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CHARGE SINGULARITY AT THE VERTEX OF A SLENDER CONE OF GENERAL CROSS-SECTION

J. A. MORRISON

ABSTRACT. Let the equation of the cone be given in spherical coordinates (r, θ, φ) by $f(\epsilon^{-1}\tan\theta\cos\varphi, \epsilon^{-1}\tan\theta\sin\varphi) = 0$, with $\theta = O(\epsilon)$, where $0 < \epsilon \ll 1$ is a parameter. The potential V satisfies Laplace's equation $\nabla^2 V = 0$ exterior to the cone, with boundary condition $V = 0$ on the cone. A solution is sought in the form $V = r^\rho U(\theta, \varphi)$, and the boundary condition leads to an eigenvalue problem for ρ. The quantity of interest is the smallest $\rho > 0$. For $0 < \rho < 1$ there is a singularity in the surface charge density of the form $r^{\rho-1}$ at the vertex of the cone, in addition to any edge singularities that may be present.

Near the cone, for $\theta = O(\epsilon)$, the complex variable $\tau = \epsilon^{-1}\tan(\theta/2)e^{i\varphi}$ is introduced, and in the inner limit the equation for U tends to the 2-dimensional Laplace equation as $\epsilon \to 0$. The eigenvalue problem is solved by a singular perturbation procedure, by matching the inner solution to the outer solution, away from the cone. Let C_ϵ denote the contour in the τ-plane which corresponds to the cone. Suppose that the domain exterior to the circle $|w| = \lambda(\epsilon)$ is mapped conformally onto the domain exterior to C_ϵ by $\tau = g(w, \epsilon)$, with $\tau - w - \alpha_0 \to 0$ as $|w| \to \infty$, so that $\lambda(\epsilon)$ is the outer radius of C_ϵ. It is found that

$$1/2\psi(-\rho) + 1/2\psi(1+\rho) - \psi(1) + \epsilon^2\kappa\rho(\rho+1)$$
$$\sim \log(1/\epsilon\lambda) + \epsilon^2\alpha_0\alpha_0^*,$$

where ψ denotes the logarithmic derivative of the gamma function, the asterisk denotes complex conjugate, and κ is determined in terms of the mapping function $g(w, \epsilon)$. Details of the derivation of the above result will be given elsewhere [1]. Examples of cones with particular cross-sections have been considered.

The charge singularity at the vertex of a sectorial flat plate of angle χ has been investigated earlier by Morrison and Lewis [2]. For that problem, Laplace's equation was solved by separation of variables in conical coordinates. This leads to a dual eigenvalue problem for ρ and for a separation constant δ. Analytic expressions for the singularity strength were derived for $0 < (2\pi - \chi) \ll 1$, for $|\pi - \chi| \ll 1$, and for $0 < \chi \ll 1$, by means of singular perturbation techniques, and numerical results were obtained for other values of χ. The present result generalizes the one for small angles to cones of general cross-section.

References to related work are given in [1] and [2].

References

1. J. A. Morrison, *Charge singularity at the vertex of a slender cone of general cross-section,* submitted to SIAM J. Appl. Math.

2. J. A. Morrison and J. A. Lewis, *Charge singularity at the corner of a flat plate,* SIAM J. Appl. Math., to appear.

Bell Laboratories, Murray Hill, New Jersey 07974