PROJECTIONS OF CANTOR SETS, SIMPLE CLOSED CURVES, AND SPHERES IN E^3

MARK D. MEYERSON*

1. **Introduction.** The main theorem in this paper is that in E^3 there is a tame 2-sphere with a Cantor subset such that all projections of the two sets are the same. There are several interesting related results. One corollary is that every tame knot class contains a knot which cannot be linked by a straight line. Compare this to the well-known result that every knot can be linked by a polygonal loop. Another implication solves a problem posed to the author by R. H. Bing, namely there does exist a simple closed curve in E^3 with Cantor subset such that all projections of the two sets are the same. The chief previous result in the area of this paper is due to K. Borsuk who showed, [4], that there is an arc whose projection in every plane has interior points.

2. **Terminology.** We say a k-sphere (or Cantor set) in E^n is tame if there is a homeomorphism of E^n onto itself which carries the k-sphere (or Cantor set) to a polyhedral k-sphere (or standard Cantor set — a subset of a line). A 2-sphere in E^3 is tame iff there is a homeomorphism of E^3 onto itself carrying the 2-sphere onto the standard unit sphere (by [1]). A Cantor set C in E^3 is tame iff for each $\epsilon > 0$, C can be covered by a finite collection of mutually exclusive open 3-cells of diameter less than ϵ (by [3, p. 435]). A knot and a straight line in E^3 will be called linked if they are disjoint and there is no isotopy of E^3 pushing the knot and line to different sides of some plane (this is stronger than some other possible definitions). A projection is defined geometrically as follows. For $X \subset E^3$, M a plane, the projection of X onto M consists of those points m of M such that the line through m perpendicular to M meets X.

3. **Main results.**

THEOREM. *In E^3 there exists a tame 2-sphere with a Cantor subset such that all projections of the two sets are the same.*

We postpone the proof of the theorem to the last section. First we summarize the idea of the proof and discuss some corollaries.

IDEA OF PROOF. Consider S_0, a P.L. 2-sphere, inside Δ_0, a 3-simplex. Any line meeting S_0 meets Int Δ_0. Triangulate Δ_0 into $\{\Delta_{1,i}\}$ and push out spikes on S_0 to get S_1 a P.L. 2-sphere which meets each Int $\Delta_{1,i}$. We can alter each $\Delta_{1,i}$ slightly so that any line meeting S_1 meets some Int $\Delta_{1,i}$. We shrink each $\Delta_{1,i}$ slightly so this still holds and now $\{\Delta_{1,i}\}$ are pairwise disjoint. Continuing we get a sequence of P.L. 2-spheres $\{S_n\}$ and P.L. 3-cells $\{\Delta_{n,i}\}$, so that any line meeting S_n meets some Int $\Delta_{n,i}$ and S_{n+1} is "pushed out" from S_n in $\bigcup_i$ Int $\Delta_{n,i}$. This can be done so that $C = \bigcap_n \bigcup_i \Delta_{n,i}$ is a Cantor set; so that S_n approaches a tame 2-sphere S (in a precise sense), $C \subset S$, and any line meeting S meets C.

COROLLARY 1. *There exists a simple closed curve in E^3 which cannot be linked by a straight line.*

PROOF. For $H : S^2 \to S$, S as in the theorem, H a homeomorphism, construct $K' \subset S^2$ a simple closed curve passing through the Cantor set $H^{-1}(C)$. Then $K = H(K')$ is the desired curve. For if l is a line disjoint from K then l is disjoint from S by the theorem. By the tameness of S, we can shrink S, and hence K, by an isotopy, toward a point inside S. This shows that l and K are not linked.

COROLLARY 2. (Bing's problem) *There exists a simple closed curve in E^3 with Cantor subset such that all projections of the two sets are the same.*

PROOF. Use K as in Corollary 1, $C \subset K \subset S$. Since all projections of C and S are the same, all projections of C and K will be the same.

COROLLARY 3. *Furthermore, in Corollaries 1 and 2, we can take K equivalent to any tame knot.*

PROOF. A small arc of K which misses C can be pulled inside S and knotted as desired there (the original K is unknotted since it lies in a tame 2-sphere). It may be easier to see that this can be done by performing the knotting on K' inside S^2 and then using H, where H now has been extended to all of E^3. Let π be an arbitrary projection. Since $\pi C = \pi S = \pi(S \cup \text{interior } S)$ and $C \subset K \subset S \cup \text{interior } S$, we immediately have $\pi C = \pi K = \pi S$. It is now clear that this knotted K is as desired.

COROLLARY 4. *Consider S^2 as a transparent material with an opaque equator loop. Then there exists an ambient isotopy H of E^3 so that there is darkness for anyone inside $H(S^2)$, even with light shining from infinity in all directions.*

PROOF. In the proof of the theorem, the homeomorphism H is actually the last stage of an isotopy. The arguments of the proof are easily adapted so that any line from infinity meets C before meeting $S - C$. In particular, in the proof of Lemma 1, if we use a mesh of $\epsilon/6$ instead of $\epsilon/4$ we see that a line from infinity meets P in (RT) before intersecting P^ϵ, length $(RT) = \min(\epsilon, \sigma)$. It follows as in the proof of Lemma 1 that the line will meet $\bigcup_n [(\Delta'_n)^\gamma]$ before P^ϵ, and the remainder of the proof is basically unchanged. Finally, it is well-known that there is an ambient isotopy taking S^2 to S^2 and any Cantor set in S^2 to a Cantor set on the equator, e.g., by methods similar to [2, pp. 307–310].

NOTE. These results can be intuitively stated in many ways: e.g., we could speak of shadows or lines of sight instead of projections.

4. **Preliminaries.** Everything is done in E^3.

DEFINITION. For $\epsilon > 0$, $X \subset E^3$, $X^\epsilon = \{x \in X : d(x, \partial X) \geqq \epsilon\}$. Equivalently, $X^\epsilon = \{x \in X : B_\epsilon(x) \subset X$, where $B_\epsilon(x) = \{y : |x - y| < \epsilon\}$. Note that X^ϵ is a closed subset of Int X.

LEMMA 1. *Given a polyhedron P, with triangulation into 3-simplices $\{\Delta_i\}_{i=1}^n$ and $\delta, \epsilon > 0$, there exists an isotopy F such that:*
  (i) *F is fixed on ∂P.*
  (ii) *F moves points by at most δ.*
  (iii) *Δ'_i, the final image of Δ_i under F, is a polyhedron.*
  (iv) *There is a $\gamma > 0$, such that if a line meets P^ϵ, then it meets* $\bigcup_i [(\Delta'_i)^\gamma]$.
Furthermore, if K is a compact subset of P which misses the 1-skeleton of $\{\Delta_i\}$, then we can assume that F is fixed on K.

PROOF. Let σ be the distance from K to the 1-skeleton of $\{\Delta_i\}$. Subdivide $\{\Delta_i\}$ into $\{\Gamma_j\}$ with mesh less than $\epsilon/4$ and $\sigma/7$. We can assume that if three vertices of $\{\Gamma_j\}$ determines a 2-face and also lie in ∂P, then the 2-face lies in ∂P (say by constructing $\{\Gamma_j\}$ using iterated barycentric subdivision). Let E be the set of vertices of $\{\Gamma_j\}$ and let K' be the closed $\sigma/7$ neighborhood of K. Then E can be shifted slightly to E' so that:
  (i) Vertices lying in $K' \cup \partial P$ don't move.
  (ii) By linear extension, the small shift $E \to E'$ defines an isotopy F taking $\Gamma_j \to \Gamma'_j$. We can assume E' is close enough to E so that points in (Int P) $- K'$ remain in (Int P) $- K'$, F is an isotopy moving points by at most δ, and the mesh of $\{\Gamma'_j\}$ is less than $\epsilon/4$ and $\sigma/7$ (A small distortion of a triangulation remains a triangulation by [6, p. 370]).

(iii) If E_1, E_2, and E_3 are disjoint unordered triples from $E - K'$, such that at least one point of each E_i is not in ∂P, then the 3 planes they determine (i.e., E_1 determines one plane, etc.) have exactly one point in common.

F induces isotopies $\Delta_i \to \Delta'_i$. Δ'_i is a polyhedron since it is the union of certain Γ'_j.

Suppose l is a line meeting P^ϵ. Then l contains an open segment $(RS) \subset \text{Int } P$ meeting P^ϵ and with R and S in ∂P.

If R is not in the 1-skeleton of $\{\Delta_i\}$, then R is a positive distance from all but one Δ_i. It then follows, since F is fixed on ∂P, that R is a positive distance from all but one Δ'_i, and hence l meets some Int Δ'_i.

If R is in the 1-skeleton of $\{\Delta_i\}$, then there is a segment $(RT) \subset (RS)$ so that $(RT) \cap K = \emptyset$ and length $(RT) = \min (2\epsilon, \sigma)$. (RT) must meet in segments at least 3 disjoint Γ'_j none of which meet K'. If (RT) met all three only in 2-faces, the faces would lie in Int P, which leads to a contradiction by the general position assumption of (iii) above. So (RT) meets some Int Γ'_j. Hence l meets some Int Δ'_i. (The above argument bears some resemblance to that of Borsuk in [4].)

On the compact space of lines meeting P^ϵ consider the continuous positive function which assigns to a line l the largest $\gamma > 0$ such that for some $x \in l$ and some n, $B_\gamma(x) \subset \Delta'_n$ (as done in [4, p. 274]). This function achieves a minimum $\gamma > 0$, and so if l meets P^ϵ it meets $\bigcup_n [(\Delta'_n)^\gamma]$. This completes the proof.

LEMMA 2. *If* $X \subset E^3$, $h_1 : X \to E^3$, $h_n : range\ (h_{n-1}) \to E^3$, *for* $n \geqq 1$ h_n *is a continuous function moving points by less than* ϵ_n *and* $\sum \epsilon_n < \infty$, *then* $h_n \circ h_{n-1} \circ \cdots \circ h_1$ *converges* (*uniformly*) *to a continuous function* h. *If* X *is closed, then* range $(h) = \{x \in E^3 : given\ \epsilon > 0,\ there\ exists\ N\ such\ that\ for\ n > N,\ dist\ (x,\ range\ (h_n)) < \epsilon\}$. (*Equivalently,* range $(h) = \lim \inf$ range (h_n).)

PROOF. Let $H_n = h_n \circ h_{n-1} \circ \cdots \circ h_1 : X \to E^3$. It is immediate that H_n converges to a continuous h which moves points by less than $\sum \epsilon_n$ ($\{H_n\}$ forms a Cauchy sequence in the complete space of continuous functions from X to E^3). We let $R(g) = $ range (g).

Clearly $R(h) \subset \lim \inf R(h_n)$. Let $x \in \lim \inf R(h_n)$. Then there exists $\{x_n\} \subset X$ such that $H_n(x_n) \to x$. $\{H_n(x_n)\}$ is a bounded sequence and each H_n moves points by less than $\sum \epsilon_i$, so $\{x_n\}$ is a bounded sequence. Hence there exists subsequence $x_{n_j} \to x_0$ for some x_0 in X (closed).

We show that $H_{n_j}(x_0) \to x$. Fix $a > 0$ arbitrarily. There exists N, such that for $m > N, h_m \circ \cdots \circ h_{N+1}$ moves points by less than $a/3$. Let

B be a compact set containing $\{x_n\}$. Then there is a $b > 0$, such that for $d(x, y) < b$ and $x, y \in B$, we have $d(H_N(x), H_N(y)) < a/3$. And so, for $x, y \in B$, $d(x, y) < b$, we have $d(H_m(x), H_m(y)) < a$. Since $x_{n_j} \to x_0$ in B and a was arbitrary, $d(H_{n_j}(x_{n_j}), H_{n_j}(x_0)) \to 0$. But $H_{n_j}(x_{n_j}) \to x$, so $H_{n_j}(x_0) \to x$.

Since $H_n(x_0) \to h(x_0)$, $x = h(x_0) \in R(h)$ and we are done.

LEMMA 3 (technical lemma). *Given a square in a horizontal plane, consider a right cylinder with the square as base and height at least double the length of a side of the square (see figure 1a). Then there exists a homeomorphism h of the plane onto the plane together with the surface of the cylinder, minus the original square's interior, such that:*

(i) *h is the identity outside the square.*

(ii) *h is a local dilation (i.e., any point in the domain of h has a neighborhood so that for x and y points in the neighborhood, $d(h(x), h(y)) \geqq d(x, y)$).*
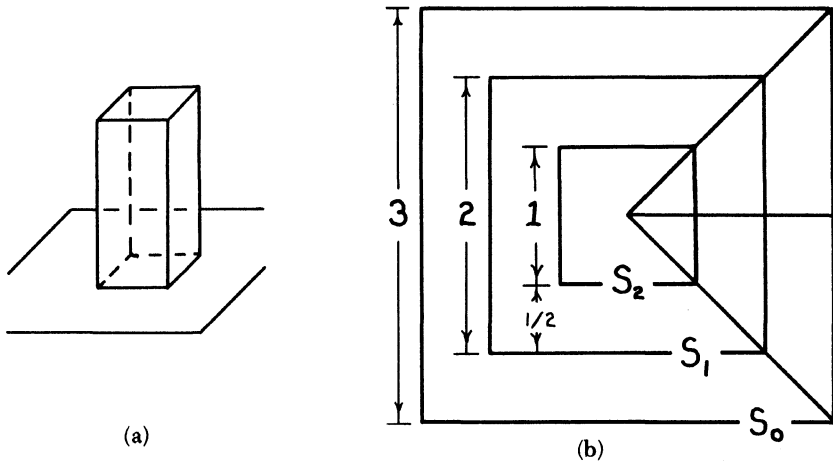

(a) (b)

Figure 1

(NOTE: Although the truth of this lemma may seem obvious, it is trickier than might be first supposed. To give some indication of the possible problems we note the following. There is no local dilation of an isoceles right triangle onto the square, two of whose sides are the legs of the triangle, such that the dilation map is the identity on the

two legs. *Proof*: Suppose there were such a map. It would not de-
crease the length of any arc. But the pre-image of the hypotenuse de-
creases in length, a contradiction. As another example, we note that
if the cylinder in the lemma were not a right cylinder, a homeomor-
phism of the type used here would fail to be a dilation.)

PROOF OF LEMMA 3. Let S_1 be the original square. We call a
square in the horizontal plane "standard" if it has the same center as
S_1 and has sides parallel to those of S_1. Assign the sides of S_1 length 2.
Let S_0 and S_2 be standard squares with sides of length 3 and 1 re-
spectively (see figure 1b). Define h to be the identity on and outside
S_1. Define h on S_2 and its interior by the linear map taking the corners
of S_2 to the corresponding corners of the top surface of the cylinder.
Note that for x and y on or inside S_2, $d(h(x), h(y)) = 2\, d(x, y)$.

Suppose S is a standard square inside the annular region bounded by
S_1 and S_2. Let t be the distance from S to S_1. Then we let h map S to
the horizontal square in the cylinder at a height of $f(t) = 2C\sqrt{t}$ from
the plane. The corners of S are mapped to the corresponding corners
of the image square and the sides are then mapped linearly. Choose
C so that $C\sqrt{2}$ equals the height of the cylinder: $C \geqq 2\sqrt{2}$. It's easy
to see h is a homeomorphism, so it remains to show that h is a local
dilation.

First we make some remarks. If x and y lie on the same standard
square then $h(x)$ and $h(y)$ are at least as far apart as x and y. Also,
$f'(t) = C/\sqrt{t}$. Now consider a segment of length a, lying in the
annular region bounded by S_1 and S_2, and lying on a line through the
center of S_1 (a radial segment). Then h maps the segment to a vertical
segment. Since the endpoints of the radial segment are on squares a
distance of at least $a/\sqrt{2}$ apart (say at distances s and t from S_1), the
image segment has length $a' = |f(t) - f(s)| = f'(w)|t - s| \geqq$
$f'(w)a/\sqrt{2}$ for some w between s and t. For $t \geqq w$, $a' \geqq Ca/\sqrt{2w} \geqq$
$2a/\sqrt{t}$, and we see the image segment has length $a' \geqq 2a$. We gen-
erally will use the prime ($'$) symbol to indicate image points or image
lengths. For x and y in the domain of h, we let $d = \text{dist}(x, y)$ and $d' =$
$\text{dist}(h(x), h(y))$.

Now we show that h is a local dilation in the annular region
bounded by S_1 and S_2. There are two cases to consider. First assume
x and y are in the same one of the four trapezoidal subregions formed
by the diagonals of S_1 (see the lower part of figure 2a). We allow x or
y to lie on the boundary of this region, but can assume $s < t$, where
$y(x)$ is on the standard square $S_s(S_t)$ which is $s(t)$ units from S_1. Con-
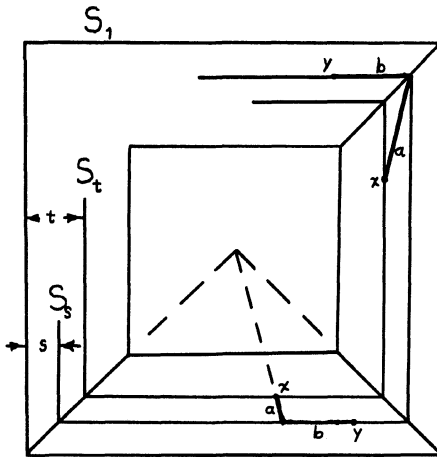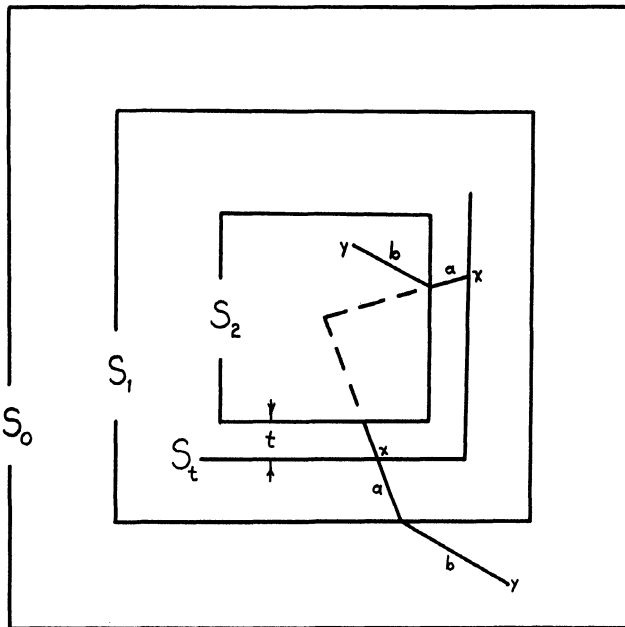struct the radial segment from x to S_s, call its length a. Let $b(\geqq 0)$

Figure 2a

Figure 2b

be the length of the segment in S_s completing the path from x to y (refer again to the lower part of figure 2a). Facts at our disposal include: $t - s \leqq a$, $0 \leqq s < t \leqq 1/2$, $1/(1 - s)^2 \geqq 1 + 2s$ (a true inequality for $s < 1$), $a' \geqq 2a/\sqrt{t} \geqq a$, $b' = b/(1 - s) \geqq b$. We can assume $b \leqq 1$ (if $b > 1$, then $d^2 \leqq a^2 + b^2 \leqq (a')^2 + (b')^2 = (d')^2$). Also $d^2 \leqq a^2 + b^2 + ab\sqrt{2}$ (law of cosines), $(d')^2 = (a')^2 + (b')^2$. If $t \leqq 2a/b$, then $(d')^2 \geqq (4a^2/t) + b^2 = (a^2/2t) + b^2 + (7a^2/2t) \geqq a^2 + b^2 + (7ab/4) \geqq d^2$. If $t \geqq 2a/b$, then $s \geqq t - a \geqq (2a - ab)/b \geqq a/b$, and $(d')^2 \geqq a^2 + (b^2/(1 - s)^2) \geqq a^2 + b^2 + 2sb^2 \geqq a^2 + b^2 + 2ab \geqq d^2$. (If $b = 0$, $d' = a' \geqq a = d$). Hence $(d')^2 \geqq d^2$.

Now assume x and y are in adjacent trapezoidal regions. As in the upper part of figure 2a, let x be in the smaller standard square, a be the length of the segment from x to the corner of S_s shared by both trapezoids, and b be the length of the segment from the corner to y. We already have shown that $a' \geqq a$, $b' \geqq b$, and since the angle between the segments of length a and b is less than $\pi/2$, we have $d^2 \leqq a^2 + b^2 \leqq (a')^2 + (b')^2 = (d')^2$. (Note: if a is the length of the segment from x to z, $a' = d(x', z')$. Hence we don't have to concern ourselves with whether the images of such segments are segments.)

It remains to look at neighborhoods of points of S_1 and S_2. We use neighborhoods of radius $1/2$. We can assume x ($\in S_t$ as before) is in the annular region bounded by S_1 and S_2, and y is outside that region. Let a be the length of the radial segment from x to S_i ($i = 1$ or 2, whichever square separates x from y), and b the length of the segment from there to y (see figure 2b). So (since $b \leqq 2$ here) $d^2 \leqq a^2 + b^2 + 2ab \leqq a^2 + b^2 + 4a$. If we are in a neighborhood of S_1, $b' = b$, and since $t \geqq a/\sqrt{2}$, $(a')^2 = f(t)^2 = 4C^2t \geqq 8a \geqq 7a + a^2$ ($a \leqq 1$). So $(d')^2 = (a')^2 + (b')^2 \geqq a^2 + b^2 + 7a \geqq d^2$. On the other hand, if we are in a neighborhood of S_2, we have $b' = 2b$, $a' \geqq 2a$ (by an earlier remark), so $(d')^2 \geqq a^2 + b^2 + 3(a^2 + b^2) \geqq a^2 + b^2 + 6ab \geqq d^2$. Note that these calculations hold even if one of the points is on S_i ($i = 1$ or 2). The proof is complete.

5. Proof of theorem. We repeat:

THEOREM. *In E^3 there exists a tame 2-sphere with a Cantor subset such that all projections of the two sets are the same.*

PROOF. (a) Let $\Delta_{0,1}$ be a 3-simplex and $S_0 \subset \text{Int } \Delta_{0,1}$ a P.L. 2-sphere.

(b) By induction on $n = 0, 1, 2, \cdots$ (steps c–h) we define polyhedral 3-cells $\{\Delta''_{n,i}\}$ (and auxiliary 3-cells $\{\Delta_{n,i}\}$, $\{\Delta'_{n,i}\}$), a P.L. 2-sphere S_n, a homeomorphism $h_n : S_{n-1} \to S_n$, and $\epsilon_n > 0$ such that:

(i) S_n meets each Int $\Delta''_{n,i}$.

(ii) For some $\delta(n) > 0 : h_n(x) \neq x$ only if $x \in (\Delta''_{n-1,i})^\delta$ for some i, and then $h_n(x) \in (\Delta''_{n-1,i})^\delta$.

(iii) h_n is a (uniform) local dilation (i.e., there exists $\gamma(n) > 0$ such that $d(x, y) < \gamma$ implies $d(x, y) \leqq d(h_n(x), h_n(y)))$.

(iv) For some $\beta(n) > 0$, any line meeting S_n meets $(\bigcup_i \Delta''_{n,i})^\beta$.

(v) diam $\Delta''_{n,i} < \epsilon_n$, $\{\Delta''_{n,i}\}$ are pairwise disjoint.

Furthermore, we will make sure that for $n > 0$,

(vi) $\Delta''_{n,i} \subset \Delta''_{n-1,j}$ for some j; $\Delta''_{n-1,j}$ contains at least two $\Delta''_{n,i}$; and $d(h_{n-1}(x), h_{n-1}(y)) < d(x, y)$ implies $d(h_{n-1}(x), h_{n-1}(y)) > 2\epsilon_n$.

(c) Fix $n > 0$ and assume the above for $n - 1$ (i–v hold for $n = 0$ trivially, if we define $S_{-1} = S_0$, $h_0 = id$, $\Delta''_{0,1} = \Delta_{0,1}$, etc.). Choose $0 < \epsilon_n < 1/2^n$ small enough so that $\epsilon_n < \min_i \text{diam}(\Delta''_{n-1,i})$ and so that if $d(h_{n-1}(x), h_{n-1}(y)) < d(x, y)$ then $d(h_{n-1}(x), h_{n-1}(y)) > 2\epsilon_n$. This latter requirement can be fulfilled as follows: by iii there is a $\gamma > 0$ such that $d(h_{n-1}(x), h_{n-1}(y)) < d(x, y)$ implies $d(x, y) > \gamma$. Cover S_{n-1} by finitely many compact $\gamma/3$ neighborhoods $N_1, \cdots, N_k$. If $d(h_{n-1}(x), h_{n-1}(y)) < d(x, y)$, then x and y are in disjoint N_i and N_j. But h_{n-1} is a homeomorphism, so $h_{n-1}(x)$ and $h_{n-1}(y)$ are in disjoint $h_{n-1}(N_i)$ and $h_{n-1}(N_j)$. Let $2\epsilon_n < \min(\text{dist}(h_{n-1}N_i, h_{n-1}N_j) : N_i$ and N_j are disjoint).

(d) Triangulate $\bigcup_i \Delta''_{n-1,i}$ into $\{\Delta_{n,i}\}$, with mesh less than ϵ_n and in general position. Note that since $\{\Delta''_{n-1,i}\}$ are pairwise disjoint, this amounts to triangulating each $\Delta''_{n-1,i}$ separately. The general position used here is with respect to S_{n-1} inside each $\Delta''_{n-1,i}$. Specifically, $\partial \Delta_{n,i} \cap S_{n-1} \cap \text{Int} \bigcup \Delta''_{n-1,j}$ is empty or 1-dimensional, and vertices of $\Delta_{n,i}$ inside Int $\bigcup \Delta''_{n-1,j}$ are not in S_{n-1}. This can be accomplished in the usual method of first taking a small triangulation, and then adjusting vertices in Int $\Delta''_{n-1,j}$ as necessary (still a triangulation by [6, p. 370]).

(e) Within each $\Delta''_{n-1,i}$ we wish to "push out" S_{n-1} to meet each $\Delta_{n,j}$ in $\Delta''_{n-1,i}$. We proceed by constructing $\{p_m\}$, a finite collection of finite polygonal trees sprouting from S_{n-1} into $\{\text{Int} \Delta_{n,j}\}$. $\{p_m\}$ is constructed by taking for the start of each p_m polygonal arcs from S_{n-1} which branch off into those $\Delta_{n,j}$ which don't meet S_{n-1} but are adjacent (by a 2-face) to a $\Delta_{n,j}$ which meets S_{n-1}. Then we consider $\Delta_{n,k}$ adjacent to those and so on. This process is used so that we can insure that $p = \bigcup p_m$ meets $\Delta_{n,i}$ in a single connected set (in fact a polygonal tree) for those $\Delta_{n,i}$ which don't meet S_{n-1}. Further, it's easy to see from our construction that we can take:

(i) $\{p_m\}$ pairwise disjoint.

(ii) $p \subset \text{Int } \Delta''_{n-1,i}$.

(iii) p misses the 1-skeleton of $\{\Delta_{n,i}\}$.

(iv) The "vertices" of p lie in $\bigcup \text{Int } \Delta_{n,i}$.

(v) $p_m \cap S_{n-1} =$ a single point lying in the interior of a 2-face of S_{n-1}.

(vi) The segment of p_m meeting S_{n-1} is perpendicular to that face.

(vii) Any two segments of p_m which meet are perpendicular, and each vertex of p_m lies in at most two segments of p_m (perhaps in the interior of one of the segments).
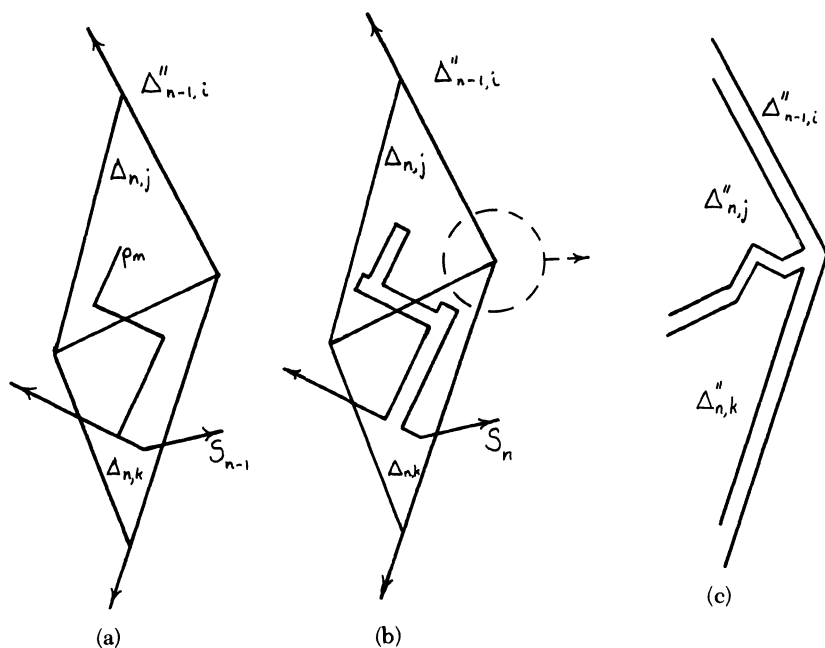

Figure 3

(See the 2-dimensional schematic in figure 3a.)

(f) The construction in (e) makes it easy to "push out" S_{n-1} — a small square patch around $p_m \cap S_{n-1}$ is pushed out so as to closely drape p_m. More precisely, by Lemma 3, a small square patch in S_{n-1} can be mapped to the sides and top of a cylinder which approximates the segment of p_m meeting S_{n-1}. The map is a local dilation of S_{n-1} (it is the identity outside the patch). Proceeding inductively on the rest of p_m we see that after a finite number of steps we have defined

h_n, a homeomorphism and local dilation with domain S_{n-1}, so that $S_n = h_n(S_{n-1})$ approximates $S_{n-1} \cup \bigcup p_m$ as close as we want. We can, in fact, insure that the set of points moved by h_n, and the image under h_n of this set, are some positive distance from $\partial \Delta''_{n-1,i}$. It follows that h_n satisfies (b ii). Since S_n meets each Int $\Delta_{n,j}$ in $\Delta''_{n-1,i}$, we have "almost" satisfied (b i). And h_n satisfies (b iii) since a composition of continuous local dilations is a local dilation (uniformity holds since the domain of h_n is compact). h_n is actually the last stage of an ambient isotopy (restricted to S_{n-1}) which moves points within a compact subset of $\bigcup$ Int $\Delta''_{n-1,i}$ (this will be proven later). Note further that if $x \neq h_n(x)$ and h_n moves x from outside $\Delta_{n,j}$ into $\Delta_{n,j}$, then $S_n \cap \Delta_{n,j} \subset h_n(\Delta_{n,i} \cap S_{n-1})$ for some i.

(g) By (b iv), any line meeting S_{n-1} meets $(\bigcup_i \Delta''_{n-1,i})^\beta$ for some $\beta > 0$. Since $h_n : S_{n-1} \to S_n$ only moves points within a compact subset of $\bigcup$ Int $\Delta''_{n-1,i}$, any line meeting S_n meets $(\bigcup_i \Delta''_{n-1,i})^\mu$ for some $\mu > 0$. So by Lemma 1, we can alter $\Delta_{n,i}$ arbitrarily slightly to $\Delta'_{n,i}$ (as in the lemma), another subdivision, so that any line meeting S_n meets $\bigcup (\Delta'_{n,i})^\gamma$ for some $\gamma > 0$ and diam$(\Delta'_{n,i}) < \epsilon_n$. In $\Delta_{n,i} \to \Delta'_{n,i}$ we can make sure points move by less than $\epsilon_n/4$. Further, by Lemma 1, since $\{\Delta_{n,i}\}$ are in general position with respect to the cylindrical neighborhoods of $p(= K$ in the lemma), we can make sure that for $x \neq h_n(x)$, $h_n(x) \in \Delta_{n,i}$ iff $h_n(x) \in \Delta'_{n,i}$.

(h) "Shrink" each $\Delta'_{n,i}$ slightly to $\Delta''_{n,i}$ so that $(\Delta'_{n,i})^\gamma \subset$ Int $\Delta''_{n,i}$ $\subset \Delta''_{n,i} \subset$ Int $\Delta'_{n,i}$ and S_n still meets each $\Delta''_{n,i}$ with general position (e.g., $\Delta'_{n,i}$ is P.L. homeomorphic to the standard 3-cell, so we just take a slightly smaller 3-cell). The $\{\Delta''_{n,i}\}$ are 3-cells and we can thus satisfy (b i–v). Only left is (vi), but this was satisfied in (c) by our choice of ϵ_n. In $\Delta_{n,i} \to \Delta'_{n,i} \to \Delta''_{n,i}$ we can make sure points move by less than $\epsilon_n/2$.

(i) We have now satisfied (b). Let $C = \bigcap_n \bigcup_i \Delta''_{n,i}$, a tame Cantor set by the preliminary remarks. Here we use the fact that each $\Delta''_{n-1,j}$ contains at least two $\Delta''_{n,i}$. Let $\Sigma = \bigcup_n (S_n - \bigcup_i \Delta''_{n,i})$ and $S = \Sigma \cup C$.

(j) S is a 2-sphere: $h_n : S_{n-1} \to E^3$. By (b ii), $d(x, h_n(x)) <$ max diam$(\Delta''_{n-1,i}) < \epsilon_{n-1} < 1/2^{n-1}$ (or $< \epsilon_0$ for $n = 1$), so by Lemma 2, $h_n \circ h_{n-1} \circ \cdots \circ h_1 \to h$ a continuous function.

Also by Lemma 2, $R(h) = S$: By (b ii), h_{n+1} is fixed on points of S_n in $\sim \bigcup_i \Delta''_{n,i}$. So $S_{n+1} \supset S_n - \bigcup_i \Delta''_{n,i}$. Since $\bigcup_i \Delta''_{n,i}$ decreases in n (by (b vi)), $S_{n+1} - \bigcup \Delta''_{n+1,i} \supset S_n - \bigcup \Delta''_{n,i}$. So $\Sigma = \bigcup_n (S_n - \bigcup \Delta''_{n,i})$ is an increasing union. If $x \in \Sigma$, then $x \in S_n$ for all n greater than some N. If $x \in C$, then for $1/2^n < \epsilon$, $\epsilon_n < \epsilon$ and $x \in \Delta''_{n,i}$ for some i, but S_n meets $\Delta''_{n,i}$, so dist$(x, R(h_{n-1})) < \epsilon_n < \epsilon$. Thus by the

lemma, $\Sigma \cup C \subset R(h)$. If $x \notin \Sigma \cap C$, then $x \notin \bigcup \Delta''_{n,i}$ for some n, and so $x \notin S_n \cup \bigcup \Delta''_{n,i}$. But for $m > n$, $S_m \subset S_n \cup \bigcup_i \Delta''_{n,i}$ (iterating (b ii) and (b vi) and so x is a positive distance from $R(h_m)$. Hence $x \notin R(h)$. So $S = \Sigma \cup C = R(h)$.

Therefore $h : S_0 \to S$ is a continuous surjection. To complete (j) we show h is 1-1. Suppose $x \neq y$, $h(x) = h(y)$. Then for some $n > 0$, $d(h_n(x'), h_n(y')) < d(x', y')$ where $x' = h_{n-1} \circ \cdots \circ h_1(x)$ and similarly for y'. By (b vi), $d(h_n(x'), h_n(y')) > 2\epsilon_{n+1}$ and so $h_n(x')$ and $h_n(y')$ are in different $\Delta_{n+1,i}$. But $h(x) = h(y)$ implies $h_{n+1}(h_n(x'))$ and $h_{n+1}(h_n(y'))$ are in the same $\Delta''_{n+1,i}$ (otherwise we can find an $r > 0$ such that for $m > n$, $h_m \circ \cdots \circ h_1(x)$ and $h_m \circ \cdots \circ h_1(y)$ are greater than r apart). In $\Delta_{n+1,i} \to \Delta''_{n+1,i}$ points move by less than $\epsilon_{n+1}/2$. But $\text{diam}(\Delta''_{n+1,i}) < \epsilon_{n+1}$, $d(h_n(x'), h_n(y')) > 2\epsilon_{n+1}$. So we must have that h_{n+1} moves one of the points (say $h_n(x')$) into $\Delta''_{n+1,i}$ from outside $\Delta''_{n+1,i}$. Then (by construction in f and g) $S_{n+1} \cap \Delta''_{n+1,i}$ consists of points pushed into $\Delta_{n+1,i}$ from some single $\Delta_{n+1,j}$. So $h_n(x')$ and $h_n(y')$ are in $\Delta_{n+1,j}$, a contradiction.

Hence $h : S_0 \to S$ is a homeomorphism.

(k) If a line meets S, it meets C, because: $1 \cap S \neq \emptyset$ implies $1 \cap C \neq \emptyset$ or $1 \cap (S_n - \bigcup_i \Delta''_{n,i}) \neq \emptyset$ for some n. We assume the latter. But by (j), $S_m \supset S_n - \bigcup \Delta''_{n,i}$ for all $m \geq n$. So 1 meets $\bigcup \Delta''_{m,i}$ for all $m \geq n$ by (b iv). $\{\bigcup_i \Delta''_{m,i}\}$ are compact and decrease to C, so by the usual argument, 1 meets C.

(l) We have now proven that there exists a tame Cantor set on a 2-sphere with the projection property of the theorem. It remains to show that the 2-sphere can be taken tame. In effect we redo the same construction with two modifications: 1. the previous construction used arbitrary triangulation (of small enough mesh) and we will now be more specific in our choice of triangulation, 2. we will be more careful in our choice of the segments forming $\{p_m\}$.

(m) Let $U_m = \{y \in E^3 : \text{dist}(y, S_0) > 1/m\}$. Going back to (e) and (f) we see that h_n was the last stage of an ambient isotopy restricted to S_{n-1} (Pushing a plane region over a right cylinder as was done in (e) can be extended to an isotopy on E^3 which is fixed outside a neighborhood of the solid cylinder. Just take a slightly larger cylinder and extend linearly "as necessary" from a point just below the center of the original base square.). Call the last stage of this isotopy H_n. Since h_n only "moves" points inside a compact subset of $\bigcup \Delta''_{n-1,i}$, we can insure that:

(1) H_n is the identity outside a compact subset of $\bigcup \Delta''_{n-1,i}$.

(2) $H_n(\Delta''_{n-1,i}) \subset \Delta''_{n-1,i}$.

(3) $H_n |_{S_{n-1}} = h_n$.

Also since $\Delta_{n,i} \to \Delta'_{n,i}$ was taken to move points away from the cylindrical neighborhoods of p_m, we can assure H_n is fixed on points moved by $\Delta_{n,i} \to \Delta'_{n,i}$.

(n) We will construct a homeomorphism H'_n with properties 2 and 3 of H_n and: $(1')$ H'_n is the identity outside $\bigcup \Delta''_{n-1,i}$. Also, we will have $H'_n \circ H'_{n-1} \circ \cdots \circ H'_1 (U_n) \subset \sim\bigcup \Delta''_{n,j}$. Now $U'_n = H'_{n-1} \circ \cdots \circ H'_1(U_n)$ is a positive distance from S_n. We will define $H'_n = H_n \circ F' \circ f$, where $F' \circ f$ pushes U'_n into a part of $\sim\bigcup \Delta''_{n,j}$ which is unaffected by H_n.

(o) f will be fixed on S_n and outside $\bigcup \Delta''_{n,i}$, so it suffices to look at one component of $\Delta''_{n,i} - S_n$. Let R be the closure of one such component. $\partial R = (R \cap S_n) \cup (R \cap \partial\Delta''_{n,i})$, neither of these two sets being empty. Remember S_n is in general position with respect to $\Delta''_{n,i}$. Let $A_1, \cdots, A_k$ be the components of $A = R \cap \partial\Delta''_{n,i}$. We show there exists $K \subset R - S_n$, a (at most) 2-dimensional polyhedron with the following property: given $\delta > 0$, there exists (the last stage of) an isotopy $g : R \to R$, g fixed on ∂R, and for $x \in R$, $d(x, S_n) > \delta$ we have $d(g(x), K \cup A) < \delta$.

(p) To get the K desired above, it suffices to show that for some triangulation we can collapse R to $K \cup A$ (rel $K \cup A$), with K as desired. This is because the collapse can be used to define an isotopy — i.e., for N a neighborhood of S_n, each collapse "pulls" $N \cap \Delta''_{n,i}$ along a little, so that in the end $N \cap \Delta''_{n,i}$ fills up all of R outside a neighborhood of $K \cup A$. In particular, we can use δ-neighborhoods so that g is as desired.

Hence it suffices to show we can collapse a triangulation of R as desired. Let M be a triangulation of R containing a subdivision of $A(\subset \partial R)$ as a subcomplex. It's clear that we can collapse M to some 2-dimensional $K^* \cup A^*$ (A^* the triangulation of A, collapse rel $K^* \cup A^*$) since each point of Int R is accessible by a general position polygonal arc from $\partial R - A$. Hence if we were unable to collapse all 3-simplices such an arc could hit some first uncollapsed 3-simplex on a face, and we could collapse it. The difficulty is that we want $K^* \subset R - S_n$.

But with the correct subdivision M^* of M, for $N = \{$simplices of M "meeting" $\partial R - A\}$ we can geometrically collapse M^* to $M^* - N$. Namely, for $\Sigma = N \cap \dot{M}$, we let $M^* = S_\Sigma^2 M$. This is defined and proven to work by Glaser [5, pp. 70–76] in his discussion of Whitehead's regular neighborhood theorem. Especially note Glaser's remark on p. 74 that for such a subdivision one can collapse a combinatorial n-manifold away from the boundary. Our result is completely analogous. And so we can, with the proper triangulation, collapse

R to $K \cup A$ (rel $K \cup A$), with $K \subset$ Int R and K 2-dimensional ($K = |K^*|, A = |A^*|$).

Thus g is defined on each such R, and f is hence defined as the union of these g's, and the identity outside $\cup \Delta''_{n,i}$.

(q) F is the ambient isotopy $\Delta_{n,j} \to \Delta'_{n,j}$. We summarize our final construction:

1. Assume we've constructed S_{n-1}, $\{\Delta''_{n-1,i}\}$.

2. Construct the 2-dimensional polyhedron K ($= K(n-1)$). The closure of each component of $\Delta''_{n-1,i} - S_{n-1}$ collapses into $K \cup \partial\Delta''_{n-1,i}$ in the component.

3. Triangulate each $\Delta''_{n-1,i}$ into $\{\Delta_{n,j}\}$ with mesh less than ϵ_n and less than $d(K, S_{n-1})/3$, and in general position with respect to S_{n-1} (as described in d). But further, we can assume a subdivision of K is contained as a subcomplex (this does not conflict with general position since K is a positive distance from S_{n-1}).

4. The collapse of each component of $\Delta''_{n-1,i} - S_{n-1}$ into part of $K \cup \partial\Delta''_{n-1,i}$ shows we can define $\{p_m\}$ from S_{n-1} to any point of Int $\Delta''_{n-1,i} - K$; in particular to each Int $\Delta_{n,j}$ with p_m disjoint from K and missing the edges of $\{\Delta_{n,j}\}$ (and $\{p_m\}$ still fulfill the conditions of (e)).

5. Define $h_n : S_{n-1} \to S_n$ using $\{p_m\}$ as in (f), but not affecting any point of K, $\partial\Delta''_{n-1,i}$ or the edges of $\{\Delta_{n,j}\}$. Define the natural extension H_n, fixed on some neighborhood of K and the edges of $\{\Delta_{n,j}\}$, and outside $\cup \Delta''_{n-1,i}$.

6. Define $\Delta_{n,j} \to \Delta'_{n,j}$ as before, by an ambient isotopy F. By Lemma 1, we can assume F doesn't move points of K outside the above (in 5) mentioned neighborhood of K, so that H_n is fixed on $F(K)$. Consider the collection of vertices of those $\Delta_{n,j}$ which meet K. The shift $E \to E'$ restricted to this set defines an isotopy F', by leaving all other vertices fixed and extending linearly (We implicitly assume here that the original shift was sufficiently small so that any such "subshift" would define an isotopy). By (q) 3 above, F' is fixed on S_{n-1}. Let $K' = F'(K) = F(K)$.

7. Define $\Delta'_{n,j} \to \Delta''_{n,j}$, a homeomorphism shrinking each $\Delta'_{n,j}$. Hence $K'' \subset$ 2-skeleton of $\{\Delta'_{n,j}\}$, is outside each $\Delta''_{n,j}$.

8. Let $U'_n = H'_{n-1} \circ \cdots \circ H'_1(U_n)$, $\sim U'_n$ is a neighborhood of S_{n-1}.

9. The collapse of each component of $\Delta''_{n-1,i} - S_{n-1}$ into $K \cup \partial\Delta''_{n-1,i}$ shows how to define f: f pushes $\sim U'_n$ out to engulf all of $\Delta''_{n-1,i}$ except a small neighborhood of $K \cup \partial\Delta''_{n-1,i}$ (Note that f is a homeomorphism fixed on $S_{n-1} \cup \partial\Delta''_{n-1,i}$). F and F' make this small neighborhood a neighborhood of $K'' \cup \partial\Delta''_{n-1,i}$. We choose

f to make the neighborhood small enough so that for $H'_n = H_n \circ F' \circ f$, $H'_n(U'_n) \cap \Delta''_{n-1,i}$ is very close to $K'' \cup \partial\Delta''_{n-1,i}$, namely $H'_n \circ H'_{n-1} \circ \cdots \circ H'_1(U_n)$ is outside $\bigcup\Delta''_{n,j}$ (H_n is fixed on $K'' \cup \partial\Delta''_{n-1,i}$).

10. Note that f, F', H_n, and hence H'_n are fixed outside $\bigcup\Delta''_{n-1,i}$. Since f and F' are each the identity on S_{n-1}, we have $H'_n|_{S_{n-1}} = H_n|_{S_{n-1}} = h_n$.

(r) H'_n only moves points within each $\Delta''_{n-1,i}$, so it moves points by less than $\epsilon_{n-1} < 1/2^{n-1}$. Hence by Lemma 2, $H'_n \circ H'_{n-1} \circ \cdots \circ H'_1$ converges to a continuous function H. Since H is the identity outside $\Delta_{0,1}$, $R(H) = E^3$ and H is a homeomorphism if it is 1-1. Suppose $H(x) = H(y)$.

Case 1: $x, y \in S_0$. Then $h(x) = H(x) = H(y) = h(y)$. By (j) h is 1-1, so $x = y$.

Case 2: $x, y \notin S_0$. Then there exists n such that $x, y \in U_n$. $H'_n \circ \cdots \circ H'_1(U_n) \subset \sim\bigcup\Delta''_{n,j}$. For $m > n$, H'_m is fixed outside $\bigcup\Delta''_{m-1,i} \subset \bigcup\Delta''_{n,j}$. So H is 1-1 on U_n and $x = y$.

Case 3: $x \in S_0, y \notin S_0$ (and vice versa). Then $y \in U_n$ for some n. But as in case 2, $H(y) \notin \bigcup\Delta''_{n,j} \cup S_n$. But $H(x) \in S \subset \bigcup\Delta''_{n,j} \cap S_n$. So this case leads to a contradiction.

(s) Hence H is a homeomorphism of E^3, $H(S_0) = h(S_0) = S$. In fact, H is the last stage of an isotopy. So S is tame. //

Further Corollaries:

COROLLARY 5. *We can construct S as in the theorem but with the unit ball inside S and the exterior of the $(1 + \epsilon)$-ball containing S. In fact, starting with $S_0 \subset \text{Int } P$, P any 3-dimensional polyhedron, we can find S as in the theorem so that $S \subset P$ and for $x \notin P$, x is inside S iff x is inside S_0.*

PROOF. Just start with $\{\Delta_{0,i}\}$ a triangulation of P. The above construction then does the job.

COROLLARY 6. The theorem provides an alternate construction of a result due to Bing ([3, Theorem 6.3, p. 443]): *There is a countable collection of tame Cantor sets in E^3 such that each straight line interval in E^3 intersects one of these Cantor sets.*

PROOF. By Corollary 5, there is a sequence S_i of spheres with the property of the theorem such that the origin is inside each S_i and diam $S_i \to 0$. Consider translations of E^3 which take the origin to elements of a countable dense set. Any line interval will meet at least one of the translated S_i's in the same non-empty set as the line containing the interval. Hence the interval will meet the corresponding Cantor set.

COROLLARY 7. *The theorem holds in E^2.*

PROOF. The construction can be so adapted. But there is an easier construction. Let $C \subset [0, 1]$ be the Cantor set formed by "deleting middle tenths". Then any line at an angle greater than $\tan^{-1}(1/10)$ with the horizontal and vertical and meeting R' also meets $C' = C \times C \subset R' = [0, 1] \times [0, 1]$. Rotate the plane $45°$ about $(1/2, 1/2)$ with function r. Then we check that $C' \cup r(C')$ is a Cantor set with the same projections as $R' \cup r(R')$. We can thread $C' \cup r(C')$ with a simple closed curve inside $R' \cup r(R')$ using two methods due to Antoine. It suffices to show that for any Cantor set, K, lying in a square, $R' (= [0, 1] \times [0, 1])$, there is a simple closed curve in R' containing K. K can be covered by a finite collection of arbitrarily small disjoint polygons and their interiors (lying in the plane [4, p. 298]). We can assume the polygons lie in R' by intersecting them with R'. If any of these polygons separate R', we can break them up into smaller disjoint polygons which don't separate R', by a theorem of Antoine's ([4, p. 296]). Now these new polygons can be connected by polygonal arcs in R' as a "first approximation" of a simple closed curve, e.g., by first using segments of $\partial R'$ to cyclically connect those polygons meeting $\partial R'$ (leaving the first and last unconnected to one another) and then using arcs in Int R' to connect the other polygons in the appropriate fashion. Each polygon and interior is homeomorphic to a square and interior, and a similar method is used inductively. In the limit we get a simple closed curve (similar to [4, p. 302-4]) as desired.

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STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305