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STABILITY THEOREMS AND HYPERBOLICITY IN DYNAMICAL SYSTEMS

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ABSTRACT. We present an approach to proving structural stability and semi-stability theorems for diffeomorphisms and flows using the idea of shadowing an ϵ-chain. We treat the cases when the whole manifold is hyperbolic and when the chain recurrent set is hyperbolic and the strong transversality condition is satisfied. The last two sections discuss the progress made on the converse to the structural stability theorem and a criterion for hyperbolicity.

0. Introduction. The first section presents the shadowing and stability results for a diffeomorphism near a single hyperbolic set. This section is very much in the spirit of C. Conley, [4] and [5]. The second section shows how these results can be carried over to several hyperbolic sets when the strong transversality condition is satisfied. The semi-stability theorem for diffeomorphisms is discussed in some detail in § 2. With this new proof the result carries over to flows as mentioned in § 3. The fourth section reviews the results of R. Mañé and V. Pliss on the converse to the structural stability theorem. Finally, the last section discusses a criterion for hyperbolicity; namely, a linear bundle isomorphism is hyperbolic if the map on the base space is chain-recurrent and the zero section is an isolated invariant set.

The results of § 1 through § 3 assume the sets are compact. We really only need uniform hyperbolicity and uniformly continuous derivatives. J. Hale has been asking how much of this theory carries over to functional differential equations and flows on infinite dimensions, [12]. In this context, see the papers of W. Oliva [24], D. Henry [13], and J. Montgomery [19].

An invariant set $\Lambda \subset M$ is called *hyperbolic* for $f: M \to M$ if there exist a continuous invariant splitting $TM \mid \Lambda = E^u \oplus E^s$ and constants $C > 0$ and $0 < \lambda < 1$ such that $\|Tf^n \mid E_x^s\| < C\lambda^n$ and $\|Tf^{-n} \mid E_x^u\| < C\lambda^n$ for $x \in \Lambda$ and $n \geqq 0$. By averaging the metric, we can take $C = 1$. This is called an adapted metric. See [22]. We use adapted metrics in this paper. If U is a small neighborhood of Λ, then the splitting extends to U, $TM \mid U = E^u \oplus E^s$. If $x, f(x) \in U$, $T_x f$ can be written as

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$$T_x f = \begin{pmatrix} A_x^{uu} & A_x^{su} \\ A_x^{us} & A_x^{ss} \end{pmatrix}$$

where $A_x^{\sigma\delta} : E_x^\sigma \to E_{f(x)}^\delta$. If U is small enough then $\|A_x^{ss}\| < \lambda$, $\|(A_x^{uu})^{-1}\| < \lambda$, $\|A_x^{us}\| < \epsilon$, and $\|A_x^{su}\| < \epsilon$. If U is such a neighborhood and $\lambda + \epsilon < 1$, then we say that U is set where f is hyperbolic. (Actually there exists an invariant splitting over U but the stable manifold theory only needs an approximate splitting.)

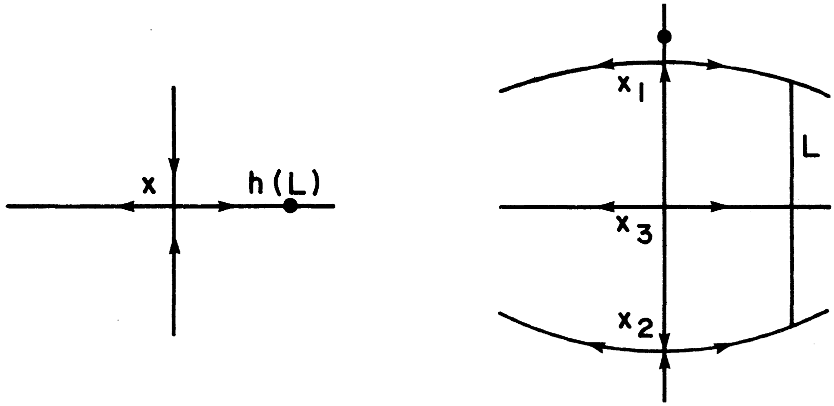
1. **A single hyperbolic set for a diffeomorphism.** In this section we prove the stability near a single hyperbolic set using the result of shadowing an ϵ-chain. R. Bowen, [2] proved the shadowing result using the stable manifold theorem. Here we construct unstable disks (like unstable manifolds) for the ϵ-chain and these easily prove the results. This proof is more readily adapted to the situations of the next two sections. It is very similar to that of C. Conley, [5]. R. McGehee has this type of proof of the stable manifold theorem in [18]. Also see [6] for the shadowing theorem.

Let $f : M \to M$ be a C^1 diffeomorphism on a compact manifold M. An (infinite) *ϵ-chain* is a doubly infinite sequence of points, $\{x_i : i \in Z\}$, such that $d(f(x_i), x_{i+1}) < \epsilon$. R. Bowen called these pseudo-orbits.

THEOREM (Shadowing). *Let $U \subset M$ be a region where f is hyperbolic. Given $\delta > 0$ there exists $\epsilon > 0$ such that if $\{x_i\}$ is an ϵ-chain in U, then there exists a unique y such that $d(f^i(y), x_i) < \delta$ for all $i \in Z$, i.e. every ϵ-chain is δ-shadowed by an f-orbit.*

PROOF. The hyperbolicity gives an almost invariant splitting $E^u \oplus E^s$ of $TM \mid U$. For each $x \in U$ take a cross product of $\delta/2$-disks in E_x^u and E_x^s. Exponentiating these down into M, this gives a neighborhood of x, $B(x) \subset M$. If δ is small enough the nonlinear map f on $B(x)$ is close to the linear map $T_x f$, more precisely to $\exp T_x f \exp_x^{-1} : B(\delta) \to M$. If ϵ is small enough and x_i is an ϵ-chain in U, then $fB(x_{j-1})$ stretches across the box $B(x_j)$ in the unstable direction and is contracted in the stable direction. Then $f^2B(x_{j-2}) \cap fB(x_{j-1}) \cap B(x_j) = f\{fB(x_{j-2}) \cap B(x_{j-1})\} \cap B(x_j)$ is an even thinner strip that crosses $B(x_j)$ in the unstable direction. Continuing, we get that $\bigcap \{f^nB(x_{j-n}) : n \geqq 0\} = D^u(x_j, \{x_i\})$ is a disk of dimension equal E_x^u that stretches across $B(x_j)$ in the unstable direction. (It does not necessarily contain x_j). We call this disk the unstable disk at x_j for the ϵ-chain. A point y is in $D^u(x_j, \{x_i\})$ if and only if $y \in f^nB(x_{j-n})$ for all $n \geqq 0$, i.e.

$f^{-n}(y) \in B(x_{j-n})$ for all $n \geqq 0$. These are the points y whose backward orbit shadows the ϵ-chain.



The stable manifold theory can show that these disks are C^1 and are almost tangent to $E_x{}^u$. The map f expands $D^u(x_j, \{x_i\})$ across $D^u(x_{j+1}, \{x_i\})$. Therefore $f^{-1}: D^u(x_{j+1}, \{x_i\}) \to D^u(x_j, \{x_i\})$ is a uniform contraction. By the contraction mapping principle, $\bigcap\{f^{-n}D^u(x_n, \{x_i\}) : n \geqq 0\}$ is a unique point y. The point y is the unique point such that $y \in f^{-n}B(x_n)$ for all n, i.e. such that $f^n(y) \in B(x_n)$, i.e. such that the orbit of y shadows the ϵ-chain.

Corollary (Expansiveness). *If $x, y \in U$ and $d(f^n(x), f^n(y)) < \delta$ for all $n \in Z$, then $x = y$.*

Proof. The orbit $x_n = f^n(x)$ is an ϵ-chain. Both x and y have orbits which δ-shadow it. By uniqueness $x = y$.

We define the chain recurrent set of f, $\mathcal{R}(f)$, to be the points $x \in M$ such that there is a periodic ϵ-chain through x for all $\epsilon > 0$, i.e. for every $\epsilon > 0$ there is an ϵ-chain $\{x_i\}$ such that $x_0 = x$ and there is an n with $x_{i+n} = x_i$ for all i.

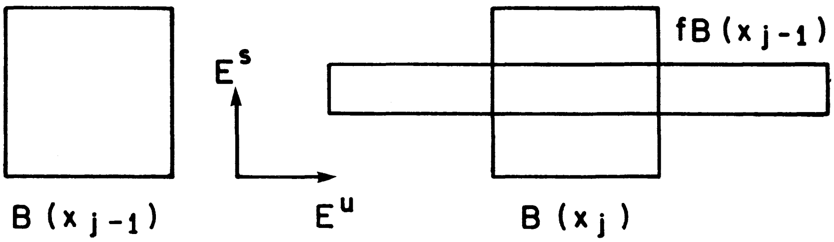
Corollary (Anosov Closing Lemma). *If the chain recurrent set of f, $\mathcal{R}(f)$, is hyperbolic, then the periodic points are dense in $\mathcal{R}$.*

Proof. Let $x \in \mathcal{R}(f)$. Given any $\delta > 0$, let $\epsilon > 0$ be given by the proposition. C. Conley has proven that $\mathcal{R}(f \mid \mathcal{R}) = \mathcal{R}(f)$. Also see the appendix of this section. Therefore, there is a periodic ϵ-chain through x, $\{x_i\}$, with $x_i \in \mathcal{R}$, $x_0 = x$, and $x_{i+n} = x_i$ for all i. By the proposition there is a unique y such that $d(f^i(y), x_i) < \delta$. In particular $d(y, x) < \delta$. Therefore the f orbits of $f^n(y)$ and y both δ-shadow $\{x_i\}$. By uniqueness $y = f^n(y)$ and y is a periodic point within δ of x.

COROLLARY (Semi-stability theorem of Walters). *Assume $\mathcal{R}(f) = M$ is hyperbolic. Then, there is an $\epsilon > 0$ such that if g is a homeomorphism with $d(g(x), f(x)) < \epsilon$ and $d(g^{-1}(x), f^{-1}(x)) < \epsilon$ for all $x \in M$, then there exists a continuous onto map $h : M \to M$ with $hg = fh$.*

PROOF. For each $x \in M$, $g^i(x)$ is an ϵ-chain for f, $d(g^{i+1}(x), f \circ g^i(x)) < \epsilon$. Therefore for each x there is a unique $y = h(x)$ such that $d(f^i \circ h(x), g^i(x)) < \delta$. Looking at $g(x)$, we get $h(g(x))$. Then $\delta > d(f^i \circ h \circ g(x), g^i \circ g(x)) = d(f^{i+1} \circ f^{-1} \circ h \circ g(x), g^{i+1}(x))$. By uniqueness of $h(x)$ we get $f^{-1} \circ h \circ g(x) = h(x)$ or $h \circ g(x) = f \circ h(x)$. The continuity of h follows from the continuous dependence on a parameter. See [32]. Since $d(h(x), x) < \delta$, we get that h is essential on the top homology class of M, so h is onto.

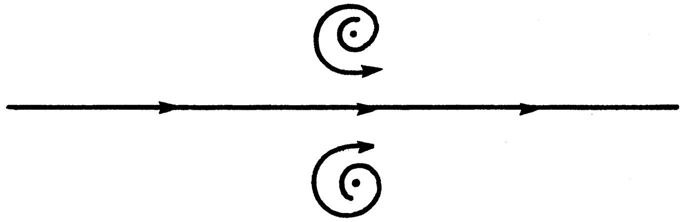
EXAMPLE. There is a diffeomorphism g constructed from a hyperbolic diffeomorphism f on the two torus called the DA diffeomorphism. See [39]. A hyperbolic fixed point x of f is made into two hyperbolic fixed points, x_1 and x_2 and a source x_3 for g. The map h given by the semi-stability theorem takes line segments L running between the unstable manifolds of x_1 and x_2 and collapses them to a point on the manifold of x.



COROLLARY (Structural stability theorem of Anosov). *If f is hyperbolic on M and g is C^1 near f, then there is a homeomorphism $h : M \to M$ such that $hg = fh$.*

PROOF. The semi-stability theorem constructs an $h : M \to M$. We only need to show that h is one to one. Assume $h(x) = h(y)$. Then $h \circ g^i(x) = f^i \circ h(x) = f^i \circ h(y) = h \circ g^i(x)$. Therefore $d(g^i(x), g^i(y)) \leqq d(g^i(x), h \circ g^i(x)) + d(h \circ g^i(y), g^i(y)) < 2\delta$, and $g^i(y)$ 2δ-shadows $g^i(x)$. If g is C^1 near f, then g is hyperbolic and so expansive. Therefore $x = y$.

Appendix on chain recurrence. In this section we give a short and direct proof that the chain recurrent set of the chain recurrent set equals the chain recurrent set, $\mathcal{R}(f \mid \mathcal{R}) = \mathcal{R}(f)$. First we note that for this to be true there needs to be some compactness. Otherwise, there is the example with fixed points at $(0, \pm 1)$, the x-axis invariant, and all other points have ω-limit points on the x-axis. Therefore $\mathcal{R}(f) = \{(0, \pm 1), (x, 0)\}$ and $\mathcal{R}(f \mid \mathcal{R}) = \{(0, \pm 1)\}$. The following theorem is due to C. Conley. The proof given here was given by the author and J. Franks.



THEOREM. *If M is compact and $f : M \to M$ is a diffeomorphism, then* $\mathcal{R}(f \mid \mathrm{R}) = \mathcal{R}(f)$.

PROOF. Let $x \in \mathcal{R}(f)$ and $C_n = \{x_i{}^n\}_{i \in Z}$ be a periodic $1/n$-chain through x. Then the set C_n is a compact subset of M. In the Hausdorff metric, there is a subsequence C_{n_k} that converge to some compact set $C \subset M$. We show below that for every $y \in C$ and $\epsilon > 0$ there is a periodic ϵ-chain $\{z_i\}$ through y with $z_i \in C$. It follows that $y \in \mathcal{R}(f \mid C) \subset \mathcal{R}(f)$. This is true for any $y \in C$ so $C \subset \mathcal{R}(f)$ and $x \in C \subset \mathcal{R}(f \mid C) \subset \mathcal{R}(f \mid \mathcal{R})$. This is true for all $x \in \mathcal{R}(f)$ so $\mathcal{R}(f) \subset \mathcal{R}(f \mid \mathcal{R})$ and so they are equal.

We have only to show that there is a periodic ϵ-chain through y with $z_i \in C$. By uniform continuity of f on M, there is a $\delta = \delta(\epsilon/3) > 0$ such that for $d(a, b) < \delta$ then $d(f(a), f(b)) < \epsilon/3$. Also take $\delta < \epsilon/3$. Because C_{n_k} converges to C, we can find an $n = n_k$ such that $1/n < \epsilon/3$ and the distance from C_n to C in the Hausdorff metric is less than δ. Assume $\{x_i{}^n\}$ has period j so $x^n_{i+j} = x_i{}^n$ for all i. For each $x_i{}^n$ take $z_i \in C$ with $d(x_i{}^n, z_i) < \delta$, $z_{i+j} = z_i$ for all i, and $z_i = y$ for some i. Then $d(f(z_i), z_{i+1}) \leqq d(f(z_i), f(x_i{}^n)) + d(f(x_i{}^n), x^n_{i+1}) + d({}^n_{i+1}, z^n_{i+1}) < \epsilon$. Therefore z_i is a periodic ϵ-chain in C through Y.

2. **Several hyperbolic sets for a diffeomorphism.** If $f : M \to M$ and $\Lambda \subset M$, then we say f is chain transitive on Λ if for $x, y \in A$ and every $\epsilon > 0$ there is an ϵ-chain $\{x_i\}$ with $x_i \in \Lambda$, $x_0 = x$, and $x_n = y$ for some n. Given a point x in $\mathcal{R}(f)$, we can take the maximal chain transitive set containing $x, \Lambda(x)$. These are closed. If $\Lambda(x) \cap \Lambda(y) \neq \emptyset$, then f is chain transitive on $\Lambda(x) \cup \Lambda(y)$, so they are equal. Assume

f is hyperbolic on $\mathcal{R}(f)$. Then there are only finitely many such maximal transitive sets. (The proof uses the local product structure. See [2], [3], or [37].) $\mathcal{R}(f) = \Lambda_1 \cup \cdots \cup \Lambda_L$ where the Λ_i are closed, disjoint, and chain transitive. Given an $\epsilon > 0$ there is an ϵ-chain $\{x_i\}$ that is ϵ dense in Λ_j. There is an orbit that δ-shadows this chain and so is $(\epsilon + \delta)$-dense in Λ_j. It follows the points, $P(2\delta)$, whose f orbits are 2δ-dense are dense in Λ_j. An easy argument shows $P(\delta')$ is a neighborhood of $P(\delta)$ if $0 < \delta < \delta'$. By the Baire category theorem the points x, whose orbits are 2δ-dense for all $\delta > 0$, are dense in Λ_j. Thus f is topologically transitive on each of the Λ_j if f is hyperbolic on $\mathcal{R}(f)$. These Λ_j are called either basic sets following S. Smale or Morse sets following C. Conley.

We define the stable manifold of a point $x \in M$ to be $W^s(x) = \{y : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$ and the unstable manifold of x to be $W^u(x) = \{y : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$. If $\Lambda \subset M$ we let $W^u(\Lambda) = \bigcup\{W^u(x)\, x \in \Lambda\}$ and $W^s(\Lambda) = \bigcup\{W^s(x) : x \in \Lambda\}$.

If f is hyperbolic on $\mathcal{R}(f)$, then $\mathcal{R}(f)$ equals the nonwandering set of f and f satisfies the nocycle property, [6]. We can number the Λ_j so that if $W^u(\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$ then $i \leqq j$. To prove the shadowing result on all of M we need to assume some compatibility of the splittings on the various Λ_i. We say that f satisfies the (*strong*) *transversality condition* if whenever $W^s(x)$ and $W^u(y)$ intersect, they intersect transversally.

THEOREM. *Assume the chain recurrent set of $f : M \to M$ is hyperbolic and f satisfies the* (*strong*) *transversality condition. Then given $\delta > 0$ there is an $\epsilon > 0$ such that any ϵ-chain on M can be δ-shadowed.* (*The orbit that shadows it is not necessarily unique*).

PROOF. The proof can be done by showing directly that $\bigcap\{f^n B(x_n) : n \in Z\}$ is nonempty. We proceed slightly differently to construct a specific point y in this intersection. This approach is more in the spirit of the semi-stability result.

Let U_j be neighborhoods of the Λ_j where f is hyperbolic. Let $\{x_i\}$ be an ϵ-chain. There is an i_1 such that for $i \leqq i_1$ $x_i \in U_{k_1}$ for some fixed k_1. For $j \leqq i_1$ $\bigcap\{f^n B(x_{j-n}) : n \geqq 0\} = D^u(x_j, \{x_i\})$ is a u_1-disk where $u_1 = \dim E_x{}^u$ for $x \in \Lambda_{k_1}$. There is an i_2 such that for $i \geqq i_2$ then $x_i \in U_{k_2}$ for some fixed k_2. Let $u_2 = \dim E_x{}^u$ for $x \in \Lambda_{k_2}$. By the transversality assumption $u_2 \leqq u_1$. Because of the transversality condition, it is possible to construct a u_2-dimensional disk $D^u(x_{i_2}, \{x_i\})$ in the direction of the unstable splitting that is contained inside $f^{i_2 - i_1} D^u(x_{i_1}, \{x_i\})$. Then $f^j D^u(x_{i_2}, \{x_i\}) \cap B(x_{i_2+j})$ is a u_2-dimensional disk near x_{i_2+j}. Then $\bigcap\{f^{-n} D^u(x_{j+n}, \{x_i\}) : n \geqq 0\}$ is a point near x_j for any j. This point has an orbit that δ-shadows the ϵ-chain.

THEOREM (Semi-stability theorem of Nitecki). *Assume the chain recurrent set of f is hyperbolic and f satisfies the (strong) transversality condition. Then there exists an $\epsilon > 0$ such that if g is a homeomorphism with $d(g(x), f(x)) < \epsilon$ and $d(g^{-1}(x), f^{-1}(x)) < \epsilon$ for all $x \in M$, then there exists a continuous onto map $h : M \to M$ such that $hg = fh$.*

PROOF. Let $\mathcal{R}(f) = \Lambda_1 \cup \cdots \cup \Lambda_L$ be a decomposition of the chain recurrent set into Morse sets.

For g C^0 near f we want to construct *compatible families of unstable disks*, i.e. families $\{D_i^u(x, g) : x \in \mathcal{O}(U_i, g)\}$ for $i = 1, \cdots, L$ where U_i is a neighborhood of Λ_i and $\mathcal{O}(U_i, g) = \bigcup\{g^n U_i : n \in Z\}$, satisfying

(1) $D_i^u(x, g)$ is a C^1 disk near x with $\dim D_i^u(x, g) = \dim E_y^u$ for $y \in \Lambda_i$,

(2) $D_i^u(x, g)$ depends continuously on x in the C^1 topology,

(3) (invariance) $fD_i^u(x, g) \supset D_i^u(g(x), g)$,

(4) (compatibility) $D_i^u(x, g) \subset D_j^u(x, g)$ if both are defined and $i > j$ (so $\dim D_i^u(x, g) \leqq \dim D_j^u(x, g)$),

(5) for $x \in U_i$, $D_i^u(x, g)$ is C^1 close to $D_i^u(x, f)$,

(6) $x \in D_i^u(x, f)$ for all x (only for f),

(7) if $x \in W^u(\Lambda_i, f)$ then $D_i^u(x, f) \subset W^u(x, f)$.

We indicate first the construction for f and then the case for g C^0 near to f. The proof proceeds by induction on k constructing disks D_i^u for $i \leqq k$ that satisfy the conditions. An easier case of this induction is introduced by J. Palis in [25]. We first take $U_1^0, \cdots, U_L^0$ to be neighborhoods of the Morse sets Λ_i where f is hyperbolic. We get $U_i \subset U_i^0$. For $k = 1$, Λ_1 is a repellor, i.e. $\Lambda_1 = \bigcap\{f^{-n}U_1^0 : n \geqq 0\}$. Therefore for $x \in U_1^0 = U_1$, $\bigcap\{f^n B(f^{-n}(x)) : n \geqq 0\} = D_1^u(x, f)$ is a C^1 disk through x satisfying the conditions. We can extend these disks to $\mathcal{O}(U_1, f)$ to satisfy condition 3.

Assume the families $\{D_i^u(x, f) : x \in \mathcal{O}(U_i, f)\}$ have been constructed for $i \leqq k - 1$. A proper fundamental domain for $W^s(\Lambda_k)$ is a closed set $F_k^s \subset W^s(\Lambda_k)$ such that $\bigcup\{f^n F_n^s : n \in Z\} = W^s(\Lambda_k) - \Lambda_k$. A fundamental neighborhood of $W^s(\Lambda_k)$, V_k^s, is a neighborhood of a proper fundamental domain such that $V_k^s \cap \Lambda_k = \emptyset$. There exists a proper fundamental domain $F_k^s \subset U_k^0$ such that if $x \in F_k^s$ then $f^n(x) \notin F_k^s$ for $n \geqq 2$. See [30, Lemma 4.3]. For a set Q let $\mathcal{O}^+(Q) = \bigcup\{f^n Q : n \geqq 0\}$ and $\mathcal{O}^-(Q) = \bigcup\{f^n Q : n \leqq 0\}$. We can construct closed sets $P_i \subset \mathcal{O}^+(U_i)$ such that $P_i \cap W^u(\Lambda_j) = \emptyset$ for $i < j \leqq k-1$ and $P^i = \bigcup\{P_j : i \leqq j \leqq k-1\}$ is a neighborhood of $F_k{}^s \cap W^u(\Lambda_i)$. Then $V_k{}^s = P^1$ is a fundamental neighborhood. We can construct disks $D_k{}^u(x, f)$ for $x \in V_k{}^s$ such that if $x \in P_i$ then $D_k{}^u(x, f) \subset D_i{}^u(x, f)$ and if x, $f(x) \in V_k{}^s$ then $fD_k{}^u(x, f) \supset D_k{}^u(f(x), f)$. See [30, Lemma 5.3].

We want $U_i' \subset U_i$ such that $\mathcal{O}^+(U_i') \cap V_k{}^s \subset P^i$. If we can do this then if $x \in \mathcal{O}^+(U_i') \cap V_k{}^s$ then $x \in P^i$ and $D_k{}^u(x, f) \subset D_i{}^u(x, f)$. Thus we have compatibility using the set U_i'. See [30, Lemma 5.3] for details of how this can be done. An easier case of the induction is introduced by J. Palis in [25]. The idea is as follows. We want $\mathcal{O}^+(U_i') \cap V_k{}^s \subset P^i$ or $U_i' \cap \mathcal{O}^-(V_k{}^s - P^i) = \emptyset$. Assume $U'_{k-1}, \cdots, U'_{j+1}$ are chosen that work. Then P^j is a neighborhood of $V_k{}^s \cap \bigcup\{W^u(\Lambda_i) : j \leqq i \leqq k-1\}$ so $\mathcal{O}^-(V_k{}^s - P^j) \subset \bigcup\{W^u(\Lambda_i) : 1 \leqq i \leqq j-1\}$. Let Q be a neighborhood of $\Lambda_1, \cdots, \Lambda_{j-1}$ so that $Q \cap \Lambda_j = \emptyset$. Then there is an n such that $f^{-i}(V_k{}^s - P^j) \subset Q$ for $i \geqq n$. Let U'_j be a neighborhood of Λ_j such that $U'_j \cap \bigcup\{f^{-i}(V_k{}^s - P^j) : 0 \leqq i \leqq n\} = \emptyset$, and so $U_j \cap \bigcup\{f^{-i}(V_k{}^s - P^j) : i \geqq 0\} = \emptyset$. This finishes the induction step and shows we can construct the U'_i.

Now we extend the disks defined on the fundamental neighborhood $V_k{}^s$ to a whole neighborhood of Λ_k. Let $\mathcal{O}^+(V_k{}^s, U_k{}^0) = \{f^n(x) : x \in V_k{}^s$ and $f^i(x) \in U_k{}^0$ for $i = 0, \cdots, n\}$, and $U_k = \mathcal{O}^+(V_k{}^s, U_k{}^0) \cup \{W^u(\Lambda_k) \cap U_k{}^0\}$. We can extend the disks on $V_k{}^s$ to all of U_k by letting $D_k{}^u(x, f) = f^n D_k{}^u(f^{-n}(x), f) \cap B(x)$ if $f^{-n}(x) \in V_k{}^s$ and by letting $D_k{}^u(x, f) = \bigcap\{f^n B(f^{-n}(x)) : n \geqq 0\}$ if $x \in W^u(\Lambda_k) \cap U_k{}^0$. These disks satisfy the conditions above using the sets U'_k. We have finished the induction step on k for f. This shows we can construct the compatible families of unstable disks.

To show we can construct the disks for g we need to look at the induction step. We have the same P_i and P^i. Since P_i is compact there is a finite n_i such that $P_i \subset \bigcup\{f^j U_i : 0 \leqq j \leqq n_j\}$. If g is C^0 near enough to f then $P_i \subset \bigcup\{g^j U_i : 0 \leqq j \leqq n_i\}$. We do this for $i = 1, \cdots, k-1$. We can construct disks $D_k{}^u(x, g)$ for $x \in V_k{}^s$ such that if $x \in P_i$ then $D_k{}^u(x, g) \subset D_i{}^u(x, g)$ and if $x, g(x) \in V_k{}^s$ then $fD_k{}^u(x, g) \supset D_k{}^u(g(x), g)$. This goes much as before.

We want to show the U_i' are neighborhoods of Λ_i giving compatibility. Let Q be a neighborhood of $\Lambda_1, \cdots, \Lambda_{j-1}$ as before for f. If g is C^0 near f we can insure that $f^{-i}(V_k{}^s - P^j) \subset Q$ for $i \geqq n$, and $U'_j \cap \bigcup\{g^{-i}(V_k{}^s - P^j) : 0 \leqq i \leqq n\} = \emptyset$. Therefore, $U'_j \cap \bigcup\{g^{-i}(V_k{}^s - P^j) : i \geqq 0\} = \emptyset$. If $x \in \mathcal{O}^+(U'_i, g) \cap V_k{}^s$ then $x \in P^i$ so $D_k{}^u(x, g) \subset D_i{}^u(x, g)$. This gives compatibility of $D_k{}^u(x, g)$ for $x \in V_k{}^s$.

We can pick a smaller neighborhood $U'_k \subset U_k$ of Λ_k such that if g is C^0 near to f and the backward g orbit of $x \in U'_k$ leaves U_k then it leaves through $V_k{}^s$, i.e. if $x \in U'_k$, $g^{-i}(x) \in U_k$ for $0 \leqq i \leqq n$, and $g^{-n-1}(x) \notin U_k$, then $g^{-n}(x) \in V_k{}^s$. Therefore for $x \in U'_k$ either (i) $g^{-n}(x) \in U_k$ for all $n \geqq 0$ or (ii) $g^{-n}(x) \in V_k{}^s$ for some n. We define disks on U'_k either (i) by letting $D_k{}^u(x, g) = \bigcap \{f^n B(g^{-n}(x)) : n \geqq 0\}$ or (ii) by letting $D_k{}^u(x, g) = f^n D_k{}^u(g^{-n}(x), g) \cap B(x)$ if $g^{-n}(x) \in V_k{}^s$. We replace the sets U_i by the U'_i for $i = 1, \cdots, k$ and we get that the disks satisfy the conditions. This completes the induction on k.

The map $f^{-1} : D_L{}^u(x, g) \to D_L{}^u(g^{-1}(x), g)$ is a contraction for $x \in U_L$. There is a unique section $h(x) \in D_L{}^u(x, g)$ for $x \in U_L$ such that $f^{-1}h(x) = hg^{-1}(x)$. Using the equation $f^{-1}h(x) = hg^{-1}(x)$ we can extend h back to a fundamental neighborhood of $W^u(\Lambda_{L-1})$. Also $h(x) \in D_L{}^u(x, g) \subset D_{L-1}{}^u(x, g)$. The map $f^{-1} : D^u_{L-1}(x, g) \to D^u_{L-1}(g^{-1}(x), g)$ is a contraction for $x \in U_{L-1}$. Therefore the section h extends back to a continuous section over U_{L-1}. Continuing we get h defined on all of M such that $f^{-1}h = hg^{-1}$ or $hg = fh$. This completes the proof of the theorem.

Remark. We do not see how to use this proof to prove the structural stability theorem. Instead in [**30**] and [**32**] we solve for a k such that $gk = kf$, i.e. solve the equation the other way around. Then we prove k is Lipschitz in a special sense and so one to one.

3. **Flows.** The reason we discussed some of the key points of the semi-stability theorem in some detail in § 2 is because with this proof it is clear that the result is true for flows. The ideas of how to make the changes in the proof are in [**31**]. The results about shadowing an ϵ-chain are also true for flows. Of course the results near a single hyperbolic set are true for flows but this is discussed in [**4**] and [**5**]. Also see [**40**].

We say that g is semi-conjugate to f if there exists a continuous onto map $h : M \to M$ and a reparameterization $\gamma : R \times M \to M$ that is continuous such that $g_1(t, x) = g(\gamma(t, x), x)$ has the group property and $f(t, h(x)) = h \circ g_1(t, x)$.

Theorem. *Assume $f : R \times M \to M$ is a flow on a compact manifold without boundary such that the chain recurrent set is hyperbolic and f satisfies the (strong) transversality condition. Then there exists an $\epsilon > 0$ such that if $g : R \times M \to M$ is a flow with $d(g(t, x), f(t, x)) \leqq \epsilon$ for all $x \in M$ and $-1 \leqq t \leqq 1$ and with $|g'(0, x) - f'(0, x)| < \epsilon$ for all $x \in M$, then g is semi-conjugate to f.*

4. **Converse to structural stability.** There are some results on the converse to the structural stability theorem but the question is not completely settled. The question is whether structural stability (or Ω-stability) implies that the closure of the periodic points is hyperbolic. We mention some results of J. Franks and then R. Mañé and V. Pliss.

A diffeomorphism f is called *time-dependently structurally stable* if there is a neighborhood of f, $\mathcal{N}$, in the C^1 topology such that if $g_1, \cdots, g_n \in \mathcal{N}$ (for any n) then there is an homeomorphism $h : M \to M$ such that $f^n h = h g_1 \cdots g_n$. J. Franks proved that the chain recurrent set (closure of the periodic points) is hyperbolic and f satisfies the transversality condition if and only if f is time-dependently structurally stable. See [9]. Also see [8] and [11] for a related result.

Let $\mathcal{P}_j$ be the periodic points with the unstable splitting of dimension j. It is known that all the periodic points are hyperbolic so the tangent bundle of M over $\mathcal{P}_j$ splits $TM \mid \mathcal{P}_j = E^u \oplus E^s$ such that $T_x f^{\pi(x)} : T_x M \to T_{f(x)} M$ expands $E_x{}^u$ and contracts $E_x{}^s$ where $\pi(x)$ is the period. Both R. Mañé and V. Pliss have proved that there exist uniform constants $C > 0$ and $0 < \lambda < 1$ such that $\|Tf^n \mid E_x{}^s\| \cdot \|Tf^{-n} \mid E^u_{f^n(x)}\| \leqq C\lambda^n$ for all $x \in \mathcal{P}_j$ and $n \geqq 0$. See [15], [26], and [27]. If f had a hyperbolic splitting on $\mathcal{P}_j$ then we would have both $\|Tf^n \mid E_x{}^s\| \leqq C\lambda^n$ and $\|Tf^{-n} \mid E^u{}_{f^n(x)}\| \leqq C\lambda^n$. As it is Tf^n may expand $E_x{}^s$ but not as much as it expands $E_x{}^u$. The above condition goes to the closure of $\mathcal{P}_j$, i.e. there are continuous bundles E^u and E^s over closure $\mathcal{P}_j$ such that $\|Tf^n \mid E_x{}^s\| \cdot \|Tf^{-n} \mid E^u{}_{f^n(x)}\| \leqq C\lambda^n$ for $x \in$ closure $\mathcal{P}_j$ and $n \geqq 0$. We don't know that f contracts E^s or expands E^u but we do know about the relative rates. If the conjecture is true then these are the correct bundles for the hyperbolicity. Let Ω be the non-wandering set of f. Assume dimensions $M = 2$, f is structurally stable, and either (i) $\Omega = M$ or (ii) measure $\Omega = 0$, then Ω is hyperbolic. This result is proved by (i) R. Mañé in [16] and (ii) by V. Pliss in [28].

5. **Criterion for hyperbolicity.** The stability theorems assume that the derivative map Tf is hyperbolic on $TM \mid \mathcal{R}$. We mention a criterion to ensure this. Assume B is a vector bundle over Λ and $F : B \to B$ is a linear bundle isomorphism covering $f : \Lambda \to \Lambda$. If $\{v \in B : |F^n v|$ is bounded for $n \in Z\} = \{0_x : x \in \Lambda\}$, then we say that the *zero section is isolated.* Variations of the following theorem are proved by R. Sacker and G. Sell [35], J. Selgrade [36], and R. Mañé [14].

THEOREM. *Let $F: B \to B$ be a linear bundle isomorphism covering $f: \Lambda \to \Lambda$. Assume the zero section is isolated for F, that f is chain recurrent on Λ, and that Λ is compact. Then F is hyperbolic on fibers (has an exponential dichotomy.)*

If f is not chain recurrent then the theorem is false as is easily seen. For example, we can construct $B \subset Z \times R^n$. Let $A: R^2 \to R^2$ be given by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

and $L(n, v) = (n + 1, Av)$. Let $v_0 = (1, 1)$ and $B = \{(n, rA^n v_0) : n \in Z, r \in R\}$. Then B is invariant by L. Let $F = L \mid B$. The zero section is isolated for F but all nonzero vectors are unbounded in both positive and negative time so F is not hyperbolic. We can make Λ compact by adding points at $\pm\infty$ to Z.

Because of connections with Anosov diffeomorphisms it was asked if it were possible to give an example as above where Λ is a manifold, $B = T\Lambda$, $F = Tf$, (f is not chain recurrent), and F is not hyperbolic. J. Franks and myself constructed such an example $f: M \to M$ where dimension $M = 3$. See [**10**]. The chain recurrent set $\mathcal{R}(f) = \Lambda_1 \cup \Lambda_2$ where $\dim E_x^u = 1$ for x in the repellor Λ_1 and $\dim E_y^u = 2$ for y in the attractor Λ_2. Also see [**34**].

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