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# EVENTUALLY $p$-VALENT FUNCTIONS

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I. **Introduction.** In this paper we introduce the concept of eventual $p$-valence and investigate properties of eventually $p$-valent functions. Let $\mathcal{A}(D)$ denote the set of all functions $f(z)$ which are analytic in $|z| < 1$. We define a function $f(z)$ in $\mathcal{A}(D)$ to be *eventually $p$-valent* if there is a neighborhood $\mathcal{W}$ of infinity such that for all $w \in \mathcal{W}, f(z) = w$ has at most $p$ roots in $|z| < 1$. Since eventual $p$-valence does not depend upon averaging properties, it is a natural concept for geometric function theory. Although the idea of eventual $p$-valence is hinted at in the literature, eventually $p$-valent functions have not been systematically investigated.

In Table 1 we summarize the necessary relations among functions which are eventually $p$-valent, areally mean $p$-valent, circumferentially $p$-valent or weakly $p$-valent. The proofs of some of these necessary relations are simplified by considering the set of all normalized locally univalent functions in $\mathcal{A}(D)$ with the real normed linear space structure introduced and developed in [11], [4]. We show that the concept of eventual $p$-valence is a meaningful linear invariant property while the notions of areal mean $p$-valence and circumferential mean $p$-valence are not meaningful linear invariant properties.

We define the *growth of an analytic function* in $\mathcal{A}(D)$ in terms of its maximum modulus $M(r,f)$ and show that *growth* $f$ is a linear invariant concept. This allows us to form a new partitioning of the (universal) families $\mathcal{U}_\alpha$ of Pommerenke [20] and leads to the definition of the (universal) families $\mathcal{U}^\gamma$. If $f(z)$ is eventually $p$-valent, we show that $M(r,f) = O((1 - r)^{-2p})$.

We extend the Asymptotic Bieberbach Conjecture of G. Wing [22] to functions which are not even locally univalent. We conclude with four open questions suggested by the Extended Wing Theorem.

II. **Definitions and Preliminary Notions.** We let $\mathcal{J}$ denote the set of all Möbius transformations of $D = \{|z| < 1\}$ onto $D$ and let $\mathcal{L.S.}$ denote the set of all functions of the form $f(z) = z + \cdots$ which are analytic and locally univalent ($f'(z) \neq 0$) in $D$. If $\mathcal{M}$ is a family of functions in $\mathcal{L.S.}$, we say that $\mathcal{M}$ is a *linear invariant family* [20, p. 112] if and only if for every $\phi(z)$ in $\mathcal{J}$, the function

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$$\Lambda_\phi[f(z)] = \frac{f(\phi(z)) - f(\phi(0))}{\phi'(0)f'(\phi(0))} = z + \cdots \tag{1}$$

is again in $\mathcal{M}$.

The *order of a linear invariant family* $\mathcal{M}$ is defined as [20, p. 115]

$$\alpha = \sup\{|f''(0)/2| : f \in \mathcal{M}\}.$$

The union of all linear invariant families of order at most $\alpha$ is denoted $\mathcal{U}_\alpha$. The (universal) family $\mathcal{U}_\alpha$ is linear invariant and if $\alpha$ is less than 1, then $\mathcal{U}_\alpha$ is empty [20, p. 117]. The family $\mathcal{U}_1$ is precisely the set of all normalized convex univalent functions. The function $(2i\gamma)^{-1}([1 + z)/(1 - z)]^{i\gamma} - 1)$, which has infinite valence, is in $\mathcal{U}_\alpha$ for $\alpha = (1 + \gamma^2)^{1/2}, \gamma > 0$.

If $f(z)$ is in $\mathcal{L.S.}$, then it generates a linear invariant family in a natural manner. We let

$$\mathcal{M}[f] = \{\Lambda_\phi[f] : \phi(z) \in \mathcal{J}\}.$$

Since $\mathcal{J}$ is a group under composition, it is easy to check that $\mathcal{M}[f]$ is indeed a linear invariant family. *The order of a function* $f(z) \in \mathcal{L.S.}$ (abbreviated order $f$) is defined to be the order of the linear invariant family which it generates and is given by [20, p. 115]

$$\text{order} f = \sup\{|t(z, f)| : z \in D\} \tag{2}$$

where $t(z,f) = -\bar{z} + (1 - |z|^2)f''(z)/2f'(z)$.

Geometric function theory gives rise to many linear invariant families. Each of the following sets of normalized analytic functions in $D$ is a linear invariant family: (a) univalent functions (order $= 2$) (b) convex univalent functions (order $= 1$) (c) Close to convex functions of order $\beta$ (order $= \beta + 1$) (d) functions whose boundary rotation is $\leqq M$(order $= M/2\pi$) [2].

The fundamental property of the families $\mathcal{U}_\alpha$ which is of interest later in this paper is that for any function $f(z)$ in $\mathcal{U}_\alpha$ [20, p. 115]

$$|f(z)| \leqq \frac{1}{2\alpha}\left(\left(\frac{1+r}{1-r}\right)^\alpha - 1\right), |z| = r < 1. \tag{3}$$

It will be useful to refer to the collection of all functions of finite order in $\mathcal{L.S.}$ and to put a normed vector space structure on this set. We let

$$X = \bigcup_{\alpha \geqq 1} \mathcal{U}_\alpha$$

and consider $X$ as a real linear space with the operations

$$[f+g](z)=\int_0^z f'(\zeta)g'(\zeta)\,d\zeta \qquad (f,g\in X)$$

$$[af](z)=\int_0^z (f'(\zeta))^a\,d\zeta \qquad (f\in X, \text{a real})$$

(square brackets will indicate the algebraic operations in $X$). These operations in $\mathcal{L.S.}$ were introduced by Hornich [11] and applied to $X$ by Campbell, Cima and Pfaltzgraff [4]. We define the norm of a function $f(z)\in X$ by

$$\|f\| = \sup\{(1-|z|)|f''(z)/f'(z)| : z\in D\}.$$

The main result of use in this paper is that the real valued function $f\to$ order $f$ is a continuous function on $(X, \|\ \|)$ and in fact [4, Lemma 3.1] for any $f, g\in X$

$$|\text{order}\,f - \text{order}\,g| \leqq \|[f-g]\|. \tag{4}$$

We now let $f(z)$ be an arbitrary analytic function in $D$ and let $n(Re^{i\theta})$ denote the number of roots counting mulitplicity of the equation $f(z)=Re^{i\theta}$ for $z$ in $D$.

1. If for each $R$, $0<R<\infty$,

$$\int_0^{2\pi} n(Re^{i\theta})\,d\theta \leqq 2\pi p,$$

where $p$ is a fixed positive real number, then we say that $f(z)$ is *circumferentially mean p-valent* [10] abbreviated *c.m.p.v.*

2. If for each $R$, $0<R<\infty$,

$$\int_0^R\int_0^{2\pi} n(re^{i\theta})r\,d\theta\,dr \leqq \pi p\,R^2,$$

where $p$ is a fixed positive real number, then we say that $f(z)$ is *areally mean p-valent* [10], abbreviated *a.m.p.v.*

3. If for each $R$, $0<R<\infty$, we have either $n(Re^{i\theta})=p$ for each $\theta$, $0\leqq\theta\leqq 2\pi$ or $n(Re^{i\theta})<p$ for some $\theta$, $0\leqq\theta\leqq 2\pi$, then we say that $f(z)$ is *weakly p-valent* [10], abbreviated *w.p.v.* In this case $p$ is necessarily a fixed positive integer.

4. If there is an $R_f$ such that for each $R$, $0\leqq R_f<R<\infty$ we have

$$n(Re^{i\theta})\leqq p, 0\leqq\theta\leqq 2\pi,$$

then we say that $f(z)$ is *eventually p valent,* abbreviated *e.p.v.* In this case $p$ is necessarily a nonnegative integer.

Clearly, if $f(z)$ is a.m.p.v., c.m.p.v., w.p.v. or e.p.v., then it is a.m.q.v., c.m.q.v., w.q.v. or e.q.v., respectively, for any $q\geqq p$.

The fundamental property of functions which are a.m.p.v., c.m.p.v. or w.p.v., which is of interest in this paper, is that for any such function

$$M(r,f) = O((1-r)^{-2p}) \tag{5}$$

where $M(r,f) = \max|f(z)|$ on $|z| = r$. (see [10] and [7, p. 159]).

The interrelations between c.m.p.v., a.m.p.v. and w.p.v. are known (see Table 1).

We conclude this section with an examination of three well known functions in $\mathcal{L.S.}$:

$$f(z) = \frac{1}{2\alpha}\left(\left(\frac{1+z}{1-z}\right)^{\alpha} - 1\right), \alpha \geqq 1, \tag{6}$$

$$f(z) = \frac{1}{2i\gamma}\left(\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1\right), \gamma \geqq 0, \tag{7}$$

$$f(z) = \tfrac{1}{2}(1 - e^{-2z/1-z}). \tag{8}$$

The function (6) maps the disc to an angular sector (which is overlapping if $\alpha > 2$) with vertex $-1/(2\alpha)$ and angular aperture $\alpha\pi$. It is in $\mathcal{U}_\alpha$ [20, p. 117] and the geometry of the mapping yields that is circumferentially mean $p$-valent and areally mean $p$-valent for $p = \max(1, \alpha/2)$. Let $q$ be the smallest integer greater than or equal to $\alpha/2$, then (6) is eventually $q$-valent and weakly $s$-valent ($s = \max(1, q - 1)$).

The function (7) is bounded, it is in $\mathcal{U}_\alpha$ for $\alpha = (1 + \gamma^2)^{1/2}$ and has infinite valence [20, p. 128]. The function (8) is bounded, is not in $X$ since $|f''(z)/f'(z)| = O((1 - |z|)^{-2})$ and it also has infinite valence.

No function in $X$ can be a.m.p.v. or c.m.p.v. for any $0 < p < 1$. This is an elementary consequence of the covering properties of functions of order $\alpha$ which always provide at least a one-sheeted covering of $|w| < 1/(2\alpha)$. Thus for $0 < R < 1/(2\alpha)$ any function of order $\alpha$ must automatically be at least c.m.l.v. and a.m.l.v.

## III. Relationships between e.$p$-valence and other concepts of $p$-valence.

In this section we show that e.p.v. neither implies nor is implied by a.m.p.v., c.m.p.v., w.p.v. or membership in $\mathcal{U}_\alpha$; consequently, e.p.v., is logically independent of these other concepts. We also examine the relations between membership in $\mathcal{U}_\alpha$ and w.p.v., a.m.p.v. or c.m.p.v.

The function (8) is bounded and covers a neighborhood of the origin infinitely often and is not in $\mathcal{U}_\alpha$ for any $\alpha$. Its boundedness assures us that it is e.p.v. for any $p$; its covering of a neighborhood of the origin infinitely often prevents it from being w.p.v., c.m.p.v. or a.m.p.v. for any $p$. Consequently e.p.v. does not imply w.p.v., c.m.p.v. or a.m.p.v. or membership in $\mathcal{U}_\alpha$.

| | $\mathcal{U}_\alpha$ | e.p.v. | c.m.p.v. | a.m.p.v. | w.p.v. |
|---|---|---|---|---|---|
| $\mathcal{U}_\alpha$ implies | | false Theorem 2 | false (7) | false (7) | false (7) |
| e.p.v. implies | false (8) | | false (8) | false (8) | false (8) |
| c.m.p.v. implies | false Remark 1 | false Theorem 1 | | true [10, p. 399] | true [10, p. 399] |
| a.m.p.v. implies | false Remark 1 | false Theorem 1 | false [10, p. 399] | | false [10, p. 399] |
| w.p.v. implies | false Remark 1 | false Theorem 1 | false [10, p. 399] | false [10, p. 399] | |

TABLE 1

THEOREM 1. *Let $0 < p < \infty$. There are functions which are c.m.p.v., a.m.p.v. or w.p.v. but which are not e.q.v. for any $q$.*

PROOF. Since a c.m.p.v. function is automatically a.m.p.v. and w.p.v., it suffices to construct for each $p > 0$ a function which is c.m.p.v. but which cannot be e.q.v. for any $q$. A c.m.p.v. function is c.m.q.v. for all $q \geqq p$, hence we may assume that $0 < p < 1/2$. Consider the following simply connected Riemann Surface of hyperbolic type. For each positive integer $n$ the surface is a smooth (unbranched) $n$ sheeted covering of the annulus

$$A_n = \{z : 2^{-1}(n+1)\sin(\pi p/(n+1)^2) < |z-n-1| < (n+1)\sin(\pi p/(n+1)^2)\}.$$

The top sheet over $A_n$ is connected to the bottom sheet over $A_{n+1}$ by means of a smooth one sheeted covering of the quadrilateral with vertices $\{n+1-i2^{-1}(n+1)\sin(\pi p/(n+1)^2),\ n+1-i(n+1)\sin(\pi p((n+1)^2),\ n+2-i2^{-1}(n+2)\sin(\pi p/(n+2)^2),\ n+2-i(n+2)\sin(\pi p/(n+2)^2)\}$. Since the projection of this simply connected Riemann Surface omits more than three points, it is of hyperbolic type. Thus there is a locally univalent analytic function $f(z)$ which maps $D$ onto the described configuration. The function cannot be e.q.v. for any $q$ and by the construction of the surface

$$\int_0^{2\pi} n(\mathrm{Re}^{i\theta})\, d\theta \leqq 2\pi p$$

for all $0 < R < \infty$.

THEOREM 2. *For each $\alpha > 3$, there exist functions in $\mathcal{U}_\alpha$ which are not eventually p-valent for any p.*

PROOF. It will suffice to show the existence of a function in $\mathcal{U}_\alpha$ which covers an entire ray infinitely often.

The function (7) covers the annulus

$$\left\{w : \frac{1}{2\gamma}\ e^{-\pi\gamma/2} < |w + (2i\gamma)^{-1}| < \frac{1}{2\gamma}\ e^{\pi\gamma/2}\right\}$$

infinitely often. The function

$$h(z) = \frac{-if(z)}{A},\ A = (e^{\pi\gamma/2} + 1)/2\gamma$$

maps the disc $D$ to an annulus which lies entirely inside the unit disc and which is internally tangent to $D$ at $z = 1$ and which covers all points of the segment $(e^{-\pi\gamma/2}, 1)$ infinitely often. Consequently the function

$$G(z) = \frac{Aih(z)}{1 - h(z)} = z + \cdots$$

covers the ray $\{iy : (2\gamma)^{-1} \coth(\pi\gamma/4) < y\}$ infinitely often and cannot be e.q.v. for any $q$.

We use (2) to compute the order of $G(z)$

$$\begin{aligned} |t(z, G)| &= \left| -\bar{z} + \frac{1 - |z|^2}{2}\,\frac{G''(z)}{G'(z)}\right| \\ &= \left|-\bar{z} + \frac{1 - |z|^2}{2}\cdot\frac{f''(z)}{f'(z)} + \frac{1 - |z|^2}{2}\cdot\frac{-2if'}{A + if}\right| \\ &\leqq |t(z,f)| + \frac{(1 - |z|^2)|f'(z)|}{A - |f(z)|}. \end{aligned}$$

However, since $|h(z)| < 1, |h'(z)| \leqq (1 - |h(z)|^2)/(1 - |z|^2)$ so $(1 - |z|^2)|f'(z)|/(A - |f(z)|) \leqq 2$. We know that $|t(z,f)| \leqq (1 + \gamma^2)^{1/2}$ because $f(z)$ has order $(1 + \gamma^2)^{1/2}$. Consequently

$$|t(z, G)| \leqq (1 + \gamma^2)^{1/2} + 2$$

which can be made as close to 3 as desired by choosing $\gamma$ sufficiently close to 0. Therefore, for each $\alpha > 3$, there exist functions in $\mathcal{U}_\alpha$ which are not eventually $p$-valent for any $p$.

The above paragraphs show that e.p.v. neither implies nor is implied by membership in $\mathcal{U}_\alpha$, w.p.v., c.m.p.v. or a.m.p.v.

REMARK 1. A function which is c.m.p.v., a.m.p.v. or w.p.v. need not be in $\mathcal{U}_\alpha$ for any $\alpha$ since the function need not even be locally univalent. As example (8) shows, there are locally univalent functions which are e.p.v. which are not in $\mathcal{U}_\alpha$.

The function (7) is bounded and covers a neighborhood of the origin infinitely often and is in $\mathcal{U}_\alpha$. It therefore is an example of a function in $\mathcal{U}_\alpha$ which is not c.m.p.v., a.m.p.v. or w.p.v. for any $p$. Therefore, membership in $\mathcal{U}_\alpha$ does not imply c.m.pv, a.m.p.v. or w.p.v.

This completes an explanation of the entries of Table 1 which list the necessary relationships among the concepts c.m.p.v., a.m.p.v., w.p.v., e.p.v. and membership in $\mathcal{U}_\alpha$.

IV. **Linear Invariant Families of Eventually $p$-valent Functions.** The concept of e.p.v. is a linear invariant property; that is, every element of the linear invariant family generated by an e.p.v. function in $\mathcal{L.S.}$ is also e.p.v. There are geometric properties which are not linear invariant; for example, if $f(z) \in \mathcal{L.S.}$ is starlike with respect to the origin, it is not necessarily true that each function in $\mathcal{M}[f]$ is starlike with respect to the origin.

THEOREM 3. *Let $p$ be a fixed nonnegative integer. If $f(z) \in \mathcal{L.S.}$ is eventually $p$-valent, then each element of $\mathcal{M}[f]$ is eventually $p$-valent.*

PROOF. We choose an arbitrary element $\Lambda_\phi[f(z)]$ in $\mathcal{M}[f]$ which is given by some $\phi(z)$, an automorphism of $D$. There is an $R_1$ such that for all $R_1 < R < \infty$

$$n(Re^{i\theta}, f(z)) \leqq p \tag{9}$$

Since $\phi(z)$ is an automorphism of $D$, it is clear that

$$n(Re^{i\theta}, f(z)) \equiv n(Re^{i\theta}, f(\phi(z))). \tag{10}$$

Let $R_1^* = (R_1 + |f(\phi(0))|)/|\phi'(0)f'(\phi(0))| < \infty$. Then for all $R_1^* < R < \infty$, we have that

$$|\phi'(0)f'(\phi(0))Re^{i\theta} + f(\phi(0))| > R_1$$

for all $0 \leqq \theta \leqq 2\pi$. Therefore,

$$\begin{aligned} n(Re^{i\theta}, \Lambda_\phi[f(z)]) &= n(\phi'(0)f'(\phi(0))Re^{i\theta} + f(\phi(0)), f(\phi(z))) \\ &= n(\phi'(0)f'(\phi(0))Re^{i\theta} + f(\phi(0)), f(z)) \leqq p \end{aligned}$$

by (1), (10) and (9). This shows that $\Lambda_\phi[f(z)]$ is e.p.v. for all $R_1^* < R < \infty$ and concludes our proof of the theorem.

Let $\mathcal{E.S.p}$ denote the set of all functions in $\mathcal{L.S.}$ which are eventually $p$-valent.

DEFINITION. A *linear invariant family* $\mathcal{M}$ *is c.m.p.v.* if and only if each function $f(z) \in \mathcal{M}$ is c.m.p.v. We make the same definition for an e.p.v., w.p.v. or a.m.p.v. linear invariant family.

Theorem 3 easily yields that $\mathcal{E}.\mathcal{S}.p$ is a linear invariant family for each $p$. It will follow as a Corollary to Theorem 5 that $\mathcal{E}.\mathcal{S}.p$ is not normal. We now state a necessary and sufficient condition for a linear invariant family to be normal which is due to Pommerenke [**20**, p. 117]:

THEOREM 4. [Pommerenke] *Let* $\mathcal{M}$ *be a linear invariant family of order* $\alpha$. $\mathcal{M}$ *is normal if and only if* $\alpha$ *is finite.*

THEOREM 5. *For each* $\alpha \geqq 1$, *there is an eventually univalent function of order* $\alpha$.

PROOF. Consider the function

$$F_a(z) = \int_0^z e^{aw}/(1 - w)\, dw, \; a \geqq 0.$$

Obvious estimates of $|t(z, F_a)|$ immediately yield that $(a + 1)/2 \leqq$ order $F_a \leqq (a + 4)/2$ and hence $F_a(z) \in X$. We use the space structure of $(X, \| \;\|)$ and write

$$F_a(z) = [\,[af] + g]\,(z)$$

where $f(z) = e^z - 1 \in \mathcal{U}_1$ and $g(z) = -\log(1 - z) \in \mathcal{U}_1$. We note (1) the mapping $k(a) : a \to$ order $F_a$ is a continuous mapping of the reals into the reals as $|\text{order } F_a - \text{order } F_b| \leqq \|[F_a - F_b]\| = |a - b|\ \|f\| = |a - b|$, (2) $k(0) =$ order $F_0(z) = 1$ (3) order $F_a \to \infty$ as $a \to \infty$. The range of $k(a)$ must be the set of all real numbers greater than one since the range of $k(a)$ is connected, contains 1 and arbitrarily large real numbers. Thus for each $\alpha \geqq 1$, there is an $F_a(z)$ whose order is precisely $\alpha$.

We now prove that $F_a(z)$ is eventually univalent for each $a \geqq 0$. We may assume $a > 0$ since $F_0(z) = -\log(1 - z)$ is convex univalent. Let $D_0 = \{z \in D : |1 - z| \leqq 1 - r_0\}$ where $r_0$ is between 0 and 1 and is large enough that $|a \operatorname{Im} z| < \pi/2$ for all $z \in D_0$. We let $R_F = e^a/(1 - r_0)$. Since $|F_a(z)| < e^a/(1 - r_0)$ for all $z \in D - D_0$, any root of $F_a(z) = Re^{i\theta}$, $R_F < R < \infty$, must lie in $D_0$. Therefore in order to show $F_a(z)$ is eventually univalent, we need only show that $F_a(z)$ is univalent in $D_0$.

It is easy to check that (1) $\operatorname{Im} F_a{}'(z) > 0$ in $D_0{}^+ = D_0 \cap \{\operatorname{Im} z > 0\}$ (2) $\operatorname{Im} F_a{}'(z) < 0$ in $D_0{}^- = D_0 \cap \{\operatorname{Im} z < 0\}$ and (3) $\operatorname{Im} F_a{}'(z) = 0$, Re $F_a'(z) > 0$ on $-1 < x < 1$. Therefore, $F_a(z)$ maps $D_0{}^+$ to the upper half plane, $D_0{}^-$ to the lower half plane and the real axis univalently onto a subset of the real axis. Since $D_0{}^+$ and $D_0{}^-$ are convex, by a result of Noshiro [**16**], $F_a(z)$ is univalent in each of $D_0{}^+$ and $D_0{}^-$. Thus it is obvious that $F_a(z)$ is univalent throughout $D_0$. This concludes our proof of the theorem.

COROLLARY 1. *For each fixed $p \geqq 1$, $\mathcal{E}.\mathcal{S}.p$ is a linear invariant family which is never normal.*

PROOF. By Theorem 3, each $f(z) \in \mathcal{L}.\mathcal{S}.$ which is e.p.v. generates an e.p.v. linear invariant family. The union of linear invariant families is linear invariant, hence $\mathcal{E}.\mathcal{S}.p$ is linear invariant for each $p$. Theorem 5 implies that the order of $\mathcal{E}.\mathcal{S}._1$ is infinite. Therefore, by Theorem 4, $\mathcal{E}.\mathcal{S}._1$, and automatically $\mathcal{E}.\mathcal{S}.p$ ($\mathcal{E}.\mathcal{S}._1 \subset \mathcal{E}.\mathcal{S}.p$), is not a normal family.

The family of all $p$-valent functions in $\mathcal{L}.\mathcal{S}.$ (denoted $\mathcal{S}p$) is a normal family [**20**, p. 119], hence of finite order. The set of eventually univalent functions in $\mathcal{L}.\mathcal{S}.$ has infinite order. Therefore there are linear invariant families of eventually univalent functions of arbitrarily large valence. It is a surprising fact that this is not the case for linear invariant families of c.m.p.v. or a.m.p.v. or w.p.v. functions.

LEMMA 1. *Let $1 \leqq p < \infty$, $[p]$ denote the integer part of $p$ and $f(z)$ be $\mathcal{L}.\mathcal{S}.$ If the family $\mathcal{M}[f]$ is c.m.p.v. or a.m.p.v., then each function in $\mathcal{M}[f]$ can be no more than $[p]$ valent.*

PROOF. Let $\mathcal{M}[f]$ be a.m.p.v. and suppose that there were a function $g(z) = \Lambda_\phi[f(z)]$ in $\mathcal{M}[f]$ with valence $\geqq [p] + 1$. Let $\{z_i\}, i = 1, 2, \cdots, [p] + 1$ be points in $D$ such that $g(z_i) = w_0$ for some $w_0$ and $i = 1, 2, \cdots, [p] + 1$. Then $\Lambda_\psi[g(z)]$ is in $\mathcal{M}[f]$ where $\psi(z) = (z + z_1)/(1 + z\bar{z}_1)$ because $\mathcal{M}[f]$ is linear invariant. Furthermore,

$$\Lambda_\psi[g(z_i{}^*)] = 0, i = 1, 2, \cdots, [p] + 1$$

for $z_1{}^* = \psi^{-1}(z_i)$. Consequently, $\Lambda\ [g(z)]$ would have at least $[p] + 1$ zeros and therefore would completely cover some neighborhood of 0 at least $[p] + 1$ times. This is absurd since $\Lambda_\phi[g(z)]$ is a.m.p.v. Thus each function in $\mathcal{M}[f]$ must have valence $\leqq [p]$. The proof is the same for c.m.p.v. functions.

THEOREM 6. *Let $f(z) \in \mathcal{L}.\mathcal{S}.$ If the family $\mathcal{M}[f]$ is c.m.p.v. or a.m.p.v., then it is c.m.$[p]$.v. or a.m.$[p]$.v. Furthermore, $\mathcal{M}[f]$ is a.m.$[p]$.v. if and only if it is c.m.$[p]$.v. if and only if it is $[p]$-valent.*

PROOF. If $\mathcal{M}[f]$ is c.m.p.v. or a.m.p.v., then Lemma 1 proves that each function is $[p]$-valent hence necessarily c.m.[p].v. or a.m.[p].v.

If $\mathcal{M}[f]$ is $[p]$ valent then $\mathcal{M}[f]$ is necessarily c.m.[p].v. which necessarily implies $\mathcal{M}[f]$ is a.m.[p].v. which in turn implies that $\mathcal{M}[f]$ is $[p]$-valent. This completes the proof of Theorem 6.

Exactly as in Theorem 6, we can prove:

COROLLARY 2. *Let $p$ be a positive integer and $f(z) \in \mathcal{L.S.}$ The linear invariant family $\mathcal{M}[f]$ is weakly $p$-valent if and only if $\mathcal{M}[f]$ is $p$-valent.*

We always consider a $p$-valent function $q$-valent for $q \geqq p$.

Theorem 6 and Corollary 2 indicate that a.m.p.v., c.m.p.v. and w.p.v. are not meaningful or significant linear invariant properties. Theorem 5 and Corollary 1 indicate that e.p.v. is a meaningful linear invariant property.

V. **Linear Invariant Properties of Growth.** We wish to define a real valued function that will measure the asymptotic growth of the maximum modulus of an arbitrary nonconstant analytic function in the unit disc. If $f(z)$ is a nonconstant analytic function in $D$, we let

$$\operatorname{growth} f = \gamma = \sup\{c \text{ real} : \limsup_{r\to 1} M(r,f)(1-r)^c = \infty\}.$$

LEMMA 2. *If $f(z) \in \mathcal{L.S.}$ and growth $f = \gamma$, then for every $\phi(z) \in \mathcal{J}$, growth $\Lambda_\phi[f] = \gamma$.*

PROOF. Any $\phi(z) \in \mathcal{J}$ has the form $e^{i\theta}(z+\zeta)/(1+\bar{\zeta}z)$ for some $|\zeta| < 1$ and $\theta \in [0, 2\pi]$. We fix a $\phi(z) \in \mathcal{J}$ and note that

$$|\phi(re^{i\theta})| \leqq \frac{r+|\zeta|}{1+r|\zeta|} \equiv s \quad (0 \leqq \theta \leqq 2\pi)$$

$$\frac{1-r}{1-s} \leqq \frac{1+|\zeta|}{1-|\zeta|} < \infty.$$

By the maximum principle, we have that $M(r, f(\phi(z))) \leqq M(s, f(z))$. Thus for any $c > \gamma \geqq 0$, we have the inequalities

$$0 \leqq \limsup_{r\to 1} M(r, \Lambda\ [f])(1-r)^c$$

$$\leqq \frac{1}{|\phi'(0)f'(\phi(0))|} \limsup_{r\to 1} M(r, f(\phi(z)))(1-r)^c$$

$$\leqq \left(\frac{1+|\zeta|}{1-|\zeta|}\right)^c \frac{1}{|\phi'(0)f'(\phi(0))|} \limsup_{s\to 1} M(s, f(z))(1-s)^c = 0.$$

Therefore, for any $\phi(z) \in \mathcal{J}$ and any $f(z) \in \mathcal{L.S.}$ we always have that

$$\text{(11)} \qquad \text{growth}\, \Lambda_\phi[f] \leqq \gamma.$$

But $\mathcal{J}$ is a group and $f(z) = \Lambda_\phi{}^{-1}[\Lambda_\phi[f(z)]]$. Therefore by applying (11) to $\phi^{-1}(z) \in \mathcal{J}$ and $\Lambda_\phi[f(z)] \in \mathcal{L.S.}$, we obtain the inequalities

$$\gamma = \text{growth} f = \text{growth}\, \Lambda^{-1}[\Lambda_\phi[f]] \leqq \text{growth}\, \Lambda_\phi[f] \leqq \gamma.$$

This concludes our proof of the theorem.

Lemma 2 shows that all of the translates $\Lambda_\phi[f]$ of a function $f(z) \in \mathcal{L.S.}$ have the same growth as $f(z)$; that is, the growth of a function in $\mathcal{L.S.}$ is a linear invariant property and allows us to partition the families $\mathcal{U}_\alpha$ into linear invariant families in a manner which is extremely useful for general function theory. We begin by recalling a definition due to Campbell, Cima and Pfaltzgraff [4]. *The family* $\hat{\mathcal{U}}_\Delta$ is the set of all $f(z)$ in $\mathcal{U}_\Delta$ which have order precisely $\Delta$. It is obvious that $\hat{\mathcal{U}}_\Delta$ is itself linear invariant and $\mathcal{U}_\alpha$, for any $\alpha \geqq 1$, is the union of all $\hat{\mathcal{U}}_\Delta$ with $1 \leqq \Delta \leqq \alpha$.

We define *the family* $\mathcal{U}(\alpha, \beta)$ to be the set of all $f(z)$ in $\hat{\mathcal{U}}_\alpha$ such that growth $f = \beta$.

THEOREM 7. *The families $\mathcal{U}(\alpha,\beta)$ are nonempty if and only if $\alpha \geqq 1$ and $0 \leqq \beta \leqq \alpha$. The families $\mathcal{U}(\alpha,\beta)$ are linear invariant. The union of all $\mathcal{U}(\alpha,\beta)$ for $0 \leqq \beta \leqq \alpha$ is precisely the family $\hat{\mathcal{U}}_\alpha$.*

PROOF. We begin by showing that if $\mathcal{U}(\alpha,\beta)$ is nonempty, then $\alpha \geqq 1$ and $0 \leqq \beta \leqq \alpha$. If $f(z)$ is in $\mathcal{U}(\alpha,\beta)$ then it has order $\alpha$ and growth $\beta$. However, the order of a function is never less than 1 and the growth is always nonnegative, so it only remains to show that $\alpha \geqq \beta$. For any function of order $\alpha$ we have from (3)

$$|f(z)| \leqq \frac{1}{2\alpha}\left(\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right), |z| = r < 1,$$

and consequently, growth $f = \beta \leqq \alpha$.

To prove the converse, it suffices to show, for each $\beta \geqq 0$ and each $\alpha \geqq \max(\beta, 1)$, that $\mathcal{U}(\alpha,\beta)$ is nonempty. Since $f(z) = (e^{az} - 1)/a$, $a = \alpha + (\alpha^2 - 1)^{1/2}$, has growth 0 and order $\alpha$ [3], we can assume that $\beta$ is positive.

The function

$$\text{(12)} \qquad f(z) = \frac{1}{2\beta}\left(\left(\frac{1+z}{1-z}\right)^{\beta} - 1\right), \beta > 0$$

has order $\beta$ if $\beta \geqq 1$. Since $f(z)$ is convex if $0 < \beta \leqq 1$, we observe that $f(z)$ has order 1 if $0 < \beta \leqq 1$ [20, p. 134]. Therefore, order $f(z) = \max(1, \beta)$.

Let us define the function

$$G_a(z) = \int_0^z e^{aw}(1+w)^{\beta-1}/(1-w)^{\beta+1}\, dw, \; a \geqq 0, \beta > 0.$$

We use the space structure of $(X, \| \ \|)$ and write

$$G_a(z) = [f + [ag]](z)$$

where $g(z) = e^z - 1 \in \mathcal{U}_1$ and $f(z)$ is (12). Exactly as in Theorem 5, the mapping $k : a \to$ order $G_a$ has the properties (1) $k(0) = \max(1, \beta)$, (2) $k(a) \to \infty$ as $a \to +\infty$, (3) $k$ is a continuous mapping of the reals to the reals. Therefore, for each $\alpha \geqq \max(1, \beta)$, there is a function $G_a(z)$ of order $\alpha$. For any $a$, it is clear that growth $G_a(z) = \beta$. This shows the existence of a function in $\mathcal{U}(\alpha, \beta)$ for every $\beta > 0$ and every $\alpha \geqq \max(1, \beta)$.

To show that $\mathcal{U}(\alpha, \beta)$ is linear invariant, it suffices to show for each $f(z) \in \mathcal{U}(\alpha, \beta)$ and each $\phi(z) \in \mathcal{J}$ that $\Lambda_\phi[f]$ is also in $\hat{\mathcal{U}}_\alpha$ and has growth $\beta$. This follows immediately from Lemma 2 and the fact that $\hat{\mathcal{U}}_\alpha$ is itself linear invariant.

Each $f(z) \in \hat{\mathcal{U}}_\alpha$ has growth $f$ between 0 and $\alpha$ by (3). Therefore $\hat{\mathcal{U}}_\alpha$ is contained in the union of all $\mathcal{U}(\alpha, \beta)$ for $0 \leqq \beta \leqq \alpha$. The converse is trivial. Thus $\hat{\mathcal{U}}_\alpha$ is the union of all the $\mathcal{U}(\alpha, \beta)$ for $0 \leqq \beta \leqq \alpha$ which concludes our proof of the theorem.

One can verify explicitly that $\mathcal{U}(\alpha, \beta)$ is nonempty for $0 < \beta \leqq \alpha$ and $2^{1/2} \leqq \alpha$ by considering the function $f_{\alpha,\beta}(z) = \int_0^z (1-z)^{\alpha-1}/(1+z)^{\beta+1}\, dz$.

Pommerenke defined the family $\mathcal{U}_\alpha$ to be the set of all $f(z) \in \mathcal{L.S.}$ such that order $f \leqq \alpha$. These families appear to be the natural setting for many questions involving the distortion theorems for functions in $\mathcal{L.S.}$ [2], but they do not seem to provide the natural setting for questions involving the growth of the modulus of $f(z)$. We therefore define *the family* $\mathcal{U}^\gamma$ to be the set of all $f(z) \in \mathcal{L.S.}$ such that growth $f \leqq \gamma$.

Theorem 8. *The families* $\mathcal{U}^\gamma$ *are nonempty if and only if* $\gamma \geqq 0$. *The families* $\mathcal{U}^\gamma$ *are linear invariant.*

Proof. If $\gamma \geqq 0$, then the family $\mathcal{U}(\gamma + 1, \gamma)$ is contained in $\mathcal{U}^\gamma$ and is nonempty by Theorem 7. Since growth $f \geqq 0$, it follows that $\mathcal{U}^\gamma$ is nonempty only if $\gamma \geqq 0$. Lemma 2 implies that the family $\mathcal{U}^\gamma$ is a linear invariant family.

The families $\mathcal{U}^\gamma$ differ in a critical manner from the $\mathcal{U}_\alpha$. The families $\mathcal{U}_\alpha$ are both normal and compact; the families $\mathcal{U}^\gamma$ are not normal and in fact even $\mathcal{U}^\gamma \cap X$ is not normal for any $\gamma \geqq 0$. Theorem 7 assures us that $\mathcal{U}(\alpha, \gamma) \subset \mathcal{U}^\gamma \cap X$ for all $\alpha \geqq \max(1, \gamma)$. Therefore the order of $\mathcal{U}^\gamma \cap X$ must be infinite, which by Theorem 4, implies that $\mathcal{U}^\gamma \cap X$, hence also $\mathcal{U}^\gamma$, cannot be normal.

As previously remarked, an arbitrary a.m.p.v. or c.m.p.v. function obeys the growth law

$$M(r, f(z)) = O((1 - r)^{-2p}), |z| = r. \tag{13}$$

The proof of (13) is heavily dependent on the fact that such functions can have no more than $p$ zeros. On the other hand, e.p.v. functions also satisfy (13) despite the fact that they can have an *infinite* number of zeros.

THEOREM 9. *If $f(z)$ is eventually $p$-valent, then*

$$M(r, f(z)) = O((1 - r)^{-2p}), |z| = r. \tag{14}$$

*If $f(z) \in \mathcal{L}.\mathcal{S}.$ is* e.p.v., *then $f(z) \in \mathcal{U}^{2p}$.*

PROOF. We may assume that $f(z)$ is unbounded, since otherwise (14) is obvious. Because $f(z)$ is e.p.v., there is an $R_f$ such that $n(Re^{i\theta}) \leqq p$ for all $R_f < R < \infty$. Therefore, we can pick a point $w_1$ such that $|w_1| > R_f$ and $f(z) = w_1$ has at least one and at most $p$ roots in $D$. The function $g(z) = f(z) - w_1$ has at most $p$ zeros in $D$ and, as is easily checked, $g(z)$ is e.p.v. However, $M(r, g(z)) = O((1 - r)^{-2p})$ implies $M(r, f(z)) = O((1 - r)^{-2p})$ and consequently it suffices to prove the theorem for unbounded e.p.v. functions with at most $p$ zeros in $D$.

We let

$$p(R) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\theta})\, d\theta,$$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

$$R_1 = \max\left( R_f, (p + 2)2^{p-1} \cdot \max_{k \leqq p} |a_k| \right),$$

$$R_2 = M(r, f(z)), |z| = r.$$

Then, by Theorem 2.4 of Hayman [9],

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} < 2 \log \frac{1}{1 - r} + A(p) \tag{15}$$

where $A(p)$ is a constant that depends only on $p$. For any $R > R_1$ we have that $p(R) \geqq p$ and hence, for $|z|$ sufficiently close to 1,

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} \geqq \frac{1}{p} \log \frac{R_2}{R_1}. \tag{16}$$

Consequently,

$$\frac{1}{p} \log \frac{M(r,f)}{R_1} < 2 \log \frac{1}{1-r} + A(p)$$

or

$$M(r,f) < A(p, R)/(1-r)^{2p},$$

where $A(p, R_1)$ is a constant depending only on $p$ and $R_1$. Therefore, $M(r,f) = O(1-r)^{-2p})$.

If $f(z) \in \mathcal{L}.\mathcal{S}.$ is e.p.v., then the above result, the definition of $\mathcal{U}^{2p}$ and Lemma 2 imply that $f(z) \in \mathcal{U}^{2p}$. This concludes the proof of Theorem 9.

Theorem 9 indicates that the growth law (13) does not require global averaging properties such as c.m.p.v. or a.m.p.v., if one places a strong enough local control on $f(z)$ in a neighborhood of infinity. Theorem 9 can also be obtained from some very deep theorems of Hayman [7, p. 159]. The proof given above is of interest because of its brevity in contrast to [7].

VI. **Wing's Theorem and the Asymptotic Bieberbach Conjecture.** Let $S = \{f \in \mathcal{L}.\mathcal{S}. : f \text{ is univalent}\}$. Bieberbach conjectured that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{S}$, then $|a_n| \leqq n$. If the conjecture is true, then it is sharp as $f(z) = z/(1-z)^2 = \sum_{n=1}^{\infty} nz^n$ indicates. The conjecture is known to be true for $n = 2$ [1], $n = 3$ [14], $n = 4$ [6], $n = 5$ [23] and $n = 6$ [17], [18]. Littlewood [13] showed that in general $|a_n| < en$. There is a long history on the improvement of the coefficient of $n$ and the best, until recently, was due to Milin [15], $|a_n| < 1.243n$. However, Fitzgerald [24] and his student Horowitz [25] have improved this to $|a_n| < 1.0691n$. Up to 1950 it appears that no one had investigated whether or not the Bieberbach conjecture was true in some average sense.

For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$, Wing [22] considered the following averages of the coefficients of $f(z)$. Let

$$S_n(k) = \sum_{j=0}^{n-1} \binom{j+k-1}{k-1} a_{n-j} \quad k \geqq 1 \tag{17}$$

and

(18) $$\sigma_n(k) = |S_n(k)| \Big/ \binom{n+k}{k+1} \quad k \geqq 1.$$

In particular,

$$\sigma_n(1) = \frac{2}{n(n+1)} \sum_{k=1}^{n} a_k.$$

If the Bieberbach conjecture is true, then it is easy to verify that this would imply $\sigma_n(k) \leqq 1$ for all $n \geqq 1$ and all $k \geqq 1$. Thus Wing formulated and proved the *Asymptotic Bieberbach Conjecture:*

*If* $f(z) \in \mathcal{S}$, *then* $\overline{\lim}_{k\to\infty} \overline{\lim}_{n\to\infty} \sigma_n(k) \leqq 1.$

In 1955 Hayman [8] proved that for each normalized $f(z)$ which is c.m.l.v., there is an $n(f)$ such that for all $n \geqq n(f)$

(19) $$\left| \frac{a_n}{n} \right| \leqq 1.$$

In 1967 Eke [5] extended (19) to the class of normalized a.m.l.v. functions. It is easy to see that if

(20) $$\overline{\lim_{n\to\infty}} |a_n/n| \leqq B \leqq 1$$

then $\overline{\lim}_{k\to\infty} \overline{\lim}_{n\to\infty} \sigma_n(k) \leqq B \leqq 1$. Thus the Asymptotic Bieberbach conjecture is true for any class which satisfies (20).

The techniques of Hayman and Eke are dependent on an area principle which is not available for functions in $\mathcal{U}_\alpha$ or $\mathcal{U}^\gamma$. It therefore seems worthwhile to point out the full strength of Wing's original method. The results suggest some open questions which will be posed at the end of this section.

We first prove a technical lemma. We shall let $\Gamma(x)$ denote the Gamma Function (see [21], p. 55–8).

LEMMA 3. *If* $k \geqq 2$ *and* $r = 1 - (k+1)/n$, *then*

(21) $$\lim_{n\to\infty} \frac{n^{1-k}}{2\pi} \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^k} = \frac{\Gamma(k-1)}{\Gamma^2(k/2)(k+1)^{k-1}2^{k-1}}.$$

*If* $k = 1$ *and* $r = 1 - 2/n$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^k}=O(\log n). \tag{22}$$

PROOF. We begin with the case $k \geqq 2$. The integral can be evaluated by first noting that

$$|1-re^{i\theta}|=(1-2r\cos\theta+r^2)^{1/2}$$

so that [19]

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^k}=\frac{1}{2\pi}\cdot\frac{1}{(1-r^2)^{k/2}}\int_0^{2\pi}\frac{(1-r^2)^{k/2}\,d\theta}{(1-2r\cos\theta+r^2)^{k/2}}$$
$$=(1-r^2)^{-k/2}P_{k/2-1}\left\{\frac{1+r^2}{1-r^2}\right\}$$

where $P_a(x)$ is the Legendre function of the first kind of order $a$. Since $\lim_{x\to\infty} P_a(x)/x^a = 2^{-a}\cdot\Gamma(2a+1)/\Gamma^2(a+1)$ for $a > -1/2$ [19], if we let $r = 1-(k+1)/n$ we obtain

$$\lim_{n\to\infty}\frac{n^{1-k}}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^k}=\frac{\Gamma(k-1)}{\Gamma^2(k/2)(k+1)^{k-1}2^{k-1}}.$$

We now assume $k = 1$.

$$\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|}=\frac{1}{r^{1/2}}\int_0^{\pi}\frac{d\theta}{[(1-r)^2/4r+\sin^2\theta/2]^{1/2}}$$
$$\leqq\frac{1}{r^{1/2}}\left[2^{3/2}\int_0^{\pi/3}\frac{d\theta}{[2(1-r)^2/r+\theta^2]^{1/2}}\right.$$
$$\left.+\int_{\pi/3}^{\pi}\frac{d\theta}{[(1-r)^2/4r+\sin^2\pi/6]^{1/2}}\right]$$
$$=O(|\log(1-r)|.$$

Thus, letting $r = 1-2/n$, we obtain

$$\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|}=O(\log n).$$

We now let $\mathcal{H}(D)$ denote the set of functions analytic in $D$ of the form $f(z) = z+\sum_{k=2}^{\infty}a_kz^k$.

THEOREM 10. *Let $\gamma \geqq 0$, $f(z) \in \mathcal{H}(D)$ and suppose that $M(r,f) \leqq A(1-r)^{-\gamma}$ for all $|z| \geqq r_0$, $0 < r_0 < 1$.*
*If $k = 1$, then*

(23) $$\sigma_n(1) = O(n^{\gamma-2}\log n)$$

*If* $k \geqq 2$, *then*

(24) $$\limsup_{n\to\infty} (\sigma_n(k)/n^{\gamma-2}) \leqq \frac{A\Gamma(k+2)\Gamma(k-1)}{(k+1)^{\gamma+k-1}2^{k-1}\Gamma^2(k/2)e^{k\gamma+1}}.$$

PROOF. By a simple application of the residue theorem,

$$S_n(k) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}(1-z)^k}\,dz,\ r<1.$$

Thus,

$$|S_n(k)| \leqq \frac{M(r,f)}{r^n}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-ze^{i\theta}|^k}.$$

We assume $k \geqq 2$, $r = 1-(k+1)/n$ and use (18) and (21) for large $n$ to write

$$\sigma_n(k) \leqq \frac{An^\gamma}{(k+1)^\gamma}\frac{1}{(1-(k+1)/n)^n}\frac{n^{k-1}\Gamma(k-1)}{\Gamma^2(k/2)(k+1)^{k-1}2^{k-1}}\cdot\binom{n+k}{k+1}^{-1}.$$

However, $\binom{n+k}{k+1} \cong n^{k+1}/\Gamma(k+2)$ for large $n$, therefore

$$\sigma_n(k) \leqq \frac{An^{\gamma-2}\Gamma(k+2)\Gamma(k-1)}{(k+1)^{\gamma+k-1}2^{k-1}\Gamma^2(k/2)e^{k\gamma+1}}$$

from which (24) follows.

The proof for $k = 1$ is identical to the above except for the use of (22) instead of (21).

Thus for functions which are c.m.p.v., c.m.p.v., w.p.v., e.p.v. or in $\mathcal{U}_\alpha$, or in $\mathcal{U}^\gamma$, Theorem 10 automatically gives asymptotic estimates for $\sigma_n(k)$. In particular:

COROLLARY 3. *If* $f(z) \in \mathcal{U}_\alpha$, *then*

(25) $$\limsup_{n\to\infty}\ (\sigma_n(k)/n^{\alpha-2}) \leqq \frac{2^{\alpha-1}\Gamma(k+2)\Gamma(k-1)}{\alpha(k+1)^{\alpha+k+1}\cdot 2^{k-1}\Gamma^2(k/2)e^{k+1}}.$$

PROOF. From (3) it follows for any $f(z) \in \mathcal{U}_\alpha$ that $M(r,f) \leqq 2^{\alpha-1}(1-r)^{-\alpha}\alpha^{-1}$ so $A = 2^{\alpha-1}/\alpha$ in Theorem 10.

COROLLARY 4. *If* $f(z) \in \mathcal{U}_2$, *then the Asymptotic Bieberbach Conjecture* $\overline{\lim}_{k\to\infty}\overline{\lim}_{n\to\infty}\sigma_n(k) \leqq 1$ *is true.*

PROOF. An application of Stirling's estimate to the right hand side of (25) (with $\alpha = 2$) yields the conclusion. Corollary 4 yields Wing's original theorem as a special case since $\mathcal{S}$ is contained in $\mathcal{U}_2$.

COROLLARY 5. *If $f(z) \in \mathcal{H}(D)$ and $M(r,f) \leqq (1-r)^{-2}$ for all $|z| \geqq r_0$ $(0 < r_0 < 1)$, then the Asymptotic Bieberbach Conjecture is true.*

PROOF. This follows from a direct application of Stirling's estimate to (24).

Thus it is the eventual behavior of $M(r,f)$ rather than univalence or even local univalence that appears to be the crucial element in the proof of the Asymptotic Bieberbach Conjecture.

The second part of Theorem 10 is best possible in the following sense: there are functions with $M(r,f) \leqq A(1-r)^{-\alpha}$, $\alpha > 2$ for which $\overline{\lim}_{n\to\infty} \sigma_n(k) = \infty$. For example, (6) is in $\mathcal{U}_\alpha$ and in $\mathcal{U}^\alpha$, and Kirwan [12] has shown that its coefficients $a_n$ are positive and satisfy $a_n \simeq n^{\alpha-1}/2\alpha\Gamma(\alpha)$. Therefore, if $M$ is an arbitrarily large positive number, we can choose $N_1$ such that $N_1{}^{\alpha-2}/8\alpha\Gamma(\alpha) > M$ and $N_2$ such that $n \geqq N_2$ implies that $a_n > n^{\alpha-1}/4\alpha\Gamma(\alpha)$. If we let $N = \max[N_1, N_2]$ and choose $n > N$ such that $\binom{N+k}{k+1}/\binom{n+k}{k+1} < 1/2$, then

$$\begin{aligned} S_n(k) &> \sum_{j=N}^{n} \binom{k+n-1-j}{k-1} \frac{j^{\alpha-1}}{4\alpha\Gamma(\alpha)} \\ &> \frac{N^{\alpha-2}}{4\alpha\Gamma(\alpha)} \sum_{j=N}^{n} \binom{k+n-1-j}{k-1} j \\ &> 2M \left[ \binom{n+k}{k+1} - \binom{N+k}{k+1} \right]. \end{aligned}$$

Hence for any $k \geqq 1$ and all $n > N$, $\sigma_n(k) > m$, which implies that $\overline{\lim}\, \sigma_n(k) = \infty$ as claimed.

The average

$$\sigma_n(1) = \frac{2}{n(n+1)} \sum_{j=1}^{n} a_j$$

is a very natural coefficient average. For this particular average, the first conclusion of Theorem 10 asserts that for functions in $\mathcal{H}(D)$ with $M(r,f) \leqq A(1-r)^{-2}$, we have $\sigma_n(1) = O(\log n)$. In light of the previous paragraph it is reasonable to ask if this is best possible in the following sense: Are there functions in $\mathcal{H}(D)$ with $M(r,f) \leqq A(1-r)^{-2}$ for which $\overline{\lim}_{n\to\infty} \sigma_n(1) = \infty$? The results of Hayman [8] and Eke [5] show that the answer is negative for c.m.l.v. or a.m.l.v. functions. The answer is negative for any class of functions with $a_n = O(n)$ since then $\overline{\lim}_{n\to\infty} \sigma_n(1) < \infty$. However, for functions which are in $\mathcal{U}_2$ or $\mathcal{U}^2$ or are e.l.v. or w.l.v., the only known coefficient estimates are the trivial estimates $a_n = O(n^2)$. Thus we pose the following four questions.

(1) Do there exist functions in $\mathcal{H}(D)$ such that $M(r,f) = O((1 - r)^{-2})$ and $\overline{\lim}_{n\to\infty} \sigma_n(1) = \infty$?

(2) Do there exist functions in $\mathcal{H}(D)$ which are eventually one valent or weakly one valent such that $\overline{\lim}_{n\to\infty} \sigma_n(1) = \infty$?

(3) Do there exist functions in $\mathcal{H}(D) \cap \mathcal{U}_2$ such that $\overline{\lim}_{n\to\infty} \sigma_n(1) = \infty$?

(4) Do there exist functions in $\mathcal{H}(D) \cap \mathcal{U}^2$ such that $\overline{\lim}_{n\to\infty} \sigma_n(1) = \infty$?

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