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SOME RESULTS ON THE POLYNOMIALS $L_n^{\alpha,\beta}(x)$

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ABSTRACT. An integral representation, a finite sum formula and a series relation are derived for the polynomials $L_n^{\alpha,\beta}(x)$ defined by

$$(*)\qquad L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n + \beta + 1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^k}{\Gamma(\alpha k + \beta + 1)k!}, \ \mathrm{Re}\,\beta > -1$$

where α is any complex number with $\mathrm{Re}\,\alpha > 0$.

1. **Introduction.** Recently, Konhauser [4], Prabhakar [5] and Srivastava [9] established several results on the polynomials $Z_n^{\alpha}(x; k)$ defined by

$$(1.1)\qquad Z_n^{\alpha}(x; k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}$$

where k is a positive integer. In [6] we considered the polynomials $L_n^{\alpha,\beta}(x)$ defined by (*) and proved results for these polynomials. Evidently, if $\alpha = k$, a positive integer, then $L_n^{k,\beta}(x^k) = Z_n^{\beta}(x; k)$ and more particularly $L_n^{1,\beta}(x) = L_n^{\beta}(x)$, where $L_n^{\beta}(x)$ is the generalized Laguerre polynomial. Thus for $\alpha = k$, the result proved in [6] as also those of the present paper yield results on $Z_n^{\beta}(x; k)$. For $\alpha = 1$, all the results for $L_n^{\alpha,\beta}(x)$ reduce to known results for $L_n^{\beta}(x)$.

2. **An integral representation.** We first show that for all β, γ with $\mathrm{Re}\,\beta > \mathrm{Re}\,\gamma > -1$

$$(2.1)\qquad L_n^{\alpha,\beta}(x^{\alpha}) = \frac{\Gamma(\alpha n + \beta + 1)x^{-\beta}}{\Gamma(\alpha n + \gamma + 1)\Gamma(\beta - \gamma)} \int_0^x (x - u)^{\beta-\gamma-1} u^{\gamma} L_n^{\alpha,\gamma}(u^{\alpha})\, du.$$

PROOF. Using (*), we are led to

$$\int_0^x (x - u)^{\beta-\gamma-1} u^{\gamma} L_n^{\alpha,\gamma}(u^{\alpha})\, du$$

$$= \frac{\Gamma(\alpha n + \gamma + 1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{\Gamma(\alpha k + \gamma + 1)k!}$$

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$$\int_0^x (x-u)^{\beta-\gamma-1}\, u^{\alpha k+\gamma}\, du$$

$$= \frac{\Gamma(\alpha n+\gamma+1)}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, x^{\alpha k+\beta}\Gamma(\beta-\gamma)}{k!\Gamma(\alpha k+\beta+1)}$$

$$= \frac{\Gamma(\alpha n+\gamma+1)\Gamma(\beta-\gamma)x^{\beta}}{\Gamma(\alpha n+\beta+1)} L_n^{\alpha,\beta}(x^{\alpha}),$$

from which (2.11) follows.

For $\alpha = k$, we get an interesting integral relation for $Z_n^{\beta}(x; k)$:

$$Z_n^{\beta}(x; k) = \frac{\Gamma(kn+\beta+1)x^{-\beta}}{\Gamma(kn+\gamma+1)\Gamma(\beta-\gamma)} \int_0^x (x-u)^{\beta-\gamma-1}\, u^{\gamma}\, Z_n^{\gamma}(u; k)\, du$$

which is new. For $\alpha = 1$, (2.1) reduces to the known result for $L_n^{\beta}(x)$ [7; (15)].

3. **A summation formula.** Since the polynomials $L_n^{\alpha,\beta}(x)$ possess the generating relation [6; (2.1)]

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha,\beta}(x)t^n}{\Gamma(\alpha n+\beta+1)} = e^t\varphi(\alpha, \beta+1, -xt) \tag{3.1}$$

where $\varphi(\alpha, \beta, z)$ is the Bessel-Wright function [3; 18.1(27)], employing the technique of Srivastava ([8], [9]), we get

$$L_n^{\alpha,\beta}(x) = \Gamma(\alpha n+\beta+1)\left(\frac{x}{y}\right)^n \sum_{k=0}^{n} \frac{L_{n-k}^{\alpha,\beta}(y)\left(\frac{y}{x}-1\right)^k}{\Gamma(\alpha n-\alpha k+\beta+1)k!}. \tag{3.2}$$

For $\alpha = 1$, (3.2) yields a multiplication formula [3; 10.12(40)] for $L_n^{\beta}(x)$, whereas for $\alpha = k$, it leads to [9; (4)].

4. **A series relation for $L_n^{\alpha,\beta}(x)$.** We now make use of the fractional differentiation operator D_ω^{λ} defined by [2]

$$\begin{aligned} D_\omega^{\lambda}\{\omega^{\mu-1}\} &= \frac{d^{\lambda}}{d\omega^{\lambda}}\{\omega^{\mu-1}\} \\ &= \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\,\omega^{\mu-\lambda-1}, \text{ for } \lambda \neq \mu, \end{aligned} \tag{4.1}$$

to show that

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha,\beta}(x)(\lambda)_n}{\Gamma(\alpha n+\beta+1)(\mu+1)_n} {}_1F_1(\mu-\lambda+1; n+\mu+1; t)t^n$$

(4.2)

$$= \frac{\Gamma(\mu+1)}{\Gamma(\lambda)} e^t \, {}_1F_2^* \left(\begin{matrix} (1, \lambda) \\ (\alpha, \beta+1), (1, \mu+1) \end{matrix} ; -xt \right)$$

where ${}_pF_q^*$ is Wright's generalized hypergeometric function [3; 4.1].

Proof. If we rewrite (3.1) as

$$\sum_{n=0}^{\infty} \frac{L_n^{\alpha,\beta}(x)\, t^n}{\Gamma(\alpha n+\beta+1)} e^{-t} = \varphi(\alpha, \beta+1, -xt),$$

multiply both the sides by $t^{\lambda-1}$, and apply the operator $D_t^{\lambda-\mu-1}$, we get

$$\sum_{n=0}^{\infty} \frac{L_n^{\beta}(x)}{\Gamma(\alpha n+\beta+1)} \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(n+r+\lambda)}{r!\Gamma(n+r+\mu+1)} t^{n+r+\mu}$$

$$= \sum_{n=0}^{\infty} \frac{(-x)^n\Gamma(n+\lambda)}{n!\Gamma(\alpha n+\beta+1)\Gamma(n+\mu+1)} t^{n+\mu}$$

which immediately leads to (4.2).

For $\lambda = \mu + m + 1$ and $\alpha = 1$, (4.2) reduces to the result on $L_n^{\beta}(x)$ due to Al-Salam [1] proved by using an operator $x(1 + xD)$.

(4.2) can also be proved directly by the use of Kummer's transformation [3; 6.3(7)]. We owe this remark to the referee. We are, indeed extremely grateful to the referee for several valuable comments and suggestions which have helped us improve the paper.

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