ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 8, Numbers 1 and 2,
Winter and Spring 1978

INTRODUCTION TO QUANTIZATION AND QUANTUM FIELD THEORY

The solitons which have been discussed in earlier sections of this volume belong to the realm of classical mechanics; in this section the quantum mechanics of solitons is discussed. The most typical difference between classical and quantum phenomena is this: quantities such as oscillation frequency or energy, which classically range over a continuum, are constrained to lie in a discrete set of possible values in certain quantum situations. For example, the masses of subnuclear particles are observed to be restricted in this sense, so it is natural to assume that some form of quantum mechanics must be the appropriate mathematical framework. Solitons, because of their extreme localization and stability, are tempting candidates for models of elementary particles; however, if they are to be useful in this respect, their description must be raised from the classical to the quantum level.

Recall that, in principle, one quantizes a system by interpreting all observable functions, such as momentum and energy, as self adjoint operators on a suitable Hilbert space. Hamilton's classical equations of motion are then replaced by Heisenberg's operator equations, and one investigates the spectrum and the temporal evolution of these operators. The point spectrum of the Hamiltonian or energy operator describes the discrete oscillation frequencies. This procedure, when applied to those classical wave equations which support solitons, is called the "quantization of solitons." (This phrase is somewhat misleading. The equations of motion, not the solitons, are actually quantized.)

Severe existence problems immediately arise because these classical wave equations which are undergoing quantization are *infinite* dimensional, *nonlinear* Hamiltonian systems. The constructive field theorist will first of all concentrate upon matters such as the existence of observables as self adjoint operators. The constructive approach, unfortunately, was not represented at the conference. Interested readers are referred to the books by Streater and Wightman [1] and by Simon [2] for the general framework, and to the article of Fröhlich [3] for the connection with solitons.

Many physicists tend to take existence of operators and Hilbert spaces for granted, and concentrate on approximating quantities of physical interest, among them: the ground state eigenvalues of the energy operator, its eigenfunctions, and certain Green's functions which describe the propagation from one state to another. To get some feeling for the physical ideas behind these calculations, we outline the quantization for two related models: the nonlinear sine-Gordon equation ($\phi_{tt} = \phi_{xx} - \sin\phi$) and the linear Klein-Gordon equation ($\phi_{tt} = \phi_{xx} - \phi$).

In addition to continuous radiation at any wavelength, the classical sine-Gordon equation possesses localized solutions of three types: solitons, antisolitons, and breathers. The solitons and antisolitons are localized waves which translate at constant speeds, while the breathers consist of envelopes which pulsate as they translate. All three types of localized waves are nonlinear effects which have no analogues in the Klein-Gordon theory. In addition to the radiation modes, the high energy physicist now has several new candidates in his search for a description of elementary particles. Let us now see how these localized waves affect the spectrum of the quantized Hamiltonian.

Both the Klein-Gordon and sine-Gordon equations can be written as classical, infinite dimensional, Hamiltonian systems,

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \pi\end{pmatrix} = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\begin{pmatrix}\delta H/\delta\phi\\ \delta H/\delta\pi\end{pmatrix}. \tag{1}$$

The Hamiltonians H are given by

$$\begin{aligned} H_{K.G.} &\equiv \int_{-\infty}^{\infty} \frac{1}{2}[\pi^2 + (\phi_x)^2 + \phi^2]\,dx,\\ H_{S.G.} &\equiv \int_{-\infty}^{\infty} \left[\frac{1}{2}(\pi^2 + (\phi_x)^2) + (1 - \cos\phi)\right] dx. \end{aligned} \tag{2}$$

The Poisson bracket structure is defined by

$$\{F, G\} = \int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\phi} - \frac{\delta F}{\delta\phi}\frac{\delta G}{\delta\pi}\right] dx.$$

In the classical cases, the functions ϕ and π satisfy the Poisson relations

$$\begin{aligned} \{\phi(x, t), \phi(x', t)\} &= \{\pi(x, t), \pi(x', t)\} = 0,\\ \{\pi(x, t), \phi(x', t)\} &= \delta(x - x'), \end{aligned} \tag{3}$$

with the temporal evolution given by (1), or equivalently, by

$$\begin{aligned} \frac{d}{dt}\phi &= \{H, \phi\},\\ \frac{d}{dt}\pi &= \{H, \pi\}. \end{aligned} \tag{4}$$

One version of the problem of quantization is to find (for each t) self adjoint operators $\hat{\phi}$ and $\hat{\pi}$ on a Hilbert space which satisfy the equal time commutation relations (3) and evolve in t according to (4). Of course, the brackets $\{\ \}$ are now interpreted as operator commutators, $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$.

For the Klein-Gordon equation, the solution of this problem is well known. (See, for example [4].) It begins with the observation that a Fourier transformation decouples the classical system into a collection of noninteracting normal modes. The Fourier transformation defines a canonical transformation from (ϕ, π) to (a, a^*),

$$a(k, t) \equiv N \int_{-\infty}^{\infty} \left[\sqrt{k^2 + 1}\, \phi(x, t) + i\pi(x, t) \right] \frac{e^{ikx}}{(k^2 + 1)^{1/4}}\, dx, \tag{5}$$

where N denotes a suitable normalization constant. Under this transformation, the Poisson bracket structure is preserved with $\{a^*(k, t), a(k', t)\} = i\delta(k - k')$, the Klein-Gordon Hamiltonian takes the form

$$H_{K.G.} \equiv \int_{-\infty}^{\infty} \sqrt{k^2 + 1}\, \rho(k)\, dk, \quad \rho(k) \equiv |a(k)|^2, \tag{6}$$

and the temporal evolution is given by

$$\frac{d}{dt} a(k, t) = \{H_{K.G.}, a(k, t)\} = i \sqrt{k^2 + 1}\ \ a(k, t).$$

Thus, the classical Klein-Gordon equation is equivalent under a canonical transformation to a collection of noninteracting harmonic oscillators, each of which is quantized separately following the prescription of single particle quantum mechanics [4, 5]. The fundamental excitations of these harmonic oscillators are then used to describe the spectrum of the Hamiltonian operator $\hat{H}_{K.G.}$ for the Klein-Gordon field.

The inverse scattering transform is also a canonical transformation which maps the classical sine-Gordon field into decoupled, nonlinear normal modes. The precise definition of this transformation is given in [6]. Here we merely state the form taken by the classical Hamiltonian in the new variables:

$$H_{S.G.} = \int_{-\infty}^{\infty} \sqrt{k^2 + m^2}\ \ \rho(k)\, dk + \sum_{a=1}^{A} \sqrt{p_a{}^2 + M^2} + \sum_{b=1}^{B} \sqrt{P_b{}^2 + M^2(\alpha_b)}. \tag{7}$$

Notice that in these variables the continuous term in the sine-Gordon Hamiltonian is indistinguishable from the normal mode Hamiltonian of the linear Klein-Gordon theory. The two discrete sums are new. They describe the solitons and breathers. Here the integer A labels the number of solitons and B the number of breathers. Just as in the linear theory, the density of radiation $\rho(k)$ can take any value between $0 \leqq \rho(k) < \infty$; the variables P_a and P_b range over $(-\infty, \infty)$, while $0 \leqq \alpha_b \leqq \pi/2$; $M \equiv 8/\gamma$ and $M(\alpha_b) \equiv (16 \sin \alpha_b)/\gamma$. γ is a dimensionless coupling constant which can be scaled out of the classical theory, but plays a role similar to Planck's constant h in the quantum theory [7]. The decoupling of the classical dynamics through a canonical transformation should permit, at least to some approximation, the problem of quantization to be solved by quantizing each nonlinear normal mode separately. That is, quantization should be achieved as in the linear theory, except that the sine-Gordon field has three basic excitations (radiation, solitons, and breathers) rather than one (radiation). For the continuous modes, quantization proceeds as outlined for the Klein-Gordon equation. Next, since the soliton behaves like a free particle, one anticipates that it will contribute only some trivial continuous spectrum. The breathers, however, give an interesting structure to the spectrum of $\hat{H}_{S.G.}$.

Recall that the breather consists of an envelope which translates at a constant speed but has an additional internal degree of freedom (the envelope beats periodically, in time, as it translates). In the variables of equation (7), P_b fixes the momentum of translation while α_b is related to the frequency of these internal oscillations. P_b quantizes trivially as a free particle (just as the soliton's momentum P_a), but the energy of internal oscillation can assume, after quantization, only certain discrete values. By any one of a variety of more or less sophisticated arguments, one can show that $M(\alpha_b)$ must belong to the discrete set

$$(8) \qquad M_n = \frac{16}{\gamma} \sin \frac{n\gamma}{16}\,; \; n = 1, 2, \cdots, \left[\frac{8\pi}{\gamma}\right],$$

where $[r]$ denotes the greatest integer less than or equal to r.

After quantization, the field consists of a collection of almost linear excitations, together with solitons of rest mass M and breathers whose "rest masses" belong to the set (8). $P_a{}^2$ describes an increase in energy of the soliton due to translation; $P_b{}^2$ the same increase in the breather's energy. The "rest mass" of two breathers can differ because of the energy of internal oscillation. Notice that to this order of approximation, *the only effect of quantization* is to make these internal oscillations energies of the breather discrete. Although the spectrum of the Hamiltonian is distinctly influenced by the breathers, this approximation is too crude to answer the natural questions: For example, are the important features of the classical soliton (locality, stability, absence of scattering) also important after quantization? To what extent will quantum effects couple the solitons and cause them to interact?

To answer such questions, one must first understand why the procedure we have just summarized for the quantization of the sine-Gordon equation is not exact. The approximation was introduced at the first step, which was the canonical transformation from $(\phi(x, t), \pi(x, t))$ to nonlinear normal modes. The canonical transformation of the classical variables is not linear; it contains products of the form $\phi(x, t)\,\pi(x', t)$. Once quantized, the functions ϕ and π become operators which do not commute when $x = x'$. The definition of the normal mode momenta $\{\hat{\rho}(k), \hat{P}_a, \hat{P}_b\}$ then becomes ambiguous; neglecting this ambiguity of product ordering yields the leading-order theory described above. This difficulty does not arise in the Klein-Gordon case, where the definition of $a(k, t)$ in equation (5) does not contain any products of the form $\phi\pi$. The meaning of the canonical variables (a, a^*) is therefore not ambiguous, even after quantization; however, the Hamiltonian $H_{K.G.}$ depends upon $\rho(k) = a(k)a^*(k)$, and the ordering of products of a, a^* becomes important in the definition of $\hat{H}_{K.G.}$. For harmonic oscillators the correct ordering is well known [4, 5],

$$\hat{H}_{K.G.} = \int_{-\infty}^{\infty} \sqrt{k^2 + 1}\left[\frac{a^*(k)a(k) + a(k)a^*(k)}{2}\right] dk.$$

In the sine-Gordon field, even the proper definition of the momentum variables is unclear.

The method of quantization described in the preceding paragraphs amounts to an extension of the old quantum theory of Bohr and Sommerfeld to quantum field theory. The next step will be to improve this approximation. There are various ways to derive higher-order corrections; most authors use the fact that the old quantum theory is the leading term in an asymptotic approximation known as the "W.K.B. method" (after Wentzel, Kramers, and Brillouin who introduced this calculational procedure into quantum mechanics). The common description of this W.K.B. method begins with the Schrödinger probabilistic representation of the quantum mechanics of particles [8]. As Planck's constant h goes to zero, the probability density is concentrated near a classical trajectory, that is, near a solution of Hamilton's equation of classical mechanics. The W.K.B. method is a systematic asymptotic procedure to calculate effects of small deviations from these classical trajectories.

In the first article of the section, Brosl Hasslacher and Andre Neveu describe extensions of such asymptotic calculations into nonlinear quantum field theory. In this setting, the quantum mechanical wave function defines a probability density in ϕ space which is concentrated, as the coupling constant γ (or Planck's constant h) goes to zero, near a solution of the classical field equation such as a breather in the sine-Gordon case. Hasslacher and Neveu use Feynman (function space) integral representations to derive the W.K.B. approximation for quantum field theory. In this manner they obtain corrections to the leading order approximation described above.

Introductory material about the Feynman integral—W.K.B. method of approximation of Dashen, Hasslacher, and Neveu may be found in the survey [9]. For general material about the Feynman (function space) integrals, we recommend the very tutorial article [10], the general survey [11], and, of course, the two original articles by Feynman [12, 13]. High energy physicists use many other related approaches to the quantization of solitons. Some of these methods are described in [7, 14, 15] and references therein.

In the second article of this section, David Cambell studies the "sigma model," a coupled system of partial differential equations which originally was introduced to describe protons and neutrons interacting with two types of mesons. The equations should really be quantized, but as we explained above, the first step is to find classical localized solutions. Cambell does this by an interesting and novel technique utilizing inverse scattering ideas. A more detailed quasiclassical quantization apparently remains to be worked out.

The emphasis we have placed on approximate quantization quite naturally raises the question: how reliable is this technique? One test case is readily available. The nonlinear Schroedinger equation,

$$iq_t - q_{xx} + 2|q|^2 q = 0, \tag{9}$$

can be quantized by the methods outlined above. The resulting formulas can be checked by entirely different means because, as is explained in Nohl's article, (9) is equivalent to the usual (linear) Schroedinger equation for an arbitrary number of particles interacting via δ function potentials. The spectrum for this latter problem was studied by Lieb and Liniger [16]. For detailed comparisons between the two approaches, and for the relevant literature, we refer the reader to [17].

The final article, by Bill Sutherland, replaces the viewpoint of the high energy physicist with that of an intuitive solid state theorist. He studies the quantum mechanics of a Toda lattice. In particular, he uses "Bethe's ansatz" to represent the ground state wave function, excitation spectrum near the ground state, and thermodynamic quantities. Sutherland identifies two types of excitations in the quantum Toda lattice, and he traces these excitations to the solitons and the periodic wave trains of the classical Toda equations. In reading Sutherland's paper, we have found Bethe's original article [18] and the article of Lieb and Liniger [16] useful background material. Also, the work of Luther [19] provides interesting, although difficult, connections between the field theorists' and the solid state physicists' descriptions of quantized solitons.

At the time of the conference, work on quantization dealt largely with the spectrum of the quantum operators. Since that time, some progress has been made on the scattering theory of the quantum states [17, 20] and research emphasis has been placed upon quantum mechanical tunneling processes [15]. Finally, the reader will have noticed that these quantum mechanical calculations are never compared with experimental observations. The main reason is that the current theoretical models are far too naive to even consider such a comparison. For example, as in all soliton research, the limitation to one spatial dimension is terribly restrictive.

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