1^3 a_0), \\ &F_5(1) = b_1 a_0^3 (b_4 a_0 + 2b_3 b_1 a_0 + 3b_2 b_1^2 a_0 \\ &\qquad + 3b_2 b_1 a_1 + b_2^2 a_0 + 2b_1^3 a_1 \\ &\qquad + b_1^2 a_2 + b_1^4 a_0), \end{aligned}$$

....

For $m = 2$

$$(30)\qquad F_{n+2}(2) = \sum_{k=1}^{n} \left[\frac{(b_2 a_0 + k b_1 a_1) a_0^{k-1}}{k!(n-k)!} \left\{ \frac{d^{n-k}}{dy^{n-k}} F(k, y) \right\}_{y=0} \right]$$

and

$$
\begin{aligned}
&F_2(2) = b_2, \quad F_3(2) = b_1(b_2a_0 + b_1a_1),\\
&F_4(2) = (b_2a_0 + b_1a_1)b_1{}^2a_0 + (b_2a_0 + 2b_1a_1)b_2a_0,\\
&F_5(2) = (b_2a_0 + b_1a_1)(b_2 + b_1{}^2)b_1a_0{}^2\\
&\qquad + (b_2a_0 + 2b_1a_1)(b_2a_0 + b_1a_1)b_1a_0\\
&\qquad + (b_2a_0 + 3b_1a_1)b_3a_0{}^2,\\
&F_6(2) = b_1a_0{}^2(b_2a_0 + b_1a_1)(b_3a_0 + 2b_2b_1a_0 + b_1{}^2a_1 + b_1{}^3a_0)\\
&\qquad + a_0{}^2(b_2a_0 + 2b_1a_1)(b_2{}^2a_0 + b_2b_1{}^2a_0 + b_2b_1a_1 + b_1{}^3a_1)\\
&\qquad + a_0{}^2b_1(b_2a_0 + 3b_1a_1)(b_3a_0 + b_2a_1 + b_1a_2)\\
&\qquad + a_0{}^3b_4(b_2a_0 + 4b_1a_1).
\end{aligned} \tag{31}
$$

Thus the coefficients of the successive terms can all be determined in a systematic manner. Possibly, the development of a computer program will substantially reduce the labour involved.

One may be naturally concerned about the convergence of such solutions expressed in bivariate infinite series. However, this concern is easily put to rest by the Fixed Point Theorem, stated by Dieudonné [3, p. 260], wherein such solutions are said to be convergent in the open circle $x < 1, y < 1$.

5. **General solution of functional equation-alternative form.** Let the functions $\psi(xy)$ and $\phi(xy)$ in (2) be expressed in the form of absolutely convergent series of xy and be given by (21) and (22), respectively.

Also, let $H(x, y)$ be absolutely convergent and be given by

$$H(x, y) = 1 + \sum_{j=1}^{\infty} y^j \left(\sum_{i=1}^{j} a_{ij}x^i \right). \tag{32}$$

The above form of the solution looks arbitrary and somewhat restricted but a careful consideration of (2) will show that x occurs in the form of powers of xy while y occurs independently as well. This implies that the powers of x in any coefficient of y^j must be equal to or less than j. Hence the supposed form of $H(x, y)$ is quite general.

By substituting the values of $\psi(xy)$, $\phi(xy)$ and $H(x, y)$ in (2), the right hand side of the equation becomes

$$1 + \left\{ \sum_{s=1}^{\infty} b_s(xy)^s \right\} \left[1 + \sum_{j=1}^{\infty} \sum_{i=1}^{j} a_{ij}y^j \quad \left\{ \sum_{r=0}^{\infty} a_r(xy)^r \right\}^i \right],$$

which should be identical with the right side of (32). On equating the coefficeints of y^n in (32) and in the above expression, one obtains

$$\sum_{i=1}^{n} a_{in}x^i \equiv b_n x^n + \sum_{k=1}^{n-1}\Big[x^{n-k}\sum_{i=1}^{k} a_{ik}\sum_{s=k}^{n-1} b_{n-s} \tag{33}$$
$$\Big\{\sum_p \frac{i!(a_0)^{p_0}(a_1)^{p_1}}{p_0!p_1!\cdots(p_{s-k})!}\cdots\Big(\frac{a_{s-k}}{(s-k)!}\Big)^{p_{s-k}}\Big\}\Big],$$

where $\sum_p$ is taken over all partitions of i such that

$$\begin{cases} p_0 + p_1 + p_2 + \cdots + p_{s-k} = i \\ p_1 + 2p_2 + 3p_3 + \cdots + (s-k)p_{s-k} = s - k. \end{cases} \tag{34}$$

Now by equating the coefficients of different powers of x on both sides of (33), for all integral values of n, we get n different relations which can, in general, be represented by the following formula,

$$a_{n,n} = b_n,$$

$$a_{n-m,n} = \sum_{i=1}^{m} a_{i,m}\Big[\sum_{s=m}^{n-1} b_{n-s}\Big\{\sum_p \frac{i!}{p_0!p_1!\cdots(p_{s-m})!} \tag{35}$$
$$(a_0)^{p_0}(a_1)^{p_1}\cdots\Big(\frac{a_{s-m}}{(s-m)!}\Big)^{p_{s-m}}\Big\}\Big],$$

where $\sum_p$ is the same as defined in (34) with $k = m$.

The result (35) gives the values of all the coefficients $a_{n-m,n}$ in (32) by giving successive integral values to m and n; however, it may be desirable to point out the presence of three summations in the expression (35) will provide a large number of terms in each coefficient as soon as $m \geqq 4$.

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