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A PERTURBATION PROBLEM IN THE SCATTERING OF WAVES*

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1. **Introduction.** There have been a great number of works done on the scattering of waves by an inhomogeneous medium, either deterministic or stochastic. A satisfactory treatment of this problem in three dimensions seems to be lacking. To be specific, consider the time-harmonic wave propagation governed by the reduced wave equation

$$\Delta u(\underline{r}) + k^2 n^2(\underline{r}, \epsilon) u(\underline{r}) = 0, \ |\underline{r}| < \infty, \tag{1}$$

where Δ is the Laplacian operator in the space variable $\underline{r}$; k is wave number with $\mathrm{Im}\{k\} \geqq 0$, and n^2 is the refractive index depending on a small parameter ϵ. Assume that

$$n^2 = \begin{cases} 1 + \epsilon\eta(\underline{r}), & x \geqq 0, \\ 1, & x < 0, \end{cases}$$

and η is a given deterministic or random function of $\underline{r} = (x, \underline{\rho})$. For a plane wave

$$u_i = A(\underline{\rho}) e^{ikx}, \ x < 0, \tag{2}$$

incident from the left half-space, we wish to determine the scattered field $u_s(\underline{r}, \epsilon)$ and the transmitted field $u_T(\underline{r}, \epsilon)$, which satisfy a radiation condition at $|\underline{r}| = \infty$, so that

$$u_\epsilon(\underline{r}) = \begin{cases} u_T(\underline{r}, \epsilon), & x \geqq 0, \\ u_i(\underline{r}) + u_s(\underline{r}, \epsilon), & x < 0, \end{cases} \tag{3}$$

is a continuously differentiable solution to (1). It is a common practice to utilize the small parameter ϵ and seek a solution by the regular perturbation in ϵ. As was pointed out in [4], in common with the initial-value problems, this method suffers the secular behavior in the perturbation series which gives rise to unphysical results. Also we presented various methods to circumvent this difficulty for stochastic problems. Among them is the so-called "parabolic" equation approximation which has been applied to the deterministic or stochastic wave propagation at high frequency ($|k| \gg 1$). By fixing $\beta = k\epsilon = O(1)$, one can rationalize this approximation as the reduced problem of a singular perturbation problem [12] (for details, see [4]). In this

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case, not only does the reduced equation have a different type, ("parabolic" vs. elliptic), but the most vital information, the scattered field u_s, is left undetermined. Other perturbation techniques have the same defect.

To illustrate the relevant points and to gain some insight into the three-dimensional problem, the deterministic problem in one dimension will be reexamined. We shall adapt the method of averaging to obtain a uniformly valid solution for the reflection coefficient. Concrete results are obtained in certain special cases. Then a brief review of recent works on the scattering by a random medium will be given.

2. **A Deterministic Reflection Problem in One Dimension.** Let us consider the problem where the inhomogeneities are confined to $0 \leqq x \leqq L$. The half-space problem can be recovered as the limit $L \to \infty$ (with the aid of the limiting absorption principle if k is real). Then the solution in $x < 0$ and $x > L$ can be written down easily

$$u_\epsilon(x) = \begin{cases} e^{ikx} + R_\epsilon e^{-ikx}, & x < 0, \\ T_\epsilon e^{ikx}, & x > L, \end{cases} \tag{4}$$

where the amplitude A in (3) is taken to be one, the reflection coefficient R_ϵ and the transmission coefficient T_ϵ are as yet to be determined. If η is a constant, then

$$u_\epsilon(x) = Ae^{ik_1x} + Be^{-ik_1x}, \quad 0 \leqq x \leqq L, \tag{5}$$

in which $k_1(\epsilon) = k\sqrt{1 + \epsilon\eta}$. In view of (4), (5) and the continuity conditions on u_ϵ and $u_{\epsilon,x}$ we get

$$R_\epsilon = \frac{m(\epsilon)(e^{2ik_1L} - 1)}{1 - m^2(\epsilon)e^{2ik_1L}} \to -m(\epsilon) \text{ as } L \to \infty, \tag{6}$$

where

$$m(\epsilon) = \frac{n-1}{n+1} = \frac{\sqrt{1+\epsilon\eta} - 1}{\sqrt{1+\epsilon\eta} + 1}. \tag{7}$$

For a nonconstant function $\eta(x)$, we prefer to work with the differential equation governing a function closely related to the reflection coefficient. A study of such equation may lead to a novel approximation of which an error estimate becomes possible.

To this end, let us seek the interior solution in the form

$$u_\epsilon(x) = A_\epsilon(x)e^{ikx} + B_\epsilon(x)e^{-ikx}, \quad 0 < x < L. \tag{8}$$

By the standard procedure of variation of constants, (1) can be reduced to a canonical form

$$\frac{d}{dx}\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix} = \frac{k\epsilon\eta}{2i}\begin{pmatrix} -1 & -e^{-2ikx} \\ e^{2ikx} & 1 \end{pmatrix}\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}, \quad 0 < x < L. \tag{9}$$

Then the continuity of u_ϵ and $u_{\epsilon,x}$ together with (4) imply that

$$\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}_{x=0} = \begin{pmatrix} 1 \\ R_\epsilon \end{pmatrix}, \quad \begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}_{x=L} = \begin{pmatrix} T_\epsilon \\ 0 \end{pmatrix}. \tag{10}$$

Define the reflection function

$$\phi_\epsilon(x) = B_\epsilon(x)/A_\epsilon(x). \tag{11}$$

In view of (10), we have

$$\phi_\epsilon(0) = R_\epsilon. \tag{12}$$

Invoking (9), it is easy to verify that ϕ_ϵ satisfies the complex Riccati equation

$$\frac{d\phi_\epsilon(x)}{dx} = \frac{k\epsilon\eta(x)}{2i}[\phi_\epsilon(x)e^{-ikx} + e^{ikx}]^2, \quad 0 < x < L, \tag{13}$$

and the terminal condition

$$\phi_\epsilon(L) = 0. \tag{14}$$

This procedure is a variant of the so-called "invariant imbedding" by Bellman and Kalaba [1], who introduced a different reflection function as a function of the thickness L. Noting (12), the solution of the terminal-value problem (13) and (14) evaluated at $x = 0$ gives R_ϵ.

For a fixed k (or rescale x by $\tilde{x} = kx$), the form (13) suggests the use of the method of averaging by Bogoliubov and Mitropolsky [2]. However, to obtain a nontrivial result ($\phi \not\equiv 0$), we must first extract the principal contribution due to the inhomogeneous, linear part (see [3]). To this end, let

$$\phi_\epsilon = \phi_{0,\epsilon} + \epsilon\phi_{1,\epsilon}, \tag{15}$$

where $\phi_{0,\epsilon}$ solves the linear problem

$$\frac{d\phi_{0,\epsilon}}{dx} = \frac{k\epsilon\eta}{2i}(2\phi_{0,\epsilon} + e^{2ikx}), \tag{16}$$

$$\phi_{0,\epsilon}(L) = 0. \tag{17}$$

Then a substitution of (15) into (13) and (14) yields

$$
\begin{aligned}
\frac{d\phi_{1,\epsilon}}{dx} = \frac{k\eta}{2i}\,&[\epsilon^2\phi_{1,\epsilon}^2 e^{-2ikx} \\
&+ 2\epsilon(1+\phi_{0,\epsilon}e^{-2ikx})\phi_{1,\epsilon} + \phi_{0,\epsilon}^2 e^{-2ikx}],
\end{aligned}
\tag{18}
$$

$$\phi_{1,\epsilon}(L) = 0. \tag{19}$$

The linear problem can be easily solved to give

$$\phi_{0,\epsilon}(x) = (1/2)ik\epsilon \int_x^L \eta(t)\exp\{2ikt + ik\epsilon \int_x^t \eta(s)\, ds\}\, dt. \tag{20}$$

In (18) and (19), if we set

$$\phi_{1,\epsilon}(x) = \mathcal{O}_\epsilon(x)e^{-ik\epsilon \int_0^x \eta(t)dt}, \tag{21}$$

they can be brought into the form

$$d\,\mathcal{O}_\epsilon/dx = (k\epsilon^2/2i)[a_1\,\mathcal{O}_\epsilon^2 + a_2\,\mathcal{O}_\epsilon + a_3],\ 0 < x < L, \tag{22}$$

$$\mathcal{O}_\epsilon(L) = 0. \tag{23}$$

Here the coefficients a_i, $i = 1, 2, 3$, are given by

$$a_1(x,\epsilon) = \eta(x)\exp\{-2ik[x + (\epsilon/2)\int_0^x \eta(t)\, dt]\}, \tag{24}$$

$$a_2(x,\epsilon) = 2\epsilon^{-1}\eta(x)\phi_{0,\epsilon}(x)e^{-2ikx}, \tag{25}$$

$$a_3(x,\epsilon) = \epsilon^{-2}\eta(x)\phi_{0,\epsilon}^2(x)\exp\left\{-2ik\left[x - (\epsilon/2)\int_0^x \eta(t)\, dt\right]\right\}. \tag{26}$$

Now we apply the method of averaging to the problem (22)–(23) to get

$$d\,\tilde{\mathcal{O}}_\epsilon/dx = (k\epsilon^2/2i)(\tilde{a}_1\,\tilde{\mathcal{O}}_\epsilon^2 + \tilde{a}_2\,\tilde{\mathcal{O}}_\epsilon + \tilde{a}_3), \tag{27}$$

$$\tilde{\mathcal{O}}_\epsilon(L) = 0, \tag{28}$$

where $\tilde{a}_i$ are the average values of a_i which, if they exist, are defined as

$$\tilde{a}_i(\epsilon) = \lim_{T\to\infty} (1/T) \int_0^T a_i(x,\epsilon)\, dx, \tag{29}$$

and $\tilde{\mathcal{O}}_\epsilon$ denotes the average value of $\mathcal{O}_\epsilon$. Since $\tilde{a}_i$ are constants, the equation (27) can be solved readily. This is why we did not set $\epsilon = 0$ in a_i. Let $\tilde{\phi}_{1,\epsilon}$ be an approximation to $\phi_{1,\epsilon}$ when $\mathcal{O}$ is replaced by $\tilde{\mathcal{O}}$ in (21). Then, by invoking the first theorem of Bogoliubov [2], we can assert that there exists a small positive number ϵ_0 such that

$$\sup_{x\in[0,L]} |\phi_\epsilon(x) - \phi_{0,\epsilon}(x) - \epsilon\tilde{\phi}_{1,\epsilon}(x)| = o(\epsilon), \quad \text{for } \epsilon^2 L = O(1) \text{ and } 0 < \epsilon \leqq \epsilon_0. \tag{30}$$

Note that $\tilde{\mathcal{O}}_\epsilon \equiv 0$ when $\tilde{a}_3 = 0$. In this case, the linearized solution $\phi_{0,\epsilon}$ given by (20) uniformly approximates ϕ_ϵ up to $o(\epsilon)$, and hence

$$R_\epsilon = (ik\epsilon/2)\int_0^L \eta(t)\exp\left\{2ik\left[t + (\epsilon/2)\int_0^t \eta(s)\,ds\right]\right\} dt + o(\epsilon), \quad \epsilon^2 L = O(1). \tag{31}$$

A sufficient condition for (31) to hold is that $\phi_{0,\epsilon}$ is square-integrable over $[0, \infty)$. However the necessary condition is not obvious.

For example, we take η to be a constant. A trivial integration of (20) gives

$$\phi_{0,\epsilon}(x) = \frac{\epsilon\eta}{2(2+\epsilon\eta)}\{e^{ik[2L+\epsilon\eta(L-x)]} - e^{2ikx}\}, \tag{32}$$

and (29) yields

$$\begin{aligned} \tilde{a}_1 &= 0, \\ \tilde{a}_2 &= -\eta^2/(2+\epsilon\eta), \\ \tilde{a}_3 &= \frac{-\eta^3}{2(2+\epsilon\eta)^2} e^{ik(2+\epsilon\eta)L}. \end{aligned} \tag{33}$$

Then the solution of (27) and (28) can be shown to be

$$\tilde{\mathcal{O}}_\epsilon(x) = \frac{-\eta}{2(2+\epsilon\eta)} e^{ik(2+\epsilon\eta)L} \cdot \left\{1 - \exp\left[\frac{-ik\epsilon^2\eta^2}{2(2+\epsilon\eta)}(L-x)\right]\right\}, \tag{34}$$

and hence, noting (30)

$$R_\epsilon \sim \frac{\epsilon\eta}{2(2+\epsilon\eta)}\left\{\exp ik\left[(2+\epsilon\eta)L - \frac{\epsilon^2\eta^2 L}{2(2+\epsilon\eta)}\right] - 1\right\}, \tag{35}$$

which agrees with the exact result up to $o(\epsilon)$ for $\epsilon^2 L = O(1)$.

As another example for which $\eta \not\equiv$ constant, let

$$\eta(x) = \eta_0 \cos bx, \tag{36}$$

where η_0 and b are some positive constants. Corresponding to (36), the expression (20) for $\phi_{0,\epsilon}$ becomes

$$\phi_{0,\epsilon}(x) = (ik\eta_0\epsilon/2)\int_x^L \cos bt \exp[2ikt + ik\eta_0 b^{-1}\epsilon(\sin bt - \sin bx)]\, dt. \tag{37}$$

To simplify the computation of the average values $\tilde{a}_i$, we make use of the large parameter k, and expand ϕ_0 asymptotically to give

$$\phi_{0,\epsilon}(x) \sim -(\epsilon/4)\eta_0\{e^{2ikx}\cos bx - \cos bL \cdot \exp[2ikL + ik\eta_0 b^{-1}\epsilon(\sin bL - \sin bx)]\}. \tag{38}$$

Based on (38), it is found that

$$\begin{aligned} \tilde{a}_1 &= 0, \\ \tilde{a}_2 &= -\eta_0^2/4, \\ \tilde{a}_3 &= -(\eta_0^3/16)\cos bL \cdot \exp[2ikL + ik\eta_0 b^{-1}\epsilon \sin bL]. \end{aligned} \tag{39}$$

Upon substituting (39) into (27), the solution to the average problem (27) and (28) is simply

$$\tilde{\mathcal{O}}_\epsilon(x) = -(1/4)\eta_0 \cos bL \cdot \exp(2ikL + ik\eta_0 b^{-1}\epsilon \sin bL)[1 - e^{-ik\eta_0^2(\epsilon 2/8)(L-x)}]. \tag{40}$$

In view of (15), (21), (38) and (40), we obtain, for large k

$$R_\epsilon \sim (1/4)\eta_0\epsilon\{\cos bL \cdot \exp[2ikL(1 - (1/16)\eta_0^2\epsilon^2) + ik\eta_0 b^{-1}\epsilon \sin bL] - 1\} + o(\epsilon), \quad \text{for } L = O(\epsilon^{-2}). \tag{41}$$

In general, we suppose that $\tilde{a}_1 \cdot \tilde{a}_3 \neq 0$, and the roots λ_1, λ_2 of the equation

$$\tilde{a}_1 r^2 + \tilde{a}_2 r + \tilde{a}_3 = 0, \tag{42}$$

are distinct. Then it is easy to verify that the solution $\tilde{\mathcal{O}}$ of the average problem (27) and (28) is given by

$$\tilde{\mathcal{O}}_\epsilon(x) = \frac{\lambda_1\lambda_2\{\exp[(1/2)ik\tilde{a}_1\lambda_0(L - x)] - 1\}}{\{\lambda_1 \exp[(1/2)ik\tilde{a}_1\lambda_0(L - x)] - \lambda_2\}}, \tag{43}$$

where $\lambda_0 = \lambda_1 - \lambda_2$.

The above method of determining the reflection wave can be generalized to the multi-mode propagation problem. To be specific, let us consider the following $n \times n$ matrix-differential equation

$$(d^2/dx^2)U_\epsilon(x) + K^2[I + \epsilon Y(x)]U_\epsilon(x) = 0, \quad 0 < x < L, \tag{44}$$

where K is a real symmetric matrix, I is the identity and Y is a given matrix-valued function integrable over $[0, L]$. We wish to seek a solution matrix $U_\epsilon(x)$ of (44) which can be extended in a continuously differentiable manner to a function on $(-\infty, \infty)$ such that

$$U_\epsilon(x) = \begin{cases} e^{ixK} + e^{-ixK}R_\epsilon, & x \leqq 0, \\ e^{ixK}T_\epsilon, & x \geqq L, \end{cases} \tag{45}$$

in which R_ϵ and T_ϵ denote, respectively, the reflection and transmission matrices, and the unitary, exponential matrix e^{ixK} has its inverse being e^{-ixK}. Again let

$$U_\epsilon(x) = e^{ixK}A_\epsilon(x) + e^{-ixK}B_\epsilon(x). \tag{46}$$

The steps (8)-(10) can be carried over to the matricial case verbatim. Here, of course, $A_\epsilon(x)$ and $B_\epsilon(x)$ are $n \times n$ matrices and the corresponding canonical equation reads

$$\frac{d}{dx}\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix} = \frac{\epsilon}{2i}\begin{pmatrix} -\bar{Y}_2(x) & -\bar{Y}_1(x) \\ Y_1(x) & Y_2(x) \end{pmatrix}\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}, \quad 0 < x < L, \tag{47}$$

and

$$\begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}_{x=0} = \begin{pmatrix} I \\ R_\epsilon \end{pmatrix}, \quad \begin{pmatrix} A_\epsilon \\ B_\epsilon \end{pmatrix}_{x=L} = \begin{pmatrix} T_\epsilon \\ 0 \end{pmatrix}. \tag{48}$$

In (47), $\bar{Y}_i$ is the complex conjugate of Y_i and

$$\begin{aligned} Y_1(x) &= Ke^{ixK}Y(x)e^{ixK}, \\ Y_2(x) &= Ke^{ixK}Y(x)e^{-ixK}. \end{aligned} \tag{49}$$

Let the reflection function ϕ_ϵ be defined as

$$\phi_\epsilon(x) = B_\epsilon(x)A_\epsilon^{-1}(x). \tag{50}$$

Then, noting the identity $A^{-1}_{\epsilon,x} = -A_\epsilon^{-1}A_{\epsilon,x}A_\epsilon^{-1}$ and (47), one can verify by a direct substitution that ϕ_ϵ satisfies the matrix-Riccati equation

$$\begin{aligned} d\phi_\epsilon/dx = (\epsilon/2i)\{&\phi_\epsilon\bar{Y}_1(x)\phi_\epsilon + Y_2(x)\phi_\epsilon \\ &+ \phi_\epsilon\bar{Y}_2(x) + Y_1(x)\}, 0 < x < L, \end{aligned} \tag{51}$$

and

$$\phi_\epsilon(0) = R_\epsilon \quad \phi_\epsilon(L) = 0. \tag{52}$$

As before, let $\phi_\epsilon = \phi_{0,\epsilon} + \epsilon\phi_{1,\epsilon}$, where $\phi_{0,\epsilon}$ is the linearized solution of (51) and (52) given by

$$\phi_{0,\epsilon}(x) = (i\epsilon/2)M_2(x)\int_x^L \bar{M}_1(t)Y_1(t)\bar{M}_2(t)\,dt\,M_1(x). \tag{53}$$

The fundamental matrices M_1 and M_2 solve the following initial-value problems

$$(d/dx)M_1 = (\epsilon/2i)M_1\bar{Y}_2, \quad M_1(0) = I, \tag{54}$$

$$(d/dx)M_2 = (\epsilon/2i)Y_2M_2, \quad M_2(0) = I, \tag{55}$$

and satisfy the relations $M_1^{-1} = \bar{M}_2$ and $M_2^{-1} = \bar{M}_1$. Analogous to (21), we set

$$\phi_{1,\epsilon} = M_2\,\mathcal{O}_\epsilon M_1. \tag{56}$$

Then the average $\tilde{\mathcal{O}}_\epsilon(x)$ is governed by the Riccati equation with constant coefficients

$$d\,\tilde{\mathcal{O}}_\epsilon/dx = (\epsilon^2/2i)(\,\tilde{\mathcal{O}}_\epsilon\bar{b}_1\,\tilde{\mathcal{O}}_\epsilon + \bar{b}_2\,\tilde{\mathcal{O}}_\epsilon + \,\tilde{\mathcal{O}}_\epsilon\bar{b}_3 + \bar{b}_4),\ 0 < x < L, \tag{57}$$

$$\tilde{\mathcal{O}}_\epsilon(L) = 0. \tag{58}$$

Here the coefficients $\bar{b}_i$ are the average values of the matrices $b_i(x)$ defined by

$$\begin{aligned} b_1(x) &= M_1(x)\bar{Y}_1(x)M_2(x),\\ b_2(x) &= \epsilon^{-1}\bar{M}_1(x)\phi_{0,\epsilon}(x)\bar{Y}_1(x)M_2(x),\\ b_3(x) &= \epsilon^{-1}M_1(x)\bar{Y}_1(x)\phi_{0,\epsilon}(x)\bar{M}_2(x),\\ b_4(x) &= \epsilon^{-2}\bar{M}_1(x)\phi_{0,\epsilon}(x)\bar{Y}_1(x)\phi_{0,\epsilon}(x)\bar{M}_2(x). \end{aligned} \tag{59}$$

If the solution of (57) and (58) is obtainable, we can again ascertain that

$$\|R_\epsilon - \phi_{0,\epsilon}(0) - \epsilon\,\tilde{\mathcal{O}}_\epsilon(0)\| = o(\epsilon), \tag{60}$$

where $\|\cdot\|$ denotes a matrix norm.

The results presented in this section are believed to be new. However, certain results on the gradual reflection of short waves by an infinite, inhomogeneous medium were reported recently by Meyer [10] at the 1975 SIAM National Meeting (June 11). His approach seems quite different from ours.

3. **Reflection by a Random Medium.** The scattering of waves by a one-dimensional random medium has been treated by many authors (see, e.g., [6, 11, 13]). In this case, the function η in (2) is a random function, and the equation (1) becomes a stochastic differential equation. To compute the statistics of the reflection coefficient R_ϵ,

one may deal with the nonlinear stochastic equation (13) directly. For example, if $\eta(x)$ is a stationary Markov diffusion process in x, then the joint process (ϕ_1, ϕ_2, η) is a three-dimensional diffusion process, where ϕ_1, ϕ_2 denote the real and imaginary parts of the reflection function ϕ. Let g be a smooth function on R^3, and let G be the conditional expectation

$$G(x, z_1, z_2, z_3) \tag{61}$$
$$= E\{g[\phi_1(t), \phi_2(t), \eta(t)] \mid \phi_1(x) = z_1, \phi_2(x) = z_2, \eta(x) = z_3\}, x < t.$$

It is known from the theory of diffusion processes [7] that G satisfies the Kolmogorov Backward Equation, with $\underline{z} = (z_1, z_2, z_3)$

$$\frac{\partial G}{\partial x} = (\epsilon^2/2) \sum_{i,j=1}^{3} b_{ij}(\underline{z}) \frac{\partial^2 G}{\partial z_i \partial z_j} + \epsilon^2 \sum_{i=1}^{3} a_i(\underline{z}) \frac{\partial G}{\partial z_i} \equiv \mathcal{L}_z G, \tag{62}$$

and the end condition

$$\lim_{x \uparrow t} G(x, \underline{z}) = g(\underline{z}). \tag{63}$$

In (62), the diffusion coefficients b_{ij} and the drift vector a_i can be computed via (13) in terms of the correlation function, if $E\eta = 0$

$$\gamma(x) = E[\eta(t + x)\eta(t)]. \tag{64}$$

Assuming that the transition probability density $p(x, \underline{z}; t, \underline{\xi})$ for the process (ϕ, η) is smooth, it satisfies the Fokker-Planck Equation

$$\partial p/\partial t = \mathcal{L}_\xi{}^* p, \tag{65}$$

and

$$\lim_{t \downarrow x} p(x, \underline{z}; t, \underline{\xi}) = \delta(\underline{z} - \underline{\xi}) \tag{66}$$

where $\mathcal{L}^*$ designates the adjoint of $\mathcal{L}$ and δ stands for the Dirac delta function. Once the density function p is determined from (65) and (66), the mean values of all functions of R, e.g., the mean reflection power $E|R|^2$, can be obtained by integrating it with respect to the marginal density in (ϕ_1, ϕ_2) after setting $x = 0$, $t = L$ and $\xi_1 = \xi_2 = 0$ in p. Results obtained this way are exact.

The perturbation theory comes into the picture through a limit theorem of Hashiminskii [8]. When η is not Markovian but satisfies a strong mixing condition, the solution process (ϕ_1, ϕ_2) of (13) and (14) will converge weakly to a Markov process as $\epsilon \to 0$, $L \to \infty$ with $L\epsilon^2$ fixed. Here the hybrid of the principle of averaging, laws of large numbers and the central limit theorem yield some rather far-reaching results in stochastic differential equations. For the detailed analysis of concrete problems, one is referred to [6, 11, 13].

In three dimensions, however, there exist no parallel results. The reflection of mean waves was treated in [9] by the method of smoothing perturbation. The resulting integro-differential equation was solved by the Wiener-Hopf technique. The second-moment problem, determining the mean reflection power, has not yet been resolved. This problem is of great interest due to its importance in radio science.

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