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FOURIER MULTIPLIERS FROM $H^1(R^n)$ TO $L^p(R^n)$

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1. **Introduction.** We say that a function m is a multiplier from $H^1(R^n)$ into $L^p(R^n)$ if the mapping

$$f \to T_m f = F^{-1}(m\hat{f})$$

takes H^1 into L^p continuously. We think of H^1 as a subspace of L^1 defined by real variable methods as in Fefferman and Stein [2] and confine our attention to the case $1 < p \leqq \infty$. We establish sufficient conditions for m to be a multiplier from H^1 into L^p which are weak enough to allow T_m to be unbounded on L^1; in particular, our methods are applicable to homogeneous functions.

Our primary tools are the atomic representation of H^1 developed by Coifman and Weiss [1] and several theorems about homogeneous Lipschitz spaces found in Herz [3]. There is a natural dichotomy in our methods, according to whether $1 < p < 2$ or $2 \leqq p \leqq \infty$. In the latter case we may simply estimate the conjugate norm of $m\hat{f}$; in the former case we make a Lipschitz estimate of $m\hat{f}$ and then use Herz's version of the Bernstein theorem to bound $\|T_m f\|_p$.

Among the results given by Johnson [4] are sufficient conditions on m in order that $\|m\hat{f}\|_2$ or $\|m\hat{f}\|_1$ be bounded by the H^1 norm of f. We give an independent proof of the sufficiency of his conditions which treats the intermediate cases as well. We also given an alternate condition which seems to be more useful when $p \neq 2$; it requires slightly more smoothness locally in $R^n \sim \{0\}$ but is less restrictive at the origin and infinity.

These investigations had their origin in a course taught at Washington University by Prof. Guido Weiss in which he presented unpublished work done jointly with Prof. R. R. Coifman. The basic idea of using atoms to prove multiplier theorems is theirs.

2. **Definitions and the statement of our main result.** Let x denote a point in R^n. The usual inner product will be denoted by $x \cdot y$; dx denotes Lebesgue measure.

We define the Fourier transform on $L^1(R^n)$ by

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$$\hat{f}(x) = (2\pi)^{-n/2} \int e^{-ix\cdot y} f(y)\, dy;$$

the inverse Fourier transform operator is F^{-1}.

The difference operator Δ_y is defined by

$$\Delta_y f(x) = f(x+y) - f(x);$$

higher order differences are given by

$$\Delta_y{}^k f(x) = \Delta_y(\Delta_y^{k-1} f)(x) = \sum_{j=0}^{k} C_{k,j}(-1)^{k-j} f(x+jy).$$

Note that $\Delta_y{}^k \hat{f} = [(e^{-ix\cdot y} - 1)^k f]\hat{}$ and $(\Delta_y{}^k f)\hat{}(x) = (e^{ix\cdot y} - 1)^k \hat{f}(x)$.

We say that f is an *atom* if $f \in L^\infty(R^n)$, $f = 0$ a.e. outside a cube Q of measure $|Q|$, $\|f\|_\infty \leqq |Q|^{-1}$, and $\int f = 0$. Coifman and Weiss [1] prove that a function ϕ is in $H^1(R^n)$ if and only if $\phi = \sum \alpha_k f_k$ where each f_k is an atom and $\sum |\alpha_k| < \infty$; moreover, the infimum of all such sums is equivalent to $\|\phi\|_{H_1}$.

Let p' denote the Hölder conjugate of p. We shall assume throughout that $m \in L^{p'}_{\text{loc}}(R^n \sim \{0\})$. Other conditions we shall impose from time to time are

1° $2 \leqq p \leqq \infty$, $1/q = 1/2 - 1/p$, and the sequence $\{b_k\}_{k=-\infty}^{\infty}$, defined by

$$b_k = \left(\int_{2^k \leqq |x| \leqq 2^{k+1}} |m(x)|^{p'}\, dx \right)^{1/p'},$$

is in ℓ^q.

2° For some q with $1/q + 1/p < 1$, there is a constant C such that

$$\left(\int_{|x| \geqq R} |m(x)|^q\, dx \right)^{1/q} \leqq C R^{n(1/p + 1/q - 1)}, \text{ all } R > 0.$$

3° Let $1 < p < 2$, $1 \leqq r < 2$, and let k be the first integer greater than $n(1/r + 1/p - 1)$. Then

$$\sup_{y \neq 0} \|\Delta_y{}^k m\|_r |y|^{-n(1/r + 1/p - 1)} < \infty.$$

Main Theorem. *Let* $T_m f = F^{-1}(m\hat{f})$ *for* $f \in H^1$. *Then* $T_m : H^1 \to L^p$ *continuously if either*

(i) $2 \leqq p \leqq \infty$ *and* m *satisfies* 1° *or* 2°.

or (ii) $1 < p < 2$ *and* m *satisfies* 2° *and* 3°.

3. Preliminary calculations.

LEMMA 1. *If m satisfies either* 1° *or* 2°, *then there is a constant C such that*

$$\int_{R \leqq |x| \leqq 2R} |m(x)|^{p'}\, dx \leqq C, \text{ all } R > 0.$$

PROOF. If 1° is satisfied the integral is dominated by $b_k^{p'} + b_{k+1}^{p'} \leqq (2\|b\|_q)^{p'}$ for some k.

When 2° is satisfied we have

$$\left(\int_{R \leqq |x| \leqq 2R} |m(x)|^{p'}\, dx \right)^{1/p'} \leqq C(2R)^{n(1/p' - 1/q)} \left(\int_{R \leqq |x|} |m(x)|^{q}\, dx \right)^{1/q} = C$$

by Hölder's inequality since $q > p'$.

LEMMA 2. *If m satisfies* 2° *with* $q = q_0$, *then* 2° *remains valid for* $1/p + 1/q_0 < 1/p + 1/q < 1$.

PROOF. By Hölder's inequality

$$\left(\int_{R_0 \leqq |x| \leqq R_1} |m(x)|^{q}\, dx \right)^{1/q} \leqq C R_1^{n(1/q - 1/q_0)} \left(\int_{R_0 \leqq |x|} |m(x)|^{q_0}\, dx \right)^{1/q_0} \leqq C R_1^{n(1/q - 1/q_0)}\, R_0^{n(1/p + 1/q_0 - 1)}.$$

Choosing $R_1 = 2R_0 = 2^{j+1}R$, we have

$$\int_{R \leqq |x|} |m(x)|^{q}\, dx \leqq \sum_{j=0}^{\infty} \int_{2^j R \leqq |x| \leqq 2^{j+1} R} |m(x)|^{q}\, dx \leqq C \sum_{j=0}^{\infty} (2^j R)^{n(1/p + 1/q - 1)} = C R^{n(1/p + 1/q - 1)}.$$

LEMMA 3. *If m satisfies* 1° *or* 2°, *then there is a constant C such that*

$$\left(\int_{|x| \leqq R} |x|^{p'} |m(x)|^{p'}\, dx \right)^{1/p'} \leqq CR, \text{ all } R > 0.$$

PROOF.

$$\int_{|x|\leqq R} |x|^{p'}|m(x)|^{p'}\,dx = \sum_{j=0}^{\infty} \int_{2^{-j-1}R\leqq|x|\leqq 2^{-j}R} |x|^{p'}|m(x)|^{p'}\,dx$$

$$\leqq \sum_{j=0}^{\infty} C(2^{-j}R)^{p'} = CR^{p'}$$

by Lemma 1.

LEMMA 4. *If m satisfies* 2° *and* $p < 2$, *then there is a constant* C *such that*

$$\left(\int_{|x|\leqq R} |m(x)|^2\,dx\right)^{1/2} \leqq CR^{n(1/p-1/2)}, \text{ all } R > 0.$$

PROOF. By Hölder's inequality and Lemma 1,

$$\left(\int_{R\leqq|x|\leqq 2R} |m(x)|^2\,dx\right)^{1/2}$$

$$\leqq C(2R)^{n(1/p-1/2)}\left(\int_{R\leqq|x|\leqq 2R} |m(x)|^{p'}\,dx\right)^{1/p'}$$

$$\leqq CR^{n(1/p-1/2)}, \text{ all } R > 0.$$

The desired conclusion follows by integrating over annuli as above.

LEMMA 5. *If m satisfies* 2° *and* 3° *with* $1 < p < 2$, *then there is a constant* C *and a* $\delta > 0$ *such that*

(a) $$\left(\int_{|x|\geqq R} |\Delta_y{}^k m(x)|^2\,dx\right)^{1/2} \leqq CR^{-\delta}|y|^{n(1/p-1/2)+\delta},$$

(b) $$\left(\int_{|x|\leqq R} |x|^2|\Delta_y{}^k m(x)^2\,dx\right)^{1/2} \leqq CR^{1-\delta}|y|^{n(1/p-1/2)+\delta},$$

(c) $$\|\Delta_y{}^k m\|_2 \leqq C|y|^{n(1/p-1/2)},$$

for all $y \in R^n$, *all* $R > 0$.

PROOF. Since $r < 2$ and $q > p' > 2$, we can choose θ with $0 < \theta < 1$ so that $1/2 = (1-\theta)/q + \theta/r$. Then

$$\left(\int_{|x|\geqq R} |\Delta_y{}^k m(x)|^2\,dx\right)^{1/2}$$

$$\leqq \left(\int_{|x|\geqq R} |\Delta_y{}^k m(x)|^q\,dx\right)^{(1-\theta)/q} \|\Delta_y{}^k m\|_r^{\theta}.$$

When $2k|y| \leqq R$, we have

$$\left(\int_{|x|\geqq R} |\Delta_y{}^k m(x)|^q\,dx\right)^{1/q} \leqq C\left(\int_{|x|\geqq R/2} |m(x)|^q\,dx\right)^{1/q}$$
$$\leqq CR^{n(1/p+1/q-1)} \text{ by } 2^\circ.$$

From 3° we have

$$\|\Delta_y^k m\|_r^\theta \leqq C|y|^{\theta n(1/p+1/r-1)}$$
$$= C|y|^{n(1/p-1/2)-(1-\theta)n(1/p+1/q-1)};$$

we take $\delta = -(1-\theta)n(1/p+1/q-1)$.

In particular, we now have

$$\left(\int_{|x|\geqq 2k|y|} |\Delta_y{}^k m(x)|^2\,dx\right)^{1/2} \leqq C|y|^{n(1/p-1/2)}.$$

But also

$$\left(\int_{|x|\leqq 2k|y|} |\Delta_y{}^k m(x)|^2\,dx\right)^{1/2} \leqq C\left(\int_{|x|\leqq 3k|y|} |m(x)|^2\,dx\right)^{1/2}$$
$$\leqq C|y|^{n(1/p-1/2)}$$

by Lemma 4; thus $\|\Delta_y{}^k m\|_2 \leqq C|y|^{n(1/p-1/2)}$, which is (c).

We have already established (a) for $2k|y| \leqq R$; when $2k|y| > R$ (b) implies (a).

To prove (*b*), we evaluate $\int_{|x|\leqq R} |x|^2|\,\theta_y{}^k m(x)^2\,dx$ by summing the integrals over annuli $2^{-j-1}R \leqq |x| \leqq 2^{-j}R$ and using the estimate in (a). The resulting series will converge to the desired bound as long as $\delta < 1$; we can achieve this by means of Lemma 2 if necessary.

Lemma 6. *Let k be a non-negative integer and let $2 \leqq s \leqq \infty$. Then there is a constant C such that*

$$\|\Delta_y{}^k \hat{f}\|_s \leqq Ca^{-n/s+k}|y|^k$$

for all $a < 0$ and all bounded measurable functions f supported in $[0, a]^n$ with $\|f\|_\infty \leqq a^{-n}$.

Proof. Note that $\Delta_y{}^k \hat{f}$ is the Fourier transform of $(e^{-ix\cdot y}-1)^k f(x)$. Since $|e^{-ix\cdot y}-1| \leqq |x\cdot y| \leqq Ca|y|$ on the support of f, the result follows immediately using the Hausdorff-Young inequality.

Lemma 7. *For a and f as in Lemma 6, set $\alpha_j = \sup\{|\hat{f}(x)| : 2^{j-1} \leqq a|x| \leqq 2^{j+1}\}$. Then $\sum_{j=0}^{\infty} \alpha_j{}^2 \leqq C$, where C is independent of a and f.*

Proof. Let ϕ be a C^∞ radial function which is unity for $1/2 \leqq |x| \leqq 2$ and vanishes for $|x| \leqq 1/4$ as well as $|x| \geqq 4$. We shall show $\sum_{j=0}^{\infty} \|\phi_j \hat{f}\|_\infty^2 \leqq C$ for $\phi_j(x) = \phi(2^{-j}ax)$.

According to the Sobolev inequality (see Herz [3])

$$\|\phi_j \hat{f}\|_\infty \leqq C \int \|\Delta_y{}^k(\phi_j \hat{f})\|_2 |y|^{-n/2-n}\, dy$$

where k is an integer greater than $n/2$.

For $0 < \epsilon < 1/(8k)$ set

$$\beta_j = \int_{|y| \geqq \epsilon a^{-1}} \|\Delta_y{}^k(\phi_j \hat{f})\|_2 |y|^{-n/2-n}\, dy$$

and

$$\gamma_j = \int_{|y| \leqq \epsilon a^{-1}} \|\Delta_y{}^k(\phi_j \hat{f})\|_2 |y|^{-n/2-n}\, dy;$$

we shall bound $\sum \beta_j{}^2$ and $\sum \gamma_j{}^2$.

Since $\|\Delta_y{}^k(\phi_j \hat{f})\|_2 \leqq C\|\phi_j \hat{f}\|_2$ we have $\beta_j \leqq Ca^{n/2}\|\phi_j \hat{f}\|_2$; thus $\sum \beta_j{}^2 \leqq Ca^n \sum \|\phi_j \hat{f}\|_2^2 \leqq Ca^n \|f\|_2^2 \leqq C$ by Lemma 6.

Leibnitz's rule for finite differences gives

$$\|\Delta_y{}^k(\phi_j \hat{f})\|_2 \leqq \sum_{i=0}^{k} C_{k,i} \|(\Delta_y{}^{k-i}\phi_j)(\cdot + iy))\Delta_y{}^i \hat{f}\|_2.$$

Note $|\Delta_y{}^{k-i}\phi_j(x + iy)| \leqq C(a|y|)^{k-i}$ since $j \geqq 0$; moreover, for $ka|y| \leqq 1/8$ this function vanishes unless $1/8 \leqq 2^{-j}a|x| \leqq 8$. Letting χ_j be the characteristic function of this set, we have

$$\|(\Delta_y{}^{k-i}\phi_j(\,\cdot + iy))\Delta_y{}^i \hat{f}\|_2 \leqq C(a|y|)^{k-i}\|\chi_j \, \Delta_y{}^i \hat{f}\|_2$$

for $|y| \leqq \epsilon a^{-1}$. We thus have

$$\gamma_j \leqq C \sum_{i=0}^{k} \delta_{ij} \text{ with } \delta_{ij} = a^{k-i} \int_{|y| \leqq \epsilon a^{-1}} \|\chi_j \Delta_y{}^i \hat{f}\|_2 |y|^{k-i-n/2-n}\, dy.$$

Then

$$\left(\sum_{j=0}^{\infty} \delta_{ij}^2 \right)^{1/2} \leqq Ca^{k-i} \int_{|y| \leqq \epsilon a^{-1}} \left(\sum_{j=0}^{\infty} \|\chi_j \Delta_y{}^i \hat{f}\|_2^2 \right)^{1/2} |y|^{k-i-n/2-n}\, dy$$

$$\leqq Ca^{k-i} \int_{|y| \leqq \epsilon a^{-1}} \|\Delta_y{}^i \hat{f}\|_2 |y|^{k-i-n/2-n}\, dy$$

$$\leqq Ca^{k-i} \int_{|y| \leqq \epsilon a^{-1}} a^{-n/2+i} |y|^{k-n/2-n}\, dy = C$$

by Lemma 6. Since $(\sum \gamma_j^2)^{1/2} \leqq C \sum_{i=0}^{k} (\sum_{j=0}^{\infty} \delta_{ij}^2)^{1/2}$, we are done.

4. **Proof of the main theorem.** Since T_m is linear it suffices to show $\|T_m f\|_p \leqq C$ whenever f is an atom; of course, C must be independent of the particular atom chosen. However, since T_m is translation invariant we may as well assume that f is supported in $[0, a]^n$ with $\|f\|_\infty \leqq a^{-n}$.

Thus, in addition to the estimates in Lemmas 6 and 7, $\hat{f}(0) = \int f = 0$ so that

$$(*) \qquad |\hat{f}(y)| = |\hat{f}(y) - \hat{f}(0)| \leqq \|\Delta_y \hat{f}\|_\infty \leqq Ca|y|.$$

First let us treat the case (i): $2 \leqq p \leqq \infty$. In view of the Hausdorff-Young inequality it suffices to show $\|m\hat{f}\|_p \leqq C$. Obviously

$$\|m\hat{f}\|_{p'}^{p'} = \int_{|x| \leqq a^{-1}} |m(x)\hat{f}(x)|^{p'}\, dx + \int_{|x| \geqq a^{-1}} |m(x)\hat{f}(x)|^{p'}\, dx;$$

the first term is bounded in view of (*) and Lemma 3. The technique used for the second term depends upon whether 1° or 2° is satisifed.

If 1° is satisfied we let J be the first integer such that $2^{J+1} > a^{-1}$; then

$$\int_{|x| \geqq a^{-1}} |m(x)\hat{f}(x)|^{p'}\, dx \leqq \sum_{j=0}^{\infty} \int_{2^{J+j} \leqq |x| \leqq 2^{J+j+1}} |m(x)\hat{f}(x)|^{p'}\, dx.$$

For $2^{J+j} \leqq |x| \leqq 2^{J+j+1}$ we have $2^{j-1} \leqq a|x| \leqq 2^{j+1}$ so that $|\hat{f}(x)| \leqq \alpha_j$. (See Lemma 7.)

Thus

$$\int_{|x| \geqq a^{-1}} |m(x)\hat{f}(x)|^{p'}\, dx \leqq \sum_{j=0}^{\infty} \alpha_j^{p'} \int_{2^{J+j} \leqq |x| \leqq 2^{J+\ \ +1}} |m(x)|^{p'}\, dx$$

$$= \sum_{j=0}^{\infty} \alpha_j^{p'} b_{J+j}^{p'}.$$

Hölder's inequality with exponents $2/p'$ and q/p' gives

$$\int_{|x| \geqq a^{-1}} |m(x)\hat{f}(x)|^{p'}\, dx \leqq \left(\sum_{j=0}^{\infty} \alpha_j^2 \right)^{p'/2} \left(\sum_{j=0}^{\infty} b_{J+1} \right)^{p'/q}$$

$$\leqq C \text{ by 1° and Lemma 7.}$$

When 2° is satisfied the argument is simpler. Choosing s so that $1/s + 1/q = 1/p'$, Hölder's inequality yields

$$\left(\int_{|x|\geqq} |m(x)\hat{f}(x)|^{p'}\,dx\right)^{1/p'} \leqq \|\hat{f}\|_s \left(\int_{|x|\geqq a^{-1}} |m(x)|^q\,dx\right)^{1/q}$$
$$\leqq Ca^{-n/s}\cdot a^{-n(1/p+1/q-1)} = C$$

by 2° and Lemma 6.

Now we turn to case (ii), where $1 < p < 2$. In view of Bernstein's theorem (see Herz [3]), it suffices to bound

$$\int \|\Delta_y{}^{2k}(m\hat{f})\|_2 |y|^{-n(1/p-1/2)-n}\,dy$$

where k is the integer in 3°. This is a slightly stronger bound than is necessary; it bounds the norm of $T_m f$ in the Lorentz space $L(p, 1) \subset L^p$. We estimate the integrals over $|y| \leqq \epsilon a^{-1}$ and $|y| \geqq \epsilon a^{-1}$ separately; it will be convenient to take $\epsilon = 1/(4k)$.

When $|y| \geqq \epsilon a^{-1}$ we have crudely $\|\Delta_y{}^{2k}(m\hat{f})\|_2 \leqq C\|m\hat{f}\|_2$; it will suffice to show $\|m\hat{f}\|_2 \leqq Ca^{-n(1/p-1/2)}$. Choose s so that $1/s + 1/q = 1/2$; we have then from Hölder's inequality

$$\|m\hat{f}\|_2^2 = \int_{|x|\leqq a^{-1}} |m(x)\hat{f}(x)|^2\,dx + \int_{|x|\geqq a^{-1}} |m(x)\hat{f}(x)|^2\,dx$$

$$\leqq \|\hat{f}\|_\infty^2 \int_{|x|\leqq a^{-1}} |m(x)|^2\,dx + \|\hat{f}\|_s^2 \left(\int_{|x|\geqq a^{-1}} |m(x)|^q\,dx\right)^{2/q}$$

The desired estimate follows at once from 2°, Lemma 4, and Lemma 6.

When $|y| \leqq \epsilon a^{-1}$ Leibnitz's rule yields

$$\|\Delta_y{}^{2k}(m\hat{f})\|_2 \leqq \sum_{j=0}^{2k} C_{k,j}\|\Delta_y{}^{2k-j}\hat{f}(\,\cdot\,+jy)\Delta_y{}^j m\|_2;$$

it suffices to bound each term in the sum by $Ca^\delta |y|^{n(1/p-1/2)+\delta}$ for some $\delta > 0$.

When $j \leqq k$ we estimate as above

$$\|\Delta_y{}^{2k-j}\hat{f}(\,\cdot\,+jy)\Delta_y{}^j m\|_2^2 \leqq \|\Delta_y{}^{2k-j}\hat{f}\|_\infty^2 \int_{|x|\leqq a^{-1}} |\Delta_y{}^j m(x)|^2\,dx$$

$$+ \|\Delta_y{}^{2k-j}\hat{f}\|_s^2 \left(\int_{|x|\geqq a^{-1}} |\Delta_y{}^j m(x)|^q\,dx\right)^{2/q}.$$

Since $j|y| < (2a)^{-1}$ we have

$$\left(\int_{|x|\leqq a^{-1}} |\Delta_y{}^j m(x)|^2\, dx\right)^{1/2} \leqq C\left(\int_{|x|\leqq 2a^{-1}} |m(x)|^2\, dx\right)^{1/2}$$
$$\leqq Ca^{-n(1/p-1/2)}$$

and

$$\left(\int_{|x|\geqq a^{-1}} |\Delta_y{}^j m(x)|^q\, dx\right)^{1/q} \leqq C\left(\int_{|x|\geqq (2a)^{-1}} |m(x)|^q\, dx\right)^{1/q}$$
$$\leqq Ca^{-n(1/p+1/q-1)}$$

by Lemma 4 and 2°. The estimates for $\hat{f}$ in Lemma 6 yield the desired conclusion; note $2k - j \geqq k > n(1/p - 1/2)$.

The simplest case occurs when $k + 1 \leqq j \leqq 2k - 1$. We have

$$\|\Delta_y{}^{2k-j}\hat{f}(\,\cdot\, + jy)\Delta_y{}^j m\|_2 \leqq \|\Delta_y{}^{2k-j}\hat{f}\|_\infty \|\Delta_y{}^j m\|_2$$
$$\leqq C\|\Delta_y \hat{f}\|_\infty \|\Delta_y{}^k m\|_2 \leqq Cay^{1+n(1/p-1/2)}$$

by Lemmas 5 and 6.

When $j = 2k$ we have

$$\|\hat{f}(\,\cdot\, + 2ky)\Delta_y{}^{2k} m\|_2 \leqq \|\Delta_{2ky}\hat{f}\Delta_y{}^{2k} m\|_2 + \|\hat{f}\Delta_y{}^{2k} m\|_2;$$

the first of these can be treated like the term for $j = 2k - 1$. For the second, we have

$$\|\hat{f}\Delta_y{}^{2k} m\|_2^2 = \int_{|x|\leqq a^{-1}} |\hat{f}(x)\Delta_y{}^{2k} m(x)|^2\, dx$$
$$+ \int_{|x|\geqq a^{-1}} |\hat{f}(x)\Delta_y{}^{2k} m(x)|^2\, dx$$
$$\leqq Ca^2 \int_{|x|\leqq a^{-1}} |x|^2 |\Delta_y{}^{2k} m(x)|^2\, dx$$
$$+ \|\hat{f}\|_\infty^2 \int_{|x|\geqq a^{-1}} |\Delta_y{}^{2k} m(x)|^2\, dx$$

by (*). Lemma 5, parts (a) and (b), and Lemma 6 conclude the proof.

5. **Example.** The function $m(x) = |x|^{-n/p'}$ satisfies 2° for $1 < p \leqq \infty$, 3° for $1 < p < 2$, and 1° for $p = 2$ but not for $p > 2$.

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