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THE DISAPPEARANCE OF SOLITARY TRAVELLING WAVE SOLUTIONS OF A MODEL OF THE BELOUSOV-ZHABOTINSKII REACTION

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ABSTRACT. We investigate a system of non-linear partial differential equations which describe spatial structure in the Belousov-Zhabotinskii chemical reaction. For various concentrations of reactants the Belousov-Zhabotinskii reaction exhibits periodic travelling waves of chemical activity, solitary travelling waves, or no waves whatsoever. It has been previously proven that there are ranges of values of f, the stoichiometric factor, over which the system has a solitary travelling wave solution and periodic travelling wave solutions. We show here that over still another range of values of f, the system cannot have a travelling wave solution.

1. **Introduction.** The Belousov-Zhabotinskii reaction is the only known chemical reaction which exhibits both temporal oscillations and spatial structure. This system is the metal ion catalyzed oxidation by Bromate ion (BrO_3^-) of easily brominated organic materials.

Temporal oscillations, first reported by Belousov [1], occur in the ratio of [Ce(IV)]/[*Ce*(III)]. A redox indicator such as Ferroin is often used to make the oscillations visible as sharp color changes. The periods of the oscillations may vary from seconds to minutes and they can persist for several hours since each cycle of the catalyst consumes very little of the principal reactants.

In 1970 Zaikin and Zhabotinskii reported the existence of travelling waves of chemical activity in a two dimensional system consisting of reagent spread in a thin layer over a flat surface such as a petri dish [9].

Winfree [7] showed experimentally that the two dimensional waves are of two general types. In the first case the reagent is oscillatory in time and the waves result from continuous phase gradients. These waves, called phase waves, are diffusion independent and appear to pass through impermeable barriers. The second variety of wave, called trigger waves, appear to be propagated by a reaction-diffusion mechanism. The reagent need not exhibit temporal oscillations for trigger waves to appear, and they are most striking when observed in such a solution. For example, as reported by Winfree [8],

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with a little less acid and a little more bromide in the solution, the temporal oscillations disappear without affecting its ability to conduct waves of chemical activity. What remains is a red solution "doing nothing." However, when it is stimulated by a droplet of a blue wave from another solution or by the touch of a heated needle, a single sharp blue ring propagates through the red medium at a steady rate of a few millimeters per minute.

In this paper we present a reaction-diffusion equation, coupled non-linearly with two first order equations which model the spatial case. Field and Troy [5] have proven that there are solitary travelling wave solutions of the model over an appropriate range of physical parameters. We prove here that there is also a range of physical parameters over which the equation has no solitary travelling wave solutions.

In the next section we present the model. § 3 contains a statement of the main result and the proof is given in § 4.

2. **The Model.** Field and Noyes [3] extracted the following simple model of the reaction:

$$A + Y \rightarrow X \tag{1}$$

$$X + Y \rightarrow P \tag{2}$$

$$B + X \rightarrow 2X + Z \tag{3}$$

$$2X \rightarrow Q \tag{4}$$

$$Z \rightarrow fY \tag{5}$$

Here, X, Y and Z represent the concentrations of $HBrO_2$ (Bromus acid), Br^- (Bromide ion) and Ce (IV), respectively. A and B denote the concentrations of the reactant (BrO_3^-), P and Q are products, and f is the stoichiometric factor.

The kinetic behavior of reactions (1)–(5) in a continuously stirred solution is described by the associated system of ordinary differential equations which result from an application of the law of mass action to reactions (1)–(5). These equations are

$$\frac{dX}{d\zeta} = k_1 AY - k_2 XY + k_3 BX - 2k_4X^2 \tag{6}$$

$$\frac{dY}{d\zeta} = -k_1 AY - k_2 XY + fk_5 Z \tag{7}$$

$$\frac{dZ}{d\zeta} = k_3 BX - k_5 Z \tag{8}$$

where ζ represents time and $k_1 - k_5$ are the reaction rates for reactions

(1)–(5), respectively. Field and Noyes [3] assume that $A = B = .06M$ and remain constant. This makes the system (6)–(8) effectively open. By analogy with the chemistry they assigned the numerical values

$$k_1 = 1.34\, M^{-1}\,\mathrm{sec}^{-1},$$
$$k_2 = 1.6 \times 10^9\, M^{-1}\,\mathrm{sec}^{-1},$$
$$k_3 = 8 \times 10^3\, M^{-1}\,\mathrm{sec}^{-1},$$
$$k_4 = 4 \times 10^7\, M^{-1}\,\mathrm{sec}^{-1}.$$

The values of f and k_5 are considered expendable although chemically reasonable arguments can be given which indicate that k_5 is small. Equations (6)–(8) are most easily handled analytically by a transformation into the system

$$\frac{dx}{d\tau} = s(y - xy + x - qx^2) \equiv sF(x, y) \tag{9}$$

$$\frac{dy}{d\tau} = \frac{1}{s}(-y - xy + fz) \equiv \frac{1}{s}\, G(x, y, z) \tag{10}$$

$$\frac{dz}{d\tau} = w(x - z) \tag{11}$$

where

$$X = \frac{k_1 A}{k_2} x,\quad Y = \frac{k_3 B}{k_2}\, y,\quad Z = \frac{k_1 k_3 AB}{k_2 k_5}\, z,$$
$$\tau = \zeta/(k_1 k_3 AB)^{1/2} = .161\, \zeta$$
$$s = (k_3 B/k_1 A)^{1/2} = 77.27$$
$$w = k_5/(k_1 k_3 AB)^{1/2} = .161\, k_5$$
$$q = \frac{2k_1 k_4 A}{k_2 k_3 B} = 8.375 \times 10^{-6}.$$

The physically reasonable region is the positive octant and the only steady state solution of (9)–(11) which lies in the positive octant is given by

$$x_0(f) = (1 - f - q + ((1 - f - q)^2 + 4q(1 + f))^{1/2})/(2q) \tag{12}$$

$$y_0(f) = f x_0(f)/(1 + x_0(f)) \tag{13}$$

$$z_0(f) = x_0(f). \tag{14}$$

Linearizing (9)–(11) about (x_0, y_0, z_0) we obtain the equation $u' = A_w u$ where

$$(15)\quad A_w = \begin{pmatrix} sF_x & sF_y & 0 \\ \frac{1}{s}G_x & \frac{1}{s}G_y & \frac{1}{s}G_z \\ w & 0 & -w \end{pmatrix} \Bigg|_{(x,y,z)=(x_0,y_0,z_0)}$$

The characteristic equation associated with A_w is given by

$$(16)\quad \begin{aligned} &\lambda^3 + \left(w - sF_x - \frac{1}{s}Gy\right)\lambda^2 + \left(F_xG_y - F_yG_x - wsF_x - \frac{w}{s}G_y\right)\lambda \\ &\qquad + w(F_xG_y - F_yG_x - F_yG_z) = 0. \end{aligned}$$

The stability of the steady state solution of (9)–(11) depends on the nature of the roots of (16). If each root has negative real part then the steady state is locally asymptotically stable. However, if at least one root has positive real part then the steady state is unstable and one may expect oscillations to appear. Field and Noyes [3], and subsequently Hastings and Murray [4] computed the region of the (f, k_5) plane over which the steady state is unstable. In particular, with $f = 1$ and $k_5 = 1$, Field and Noyes [3] computed what appears to be a limit cycle solution of (9)–(11). Hastings and Murray [4] showed analytically that if the steady state solution is unstable then (9)–(11) has at least one periodic solution.

Our interest is in the region of the (f, k_5) plane where the steady state is stable. Numerically, for $0 < f < .5$ or $f > 1 + \sqrt{2}$, all roots of (16) have negative real parts (for all $k_5 > 0$) and the steady state solution is stable to small perturbations. For sufficiently large or small values of f, Troy and Field [6] have shown analytically that the steady state solution is globally stable to any initial perturbation which lies in the positive octant. That is, all solutions with initial values in the positive octant return to the steady state solution. This means that there can be no periodic solutions and the reagent remains red.

Next we discuss the spatial case in which the *unstirred* reagent is spread in a thin layer over a flat surface. In particular we wish to investigate trigger waves. As pointed out by Field [2], the most important contribution to the propagation of a trigger wave of chemical activity is the autocatalytic formation (step (3)) of x (Bromous Acid), and its subsequent diffusion through the medium. Thus one sees a blue solitary plane wave diffusing through the red reagent. We model this phenomenon with the system

(17) $$\frac{\partial x}{\partial \tau} = sF(x, y) + D_x \frac{\partial^2 x}{\partial u^2}$$

(18) $$\frac{\partial y}{\partial \tau} = \frac{1}{s} G(x, y, z)$$

(19) $$\frac{\delta z}{\delta \tau} = w(x - z)$$

where F and G are defined in (9) and (10), and u denotes the one dimensional space variable.

A travelling wave solution of (17)–(19) is a solution of the form $(x(u/\alpha + \tau), y(u/\alpha + \tau), z(u/\alpha + \tau))$ and which, upon substitution into (17)–(19), leads to the system of ordinary differential equations

(20) $$\frac{dx}{dt} = sF(x, y) + \frac{1}{\theta}\frac{d^2x}{dt^2} \quad ,$$

(21) $$\frac{dy}{dt} = \frac{1}{s} G(x, y, z),$$

(22) $$\frac{dz}{dt} = w(x - z),$$

where $t = u/\alpha + \tau$ and $\theta = \alpha^2/D_x$.

Next, transforming (20)–(22) into a system of first order equations, we obtain

(23) $$\frac{dx}{dt} = v,$$

(24) $$\frac{dv}{dt} = \theta[v - sF(x, y)],$$

(25) $$\frac{dy}{dt} = \frac{1}{s} G(x, y, z)$$

(26) $$\frac{dz}{dt} = w(x - z).$$

It is easily seen that the only steady state solution of (23)–(26) which lies in the region $x > 0$, $y > 0$, $z > 0$ is given by $\Pi_0 = (x_0(f), 0, y_0(f), z_0(f))$.

3. **Statement of Results.** A solitary travelling wave solution of (23)–(26) is a solution $\Pi(t) = (x(t), v(t), y(t), z(t))$ which is non-constant and satisfies $\lim_{t\to\pm\infty}\Pi(t) = \Pi_0$.

In addition we require that $x > 0$, $y > 0$, $z > 0$ for all $t > 0$ since chemicals cannot have negative concentrations.

Field and Troy [5] have proven that there is a range of values of f and q in the region $f > 1 + \sqrt{2}$ and $q \in (0, 1)$ over which the system (23)–(26) has a solitary travelling wave solution satisfying $1 < x(t), z(t), < 1/q$ and $y(t) > 0$ for all $t > 0$. Numerical calculations indicate that for values of f and q in this region the steady state (x_0, y_0, z_0) of the kinetic equations is globally asymptotically stable.

In this paper we prove the following

THEOREM. *There is an open interval* $(f_1, f_2) c (1 + \sqrt{2}, +\infty)$ *such that for each* $f \in (f_1, f_2)$ *there is a value* $w_f > 0$ *such that if* $0 < w < w_f$ *then the system* (23)–(26) *has no non-constant solitary wave solution* $\Pi(t)$ *which remains in the region* $x > 0$, $y > 0$, $z > 0$ *for all* $t > 0$ *and satisfies* $\lim_{t \to \pm\infty} \Pi(t) = \Pi_0$.

4. **Outline of Proof.** We first consider the reduced system derived from (23)–(26) by setting $w = 0$ and keeping $z \equiv x_0(f)$, its steady state solution.

An analysis of the steady state solution $(x_0, 0, y_0)$ of the reduced system shows that there is a one dimensional unstable manifold of solutions. We project the unstable manifold onto the x, y plane and compute the sign of its components. Then, our analysis shows that the full system (23)–(26) also has a one dimensional unstable manifold $\gamma_{w,\theta}$ of solutions which tend to Π_0 as $t \to -\infty$ for $w > 0$ sufficiently small.

We let the initial value $\Pi(0)$ of a solution lie on $\gamma_{w,0} - \{\Pi_0\}$ and use an energy method to show that either (i) $x(t)$ enters into the region $x < 0$ or (ii) $x(t)$ crosses the line $x = 1/q$ and $\lim_{t \to \infty} x(t) = +\infty$. Thus cases (i) and (ii) show that the system (23)–(26) cannot have a physically meaningful solitary travelling wave solution.

PROOF OF THEOREM. Before proceeding with the proof of our theorem we first analyze the properties of the reduced system obtained from (23)–(26) by setting $w = 0$ and keeping $z \equiv x_0(f)$, its steady state value. That is, we consider

$$\frac{dx}{dt} = v, \tag{27}$$

$$\frac{dv}{dt} = \theta[v - sF(x, y)], \tag{28}$$

$$\frac{dy}{dt} = \frac{1}{s} G(x, y, z_0). \tag{29}$$

From (9) and (10) we obtain

$$F(x, y) = 0 \Longleftrightarrow y = \frac{qx^2 - x}{1 - x} \equiv h(x) \tag{30}$$

and

$$G(x, y, z_0) = 0 \Longleftrightarrow y = \frac{fz_0}{1 + x} \equiv k(x, z_0). \tag{31}$$

It follows from a consideration of (27)–(31) that there is a steady state solution of (27)–(29) corresponding to each positive solution of the equation

$$x^3 + \frac{(q-1)}{q}x^2 + \frac{(fx_0(f) - 1)x}{q} - \frac{fx_0(f)}{q} = 0. \tag{32}$$

Troy and Field [6] have analyzed (32) and have shown that there is an open interval $(f_1, f_2) \subset (1, \infty)$ such that for each $f \in (f_1, f_2)$, (32) has three positive solutions $x_0(f)$, $\mu(x_0(f))$, $\lambda(x_0(f))$ which satisfy

$$1 < x_0(f) < \mu(x_0(f) \leqq \lambda(x_0(f)) < 1/q, \tag{33}$$

and

$$\int_{x_0(f)}^{x} F(\mu, k(\mu, x_0(f)))\, d\mu < 0 \tag{34}$$

for each $x > x_0(f)$. In addition, with $(x, y, z) = (x_0(f), y_0(f), z_0(f))$, the functions F and G satisfy

$$F_x < 0,\ F_y < 0,\ G_x < 0,\ G_y < 0 \tag{35}$$

and

$$F_x/F_y > G_x/G_y. \tag{36}$$

Next, linearizing (27)–(29) around the steady state solution $(x_0(f), 0, y_0(f))$, we obtain the linear equation $d\mu/dt = A_0\mu$ where

$$A_0 = \left.\begin{pmatrix} 0 & 1 & 0 \\ -\theta s F_x & \theta & -\theta s Fy \\ \frac{1}{s} G_x & 0 & \frac{1}{s} Gy \end{pmatrix}\right|_{(x, y, z) = (x_0(f), y_0(f), z_0(f))}$$

The characteristic equation associated with A_0 is given by

$$\lambda^3 - (\theta + G_y/s)\lambda^2 + \theta(G_y/s + sF_x)\lambda + \theta(F_yG_x - F_xG_y) = 0. \tag{38}$$

From (33)–(36) it follows that (38) has one positive eigenvalue, λ_1, while the other two eigenvalues have negative real parts. Let $\bar{a} = (u_1, u_2, u_3)^t$ be a non-zero eigenvector of A_0 corresponding to the positive eigenvalue λ_1. Then the equation $A_0\bar{a} = \lambda_1\bar{a}$ implies that

(39) $u_1 > 0, u_2 > 0, u_3 < 0, u_3/u_1 > (-G_x/G_y)|_{(x,y,z)=(x_0,y_0,z_0)}$

Thus there is a one dimensional unstable manifold of solutions which tend to $(x_0, 0, y_0)$ as $t \to -\infty$, and a two dimensional stable manifold of solutions which tend to $(x_0, 0, y_0)$ as $t \to +\infty$.

We now return to the full system (23)–(26) for non zero values of w. The linearization of (23)–(26) about the steady state $\Pi_0 = (x_0, 0, y_0, x_0)$ is given by $du/dt = A_w u$ where

$$(40)\quad A_w = \left.\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\theta s F_x & \theta & -\theta s F_y & 0 \\ \frac{1}{s} G_x & 0 & \frac{1}{s} G_y & \frac{1}{s} G_z \\ w & 0 & 0 & -w \end{pmatrix}\right|_{x, v, y, z) = (x_0 0, y_0, z_0)}$$

The characteristic equation associated with A_w is given by $\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0$ where

$$c_1 = w - \theta - \frac{1}{s} G_y,$$

$$c_2 = \theta\left(sF_x + \frac{1}{s}G_y - \frac{w}{s}G_y - w\right),$$

$$c_3 = \theta\left(F_y G_x - F_x G_y + ws\, F_x + \frac{w}{s} Gy\right)$$

$$c_4 = w\theta(F_y G_x - F_x G_y + F_y G_z).$$

It follows from consideration of the case $w = 0$, and properties (33)–(36) that for each $f \in (f_1, f_2)$ there is a value $w_f > 0$ such that if $0 < w < w_f$ then one eigenvalue of A_w is positive and the other three eigenvalues have negative real parts. We assume hereafter that $f \in (f_1, f_2)$ and $0 < w < w_f$ are held fixed. Thus, there is a one dimensional unstable manifold $\gamma_{w,\theta}$ of solutions which tend to Π_0 as $t \to -\infty$ and a three dimensional stable manifold of solutions tending to Π_0 as $t \to +\infty$.

Let $\bar{u} = (u_1, u_2, u_3, u_4)^t$ be a nonzero eigenvector of A_w corresponding to the positive eigenvalue $\lambda_1 = \lambda_1(w)$. Then from the equation $A_w\bar{u} = \lambda_1\bar{u}$ we obtain the system

$$(41)\quad u_2 = \lambda_1 u_1,$$

$$(42)\quad -\theta s F_x u_1 + \theta u_2 - \theta s F_y u_3 = \lambda_1 u_2,$$

$$\frac{1}{s}G_x u_1 + \frac{1}{s}G_y u_3 + \frac{1}{s}G_z u_4 = \lambda_1 u_3, \tag{43}$$

$$w u_1 - w u_4 = \lambda_1 u_4, \tag{44}$$

It follows from (41)–(44) that if $u_1 > 0$ then

$$u_2 > 0,\, u_3 < 0,\, u_3/u_1 > (-G_x/G_y)|_{(x,y,z)=(x_0,y_0,z_0)}$$

and $0 < u_4 < u_1$. Therefore there is a component $\gamma^+_{w,\theta}$ of $\gamma_{w,\theta} - \{\Pi_0\}$ which points into the region $0 < z - z_0 < x - x_0$, $y - y_0 < 0$, $(y - y_0)/(x - x_0) > -G_x/G_y$, and $v > 0$. In addition, substitution of $u_2 = \lambda_1 u_1$ in (42) together with (36), (43) and (44) lead to

$$\lambda_1{}^2 > \theta s(x_0 - 1)(k_x(x, x_0) - h'(x))|_{x=x_0} \tag{45}$$

Next, consider the energy function defined by

$$M = \frac{v^2}{2} + \theta s \int_{x_0(f)}^{x} F(u, k(u, x_0))\, du \tag{46}$$

where $x \geqq x_0$ and $v \geqq 0$.

Define the function

$$1(x) = -2\theta s \int_{x_0(f)}^{x} F(u, k(u, x_0))\, du. \tag{47}$$

Then, from the definitions of F and k we obtain

$$1(x_0) = \frac{dl}{dx}(x_0) = 0 \tag{48}$$

and

$$\frac{d^2 1}{dx^2}(x_0) = -2\theta s(F_x + F_y k_x)|_{(x,y,z)=(x_0,y_0,z_0)} \tag{49}$$

From (46) it follows that

$$M = 0 \Longleftrightarrow v = (1(x))^{1/2} \tag{50}$$

where $x \geqq 0$ and $v \geqq 0$. (See Figure 1 below).

We wish to show that the projection of $\gamma^+_{w,\theta}$ onto the (x, v) plane points into the region $x > x_0$, $v > 1(x)$. Since $\gamma^+_{w,\theta}$ is tangent to the eigenvector $\bar{u}$ at $x = x_0$ and $v > \ell(x)$ it suffices to show that $\bar{u}$ points into the region $x > x_0$ and $v > 1(x)$.

From (41) it follows that the projection of $\gamma^+_{w,\theta}$ onto the (x, v) plane lies along the straight line $g(x) = \lambda_1(x - x_0)$ which passes through the point $(x_0, 0)$ and has slope λ_1. Since $g^2(x) > 0$ and $1^2(x) > 0$ for all

$x > x_0$ we need only show that $g^2(x) > 1^2(x)$ for $(x - x_0) > 0$ and sufficiently small. Note that

$$g(x_0) = \frac{d}{dx}(g(x_0)) = 1(x_0) = \frac{d}{dx}(1(x_0)) = 0.$$

Thus we need to show that

$$\left.\frac{d^2g^2(x)}{dx^2}\right|_{x=x_0} > \left.\frac{d^2 1(x)}{dx^2}\right|_{x=x_0} \tag{51}$$

From the definition of $g(x)$ it follows that

$$\left.\frac{d^2g^2(x)}{dx^2}\right|_{x=x_0} = 2\lambda_1{}^2. \tag{52}$$

From (47), (9), and (31) we obtain (49).

From (10), (45), (49), and (52) we obtain (51), the desired result.

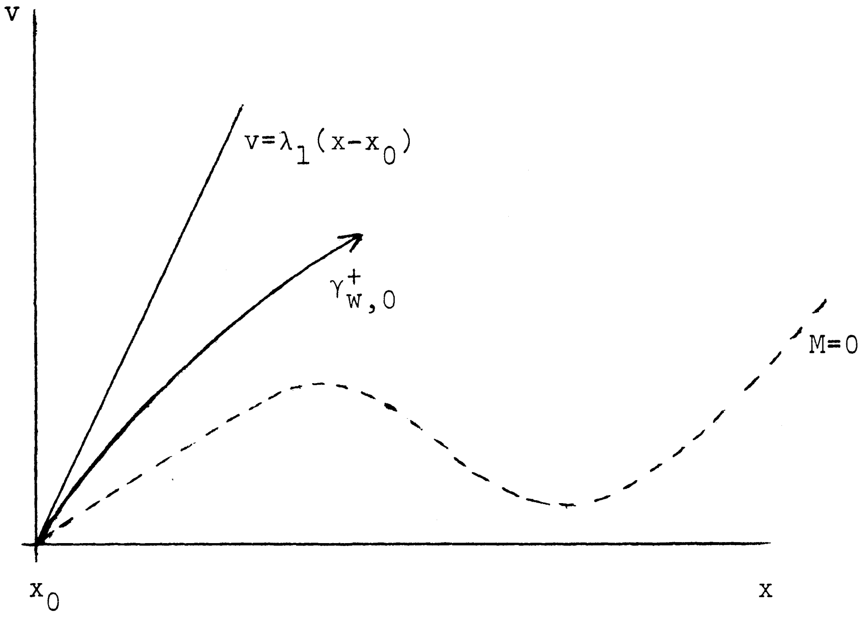

Figure 1. The dotted curve is the curve $M = 0$. The curve with the arrow represents the unstable manifold which must be tangent to the straight line $v = \gamma_1(x - x_0)$.

We may assume, therefore, that $\Pi(0) \in \gamma_{w,\theta}^{+}$. This assures that $\lim_{t\to-\infty}\Pi(t) = \Pi_0$, a condition which a solitary travelling wave solution must satisfy. Since $\gamma_{w,\theta}^{+}$ points into the region $x > x_0$ and $v > 1(x)$ it follows from (46) that we may assume that $M(0) > 0$.

Suppose now that there is a first $T > 0$ where $M(T) = 0$. Then

$$\dot{M}(T) \leqq 0. \tag{53}$$

From (46), and (23)–(26) we obtain

$$\begin{aligned}\frac{dM}{dt}(T) &= v(T)[\theta v(T) - \theta sF(x(T), y(T)) \\ &\qquad + \theta sF(x(T), k(x(T), x_0))] \\ &= \theta(v^2(T) - v(T)(y(T) - k(x(T), x_0)(1 - x(T))s).\end{aligned}$$

Note that $v(T) > 0$ since the curve $M = 0$ is entirely contained in the positive octant for all $x > x_0$. Therefore, if we show that $y(T) - k(x(T), x_0) \geqq 0$ then we obtain a contradiction to (53). Suppose that there is a first $\hat{t} \in (0, T)$ where $y(\hat{t}) - k(x(\hat{t}), x_0) = 0$. Then, since $y(0) > k(x(0), x_0)$ it follows that

$$\frac{d}{dt}(y(t) - k(x(t), x_0))\Big|_{t=t} \leqq 0. \tag{54}$$

However, since $x(t) > x_0$ for all $t \in [0, \hat{t}]$ then the comments following (44), together with (26), imply that $z(t) > x_0$ for all $t \in [0, \hat{t}]$. Hence

$$k(x(\hat{t}), x_0) = \frac{fx_0}{1 + x(t)} < \frac{fz(\hat{t})}{1 + x(\hat{t})} = k(x(\hat{t}), z(\hat{t})).$$

Thus, since $y(\hat{t}) = k(x(\hat{t}), x_0) < k(x(\hat{t}), z(\hat{t}))$ then (25) implies that $\ddot{y}(\hat{t}) > 0$. Also, from the definition of k it follows that $k_x < 0$. Therefore, $d/dt(y(\hat{t}) - k(x(\hat{t}), x_0)) = \dot{y}(\hat{t}) - k_x\dot{x}(\hat{t}) > 0$, contradicting (55). Therefore we must conclude that $M(t) > 0$ for all $t \geqq 0$ which implies that $\dot{x}(\hat{t}) > 0$ for all $t \geqq 0$ and $\lim_{t\to\infty}x(t)$ exists. If $\lim_{t\to\infty}x(t)$ is finite then there is a steady state solution of (23)–(26) in the region $x > x_0$, a contradiction. Therefore $\lim_{t\to\infty}x(t) = +\infty$ and $\Pi(t)$ cannot return to Π_0 as $t \to \infty$.

Next, we assume that $\Pi(0)$ lies on the component $\gamma_{w,\theta}^{-} - \{\Pi_0\}$ of $\gamma_{w,\theta} - \{\Pi_0\}$ which points into the region $(y - y_0)/(x - x_0) > -G_x/G_y$, $v < 0$ and $x - x_0 < z - z_0 < 0$.

Suppose that there is a first $t > 0$ where $\dot{x}(t) = 0$ and $0 \leqq x(t) < x_0$. Then $\ddot{x}(t) \geqq 0$ and, from (24), we conclude that $F(x(t), y(t)) \leqq 0$. Thus $y(t) - k(x(t), x_0) > 0$. Since the unstable manifold points into the region $x < x_0$, $y < k(x, x_0)$ then there must be a first $\hat{t} \in (0, t)$ with $y(\hat{t}) - k(x(\hat{t}), x_0) = 0$ and

$$\frac{d}{dt}(y(\hat{t}) - k(x(\hat{t}), x_0) \geqq 0. \tag{55}$$

Again, $z(\hat{t}) < x_0$ and, since $k_z > 0$ then $k(x(\hat{t}), z(\hat{t})) < k(x(\hat{t}), x_0)$ and it follows from (25) that $\dot{y}(\hat{t}) < 0$.

Therefore, since $k_x < 0$ for all $x > 0$, we obtain $d/dt(y(\hat{t}) - k(x(\hat{t}), x_0) = \dot{y}(\hat{t}) - k_x\dot{x}(\hat{t}) < 0$, contradicting (55). This implies that any solution $\Pi(t)$ with $\Pi(0) \in \gamma_{\bar{w},\theta}^{-}$ must enter the region $x < 0$ and therefore cannot represent a physically meaningful solitary travelling wave solution.

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