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A CATEGORICAL APPROACH TO THE CLOSED GRAPH THEOREM

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0. **Introduction.** The purpose of this note is to explore once again, this time from a "natural" or categorical point of view, the class of spaces which act as targets for the closed graph theorem with respect to maps from barrelled spaces. This problem has provoked over the years a series of articles in which successively broader classes of spaces are proposed as candidates for this role. Chiefly, there are the B_r-complete (= *Pták* = weakly polar) spaces of Pták [7], the slightly more general weakly t-polar spaces of Persson [6], and the still more general infra-s-spaces of Adasch [1]. The original definitions of these three classes of spaces are very similar, each being a refinement of the preceding. In the case of infra-s-spaces, however, a best-possible result is obtained — a characterization of precisely those spaces which are targets of the stated closed graph theorem.

Our approach to this well-known problem is categorical. Using universal constructions, we characterize what is meant for a space to satisfy the closed graph theorem with respect to barrelled spaces. This setting elucidates many previous results in the literature and simplifies their proofs. We are able to derive, for example, Adasch's characterization of infra-s-spaces from our criteria and some relations between infra-s-spaces and B_r-complete spaces. The resulting clarity and simplified proofs should, we hope, justify the approach and encourage its use elsewhere.

1. **Preliminaries.** We shall assume the terms *category* and *functor* as known. We denote the set of morphisms from A to B in the category $\mathcal{A}$ by $\mathcal{A}(A, B)$. We now state some of the basic categorical notions which are necessary in this note, a reference for which is [5].

Let $F : \mathcal{A} \to \mathcal{X}$ be a functor. The functor $U : \mathcal{X} \to \mathcal{A}$ is said to be the *right adjoint* of F (F the left adjoint of U) if for every $X \in \mathcal{X}$ there is a map in $\mathcal{X}$, $\epsilon_X : FUX \to X$, such that for every map $f \in \mathcal{X}(FA, X)$, $A \in \mathcal{A}$, there exists a unique map $\bar{f} \in \mathcal{A}(A, UX)$ such that $\epsilon_X \circ F\bar{f} = f$. The important properties of adjoints used in this note are: (1) adjoints are unique up to isomorphism; (2) a right (left) adjoint preserves inverse (direct) limits. A map $f \in \mathcal{A}(A, B)$ is an *epimorphism* if

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for every pair of maps $u, v \in \mathcal{A}(B, C)$ such that $u \circ f = v \circ f$, we have $u = v$. Given a diagram in $\mathcal{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

the *pushout* of this diagram is an object $P \in \mathcal{A}$ together with maps $f' : C \to P$, $g' : B \to P$ such that $g' \circ f = f' \circ g$ and universal with respect to this property, i.e., if $Q \in \mathcal{A}$ and $f'' : C \to Q$, $g'' : B \to Q$ are such that $g'' \circ f = f'' \circ g$, then there is a unique map $p : P \to Q$ with $p \circ g' = g''$, $p \circ f' = f''$. Again, pushouts are unique up to isomorphism. Furthermore, if $f(g)$ is an epimorphism, then so is $f'(g')$.

Let $\mathcal{L}$ denote the category of locally convex (Hausdorff) topological vector spaces and continuous linear maps. A linear map $f: X \to Y$, $X, Y \in \mathcal{L}$, is said to be *open* if $f(U)$ is open in Y for every open subset $U \subset X$; f is *almost open* if $\overline{f(U)}$ is a 0-neighborhood in Y whenever U is a 0-neighborhood in X; f is said to have a *closed graph* if the set $\{(x, f(x)) : x \in X\}$ is closed in the product $X \times Y$; f is said to be *almost continuous* if for every 0-neighborhood V of Y, $\overline{f^{-1}(V)}$ is a 0-neighborhood in X.

We shall be concerned not only with the category $\mathcal{L}$ but also with its full subcategory $\mathcal{B}$ of barrelled spaces. A space B is said to be *barrelled* if every continuous linear map from $X \in \mathcal{L}$ onto B is almost open. (We note that this definition is equivalent to demanding that every barrel of B, i.e. every subset which is closed, convex, circled, and absorbing, is a neighborhood of 0. See [9].) We will assume two easily proved facts about $\mathcal{B}$: (1) Every linear map $f: B \to X$, $B \in \mathcal{B}$, $X \in \mathcal{L}$, is almost continuous; (2) the image of a barrelled space under a map which is linear, continuous, and almost open, is barrelled. We say that $X \in \mathcal{L}$ is a *target of the closed graph theorem with respect to* $\mathcal{B}$ if every linear map $f: B \to X$ with a closed graph, $B \in \mathcal{B}$, is continuous.

Finally, since $X \in \mathcal{L}$ may be endowed with several topologies other than its original topology, we make the convention to designate the topology in question when giving the identity map as follows: $1_{\tau,\sigma} = 1 : X_\tau \to X_\sigma$.

2. Characterization of targets of the closed graph theorem. We begin by describing two functors from $\mathcal{L}$ to $\mathcal{B}$ which play an important role in the characterization of the targets of closed graph theorem.

Let $S : \mathcal{L} \to \mathcal{B}$ be the functor which assigns to X the space X_ω, where X_ω has the same underlying set as X and is endowed with the strongest locally convex topology possible on X (i.e., all seminorms on X are continuous on X_ω). Clearly, $X_\omega \in \mathcal{B}$ since the gauge of any barrel will be continuous. Moreover, since any linear map on X_ω is continuous, the map $f \in \mathcal{L}(X, Y)$ makes sense from X_ω to Y_ω.

Let $T : \mathcal{L} \to \mathcal{B}$ be the functor which assigns to X the space X_t, where X_t has the same underlying set as X and is endowed with the weakest barrelled topology stronger than the topology on X. The topology on X_t is generated by the continuous seminorms of X and the seminorms which were lower semi-continuous on X. (A seminorm p is lower semi-continuous if $\{x : px \leqq a\}$ is closed for all $a \in \mathbf{R}$.) If $f \in \mathcal{L}(X, Y)$, f will be continuous from X_t to Y_t, since, if p is a lower semi-continuous seminorm on Y, then $p \circ f$ is a lower semi-continuous seminorm on X. We note that $X_t = X$ if $X \in \mathcal{B}$.

2.1. Proposition. *T is the right adjoint of the forgetful functor $F : \mathcal{B} \to \mathcal{L}$.*

Proof. Let $X \in \mathcal{L}$, $\epsilon_X = 1_{t,X} : X_t \to X$. If $f \in \mathcal{L}(B, X)$, $B \in \mathcal{B}$, then we want to find a unique map $\bar{f} \in \mathcal{B}(B, X_t)$ such that $\epsilon_X \circ \bar{f} = f$. Clearly, we must have $\bar{f} = f$, and thus it suffices to show that $\bar{f} : B \to X_t$ is continuous. However, $\bar{f} = Tf : B_t \to X_t$ since $B_t = B$. Hence, $\bar{f}$ is continuous.

We remark that X_t is precisely the "associated barrelled space" for X, $X(\mathcal{T}_b(\overline{X(\mathcal{T})'}))$, constructed in a less transparent manner by Adasch [1].

It is well known that a linear map $f: X \to Y$, $X, Y \in \mathcal{L}$, has a closed graph if and only if there exists a Hausdorff topology τ on Y weaker than the topology on Y such that $f \in \mathcal{L}(X, Y_\tau)$. A proof of this fact is given in [3] by Komura, where the topology generated by the system of 0-neighborhoods $\{U + fV : U$ is a 0-neighborhood in Y and V is a 0-neighborhood in $X\}$ is offered as an example. We denote Y endowed with this topology as Y_f. Using this fact and Proposition 2.1, we can easily prove the following result of Komura.

2.2 Proposition. *X is a target for the closed graph theorem with respect to $\mathcal{B}$ if and only if $X_t = (X_\tau)_t$ for any Hausdorff topology τ weaker than the topology on X.*

Proof. ($\Rightarrow$) Let τ be a Hausdorff topology on X with $X_\tau < X$. Let $B \in \mathcal{B}$, and let $f \in \mathcal{L}(B, X_\tau)$. Then $1_{\tau,X} \circ f : B \to X$ has a closed graph and is, hence, by assumption, continuous. By definition of X_t, then, we have $\overline{1_{\tau,X} \circ f} \in \mathcal{B}(B, X_t)$. Since $1_{t,\tau} \circ \overline{1_{\tau,X} \circ f} = f$, this means $X_t = (X_\tau)_t$:

$$\begin{tikzcd} X_t \arrow[d, "1_{t,\tau}"'] & \\ X_t \arrow[r] & B \arrow[ul, "\overline{1_{\tau,X} \circ f}"'] \end{tikzcd}$$

($\Leftarrow$) Conversely, if $f: B \to X$ is any linear map with closed graph, $B \in \mathcal{B}$, then there is a Hausdorff topology τ on X with $f: B \to X_\tau$ continuous and $X_\tau < X$. Hence, $\bar{f} = f: B \to (X_\tau)_t = X_t$ is continuous, so $f = 1_{t,X} \circ \bar{f} : B \to X$ is continuous.

2.3. COROLLARY. *Let $X \in \mathcal{B}$. X is a target for the closed graph theorem with respect to $\mathcal{B}$ if and only if there is no barrelled topology on X weaker than that of X.*

For later use we wish to record the following lemma.

2.4. LEMMA. *If $f: B \to X$ is a linear map with a closed graph, $B \in \mathcal{B}$, then $1_{X,f}: X \to X_f$ is almost open.*

PROOF. Let U be a 0-neighborhood in X. Let N be a 0-neighborhood in X such that $N + N \subset U$. Since f is almost continuous, $\overline{f^{-1}(N)}$ is a 0-neighborhood of 0 in B. By definition of X_f, $N + f\overline{f^{-1}(N)}$ is a 0-neighborhood in X_f. Moreover, since $f: B \to X_f$ is continuous, we have

$$N + f\overline{f^{-1}(N)} \subset N + \mathrm{cl}_{X_f} ff^{-1}(N) \subset \mathrm{cl}_{X_f}(U).$$

We now describe a construction which will lead to our characterization theorem.

We fix $X \in \mathcal{L}$. For every $B \in \mathcal{B}$ and every linear map $f: B \to X$ with a closed graph, we form the following pushout in $\mathcal{L}$:

$$\begin{tikzcd} B_\omega \arrow[r, "f_\omega"] \arrow[d, "1_{\omega,B}"'] & X_\omega \arrow[d, dashed, "p_{(B,f)}"] \\ B \arrow[r, dashed, "q_{(B,f)}"'] & P_{(B,f)} \end{tikzcd}$$

We note that f_ω makes sense because $f: B \to X_\tau$ is continuous, where τ is some Hausdorff topology on X weaker than that of X, and $(X_\tau)_\omega = X_\omega$. Since both $1_{\omega,B}$ and f_ω are maps in $\mathcal{B}$ and since the forgetful functor $F: \mathcal{B} \to \mathcal{L}$ has a right adjoint, the pushout in $\mathcal{B}$ (all direct limits exist in $\mathcal{B}$) must be preserved by F. Hence, $P_{(B,f)} \in \mathcal{B}$.

2.5. Proposition. *$P_{(B,f)}$ has the same underlying set as X and is endowed with the strongest barrelled topology on X such that $q_{(B,f)} = f$: $B \to B_{(B,f)}$ is continuous.*

Proof. For simplicity, we omit the subscripts (B,f). Let b be a barrelled topology on X such that $f: B \to X_b$ is continuous. (It suffices to let $X_b = (X_\tau)_t$, where τ is any Hausdorff topology on X such that f: $B \to X_\tau$ is continuous.) By definition of the pushout, there exists a unique map $h: P \to X_b$ such that $h \circ p = 1_{\omega,b}$ and $h \circ q = f$:

$$\begin{tikzcd}
B_\omega \arrow[r, "f_\omega"] \arrow[d, "1_{\omega,B}"'] & X_\omega \arrow[d, "p"] \arrow[dd, bend left, "1_{\omega,b}"] \\
B \arrow[r, "q"] \arrow[dr, "f"'] & P \arrow[d, "h"] \\
 & X_b
\end{tikzcd}$$

The first of these relations tells us that p is one-one. Moreover, we have $p \circ 1_{b,\omega} \circ h \circ p = p$. But p is an epimorphism since $1_{\omega,B}$ is, so $p \circ 1_{b,\omega} \circ h = 1_P$, which shows that p is onto. Under the identification of P with X, clearly p and h are identity maps and q is f. Finally, the continuity of h implies that $X_b < P$.

2.6. Corollary. *Let $f: B \to X$ be a linear map with a closed graph, $B \in \mathcal{B}$. Then f is continuous if and only if $1: P \to X$ is continuous.*

Proof. If f is continuous, the definition of the pushout guarantees a continuous map $h: P \to X$ such that $h \circ p = 1_{\omega,X}$. Clearly, $h = 1$.
If $1: P \to X$ is continuous, so is $f: B \to P \to X$.

We define X_α to be the direct limit (generalized pushout) of the spaces $P_{(B,f)}$ taken over all linear maps $f: B \to X$ with a closed graph, $B \in \mathcal{B}$; i.e., α is the direct limit topology of the topologies on the $P_{(B,f)}$'s. Since the direct limit of barrelled spaces is barrelled, X_α is X endowed with the strongest barrelled topology weaker than all the $P_{(B,f)}$'s.

2.7. Lemma. $X_t = P_{(X_t, 1_{t,X})}$.

Proof. $1_{t,X}: X_t \to X$ is continuous, hence closed, and the following diagram is clearly a pushout:

$$\begin{tikzcd}
X_\omega \arrow[r] \arrow[d] & X_\omega \arrow[d] \\
X_t \arrow[r] & X_t
\end{tikzcd}$$

2.8. Proposition. *If $X \in \mathcal{B}$, then $1_{X,\alpha} : X \to X_\alpha$ is continuous.*

Proof. It follows from the above lemma that $X = P_{(X,1_X)}$. Hence, by definition of X_α, $1_{\alpha,X}$ is continuous.

2.9. Theorem. The following statements are equivalent.
(1) *X is a target of the closed graph theorem with respect to $\mathcal{B}$.*
(2) *$1_{\alpha,X} : X_\alpha \to X$ is continuous.*
(3) $X_\alpha \cong X_t$.

Proof.
$1 \Rightarrow 2$. By Corollary 2.6, we have $1 : P_{(B,f)} \to X$ is continuous for each (B,f). Thus by definition of X_α, $1_{\alpha,X}$ is continuous.
$2 \Rightarrow 3$. If $1_{\alpha,X}$ is continuous, so is $1_{\alpha,t} : X_\alpha \to X_t$ by definition of X_t, since $X_\alpha \in \mathcal{B}$. However, $X_t = P_{(X_t,1_{t,X})}$ by Lemma 2.7, so $1_{t,\alpha}$ is also continuous. Thus, $X_\alpha \cong X_t$.
$3 \Rightarrow 1$. Let $f : B \to X$ be a linear map with a closed graph, $B \in \mathcal{B}$. Then $f : B \to P_{(B,f)} \to X_\alpha \cong X_t$ is continuous, so $f : B \to X_t \to X$ is also.

2.10. Corollary. *The assignment $X \mapsto X_\alpha$ is functorial on the subcategory of $\mathcal{L}$ consisting of spaces which are targets of the closed graph theorem with respect to $\mathcal{B}$.*

2.11. Proposition. *If $X \in \mathcal{B}$, then X_α is a target of the closed graph theorem with respect to $\mathcal{B}$.*

Proof. Let $f : B \to X_\alpha$ be a linear map with a closed graph, $B \in \mathcal{B}$. Since $1_{X,\alpha}$ is continuous by Proposition 2.8, the map $f = 1_{\alpha,X} \circ f : B \to X_\alpha \to X$ has a closed graph. Hence, $f : B \to P_{(B,f)}$ is continuous.

3. **Infra-s- and B_r-complete spaces.** In this section we make a comparison between Theorem 2.9 and certain known results. We first give the definitions of the spaces formerly studied in connection with the closed graph theorem.

A space $X \in \mathcal{L}$ is B_r-complete if any weakly dense subspace of X' (continuous dual of X) is weakly closed whenever its intersection with the polar of any 0-neighborhood of X is weakly closed. A subspace S of X^* (the algebraic dual of X) is said to be *quasi-closed* if $S \cap B$ is closed in X^* whenever B is a bounded subset of X^*. We say that X is an *infra-s-space* if for every weakly dense subspace H of X', we have $\widehat{H} \cap X' = X'$, where $\widehat{H}$ is the intersection of all weakly quasi-closed subspaces S of X^* containing H. The first definition is due to Pták [7], the second to Adasch [1]. We remark that for B_r-complete spaces we prefer to use the equivalent definition that X is B_r-complete if every one-one continuous linear map from X onto $Y \in \mathcal{L}$ is open whenever it is almost open (see [7]).

Pták has proved that any B_r-complete space X is a target for the closed graph theorem with respect to $\mathscr{B}$. This result follows from the facts we have developed in a more simple manner than found in the literature.

3.1. THEOREM. *If X is a B_r-complete space, then X is a target for the closed graph theorem with respect to $\mathscr{B}$.*

PROOF. Let $f: B \to X$ be a linear map with a closed graph, $B \in \mathscr{B}$. By Lemma 2.4, $1_{X,f}: X \to X_f$ is almost open and continuous. Hence, by hypothesis, $1_{X,f}$ is open, and thus, $X \cong X_f$. Therefore, $f: B \to X$ is continuous.

The result of Adasch [1] that the category of infra-s-spaces comprises precisely the targets of the closed graph theorem with respect to $\mathscr{B}$ can be derived using Theorem 2.9. We shall employ the well-known criteria: (1) a linear map $f: X \to Y$, has a closed graph if and only if $\mathrm{dom}(f')$ is weakly dense in Y'; (2) a linear map $f: X \to Y$ with a closed graph, $X \in \mathscr{B}$, is continuous if and only if $\mathrm{dom}(f') = Y'$ (see [4]).

3.2. LEMMA. *If $f: X \to Y$ is a linear map, $X \in \mathscr{B}$, then* $\mathrm{dom}(f') = \overline{\mathrm{dom}(f')} \cap Y'$.

PROOF. See [1], § 3, proof of (A).

3.3. LEMMA. *Let H be a weakly dense subspace of X'. Let τ be the weak topology on X which is generated by H. Then* (1) $1_{\tau t,X} : (X_\tau)_t \to X$ *has a closed graph, and* (2) $1_{\tau t,\alpha} : (X_\tau)_t \to X_\alpha$ *is continuous.*

PROOF. The first observation is made by Adasch since $\mathrm{dom}(1'_{\tau t,X}) = Y' \cap \overline{H} \supset H$.

The second statement follows from the fact that $(X_\tau)_t = P_{((X_\tau)_t, 1_{\tau t,X})}$.

3.4. LEMMA. *If X is an infra-s-space, then $1_{X,\alpha} : X \to X_\alpha$ has a closed graph.*

PROOF. Let H be a weakly dense subspace of X', τ as in Lemma 3.3. We have

$$1_{\tau t,\alpha} = 1_{X,\alpha} \circ 1_{\tau t,X} : (X_\tau)_t \to X \to X_\alpha.$$

Since X is an infra-s-space, $\mathrm{dom}(1'_{\tau t,X}) = Y' \cap \overline{H} = Y'$. Therefore, since $\mathrm{ran}(1'_{X,\alpha}) \subset \mathrm{dom}(1'_{\tau t,X})$, $\mathrm{dom}(1'_{X,\alpha}) = \mathrm{dom}(1'_{\tau t,\alpha})$. But $\mathrm{dom}(1'_{\tau t,\alpha}) = X_\alpha{}'$ since $1_{\tau t,\alpha}$ is continuous by Lemma 3.3. Hence, $1_{X,\alpha}$ has a closed graph.

3.5. THEOREM. *The following statements are equivalent.*

(1) *X is an infra-s-space.*

(2) *$X_\alpha \to X$ is continuous.*

(3) *If $f: B \to X$ is a linear surjection with a closed graph, $B \in \mathscr{B}$, then f is continuous.*

PROOF.

$1 \Rightarrow 2$. By Lemma 3.4, $1_{X,\alpha}$ has a closed graph. Thus, so does $1_{\alpha,X}$: $X_\alpha \to X$. Hence, $\mathrm{dom}(1'_{\alpha,X})$ is weakly dense in X'. By Lemma 3.2 and the hypothesis, $\mathrm{dom}(1'_{\alpha,X}) = \mathrm{dom}(1'_{\alpha,X}) \cap Y' = Y'$. Hence $1_{\alpha,X}$ is continuous.

$2 \Rightarrow 3$. By Theorem 2.9, the continuity of $1_{\alpha,X}$ is equivalent to X being a target for the closed graph property with respect to $\mathcal{B}$. A fortiori, X must be a target for the closed graph property with respect to onto maps originating in $\mathcal{B}$.

$3 \Rightarrow 1$. Let H be a weakly dense subset of X'. Let τ be as in Lemma 3.3. Then $1_{\tau t,\alpha}$ is closed. Since $1_{\tau t,\alpha}$ is also surjective, it is continuous by hypothesis. Thus, $\mathrm{dom}(1'_{\tau t,X}) = X' \cap \widehat{H} = X'$, and X is an infra-s-space.

3.6. COROLLARY. *Being a target for the closed graph property with respect to all linear maps with source in $\mathcal{B}$ is equivalent to being a target for the closed graph property with respect to surjective maps with source in $\mathcal{B}$.*

There are examples to show that the category of infra-s-spaces is strictly larger than the category of B_r-complete spaces (see [1] for reference). However, we do have the following connection between infra-s-spaces and B_r-complete spaces in the context of $\mathcal{B}$.

3.7. THEOREM. *The following assertions are equivalent for $X \in \mathcal{B}$.*

(1) *X is B_r-complete.*

(2) *X is an infra-s-space.*

(3) $X_\alpha \cong X$.

PROOF.

$1 \Rightarrow 2$. This is a consequence of Theorems 3.1 and 3.5.

$2 \Rightarrow 3$. By Proposition 2.8, $1_{X,\alpha} : X \to X_\alpha$ is continuous. But by Theorem 3.5, so is its inverse. Hence, $X_\alpha \cong X$.

$3 \Rightarrow 2$. This is trivial by Theorem 3.5.

$2 \Rightarrow 1$. Let $f : X \to Y$ be a one-one, almost open, surjective, continuous linear map. Then f^{-1} has a closed graph. Since X is an infra-s-space and $Y \in \mathcal{B}$ (since Y is the image of a barrelled space under an almost open map), f^{-1} is continuous. Hence, f is open.

3.8. COROLLARY. *X_α is an infra-s-space if and only if X_α is B_r-complete.*

Finally, we close with a remark concerning the open mapping theorem. We say that $X \in \mathcal{L}$ is a *source for the open mapping theorem with respect to* $\mathcal{B}$ if every linear map with a closed graph of X onto B, $B \in \mathcal{B}$, is open. The closed graph theorem and the open mapping theorem have long been compared and studied together. Their relation is given in the following proposition.

3.9. PROPOSITION. *X is a source for the open mapping theorem with respect to $\mathcal{B}$ if and only if X/M is a target for the closed graph theorem with respect to $\mathcal{B}$ for every closed subspace M of X.*

PROOF. By Corollary 3.6, it suffices to show that X/M is a target for the closed graph theorem with respect to surjective maps originating in $\mathcal{B}$. Let $f: B \to X/M$ be a linear surjection with a closed graph, $B \in \mathcal{B}$. We may assume that f is one-one since otherwise the kernel (which is closed because f has a closed graph) could be factored out and $B/\mathrm{Ker}(f) \in \mathcal{B}$. Thus, $f^{-1}: X/M \to B$ is one-one onto and likewise has a closed graph. Hence, $f^{-1} \circ \pi : X \to X/M \to B$ has a closed graph, so, by hypothesis, it is open. It is easy to see that f^{-1} must be open, and, hence, f is continuous.

Let $f: X \to B$ be a linear surjection with closed graph, $B \in \mathcal{B}$. Then $\bar{f} : X/\ker(f) \to B$ is one-one onto and has a closed graph where $\bar{f}$ is the unique map such that $f = \bar{f} \circ \pi$. Hence, $\bar{f}^{-1}$ is one-one and has a closed graph; by hypothesis, it is continuous. Therefore, $\bar{f}$ is open and so is f.

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