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CANTOR SETS IN 3-MANIFOLDS[1]

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1. **Introduction.** We answer the following 3-dimensional questions posed by Bing and Daverman which show that wild Cantor sets in 3-manifolds behave essentially like a 1-dimensional polyhedron. Furthermore, any compactum in the interior of a 3-manifold can be approximated by a Cantor set. The questions below are unsolved for $n > 3$; however, some partial solutions are known and pointed out.

QUESTION 1.1 (BING). [4, Question 1, p. 17]. What are necessary and sufficient conditions on an n-manifold M^n without boundary in order that it have the property that each Cantor set in M^n lies in an open n-cell in M^n?

If we stipulate that the M^n in Bing's question is closed, then an answer to Bing's question is: M^n is homeomorphic to the n-sphere for $n = 3$ [8] and $n > 4$ [13].

DEFINITION 1.1. A compactum K in an n-manifold M^n is said to be *approximable by Cantor sets* if for each neighborhood U of K there exists a Cantor set C in U such that a loop γ in $M^n - U$ is inessential in $M^n - K$ if and only if γ is inessential in $M^n - C$. We say that the Cantor set C *approximates* K *with respect to* U.

QUESTION 1.2 (DAVERMAN). Is every compactum in the interior of an n-manifold approximable by Cantor sets?

Recent work of Daverman and Edwards [7] has shown that the answer to Question 1.2 is affirmative if K is a closed, flat, PL $(n - 2)$-dimensional manifold.

I would like to express my indebtedness to the friendship and instruction of J. W. Cannon. I would also like to thank R. J. Daverman for a helpful discussion.

2. **Approximating compacta by Cantor sets.**

LEMMA 2.1. *Suppose P is a polyhedral finite graph in the interior of a 3-manifold M. Then P is approximable by Cantor sets.*

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[1]Some of this work is found in the author's doctoral thesis under Professor J. W. Cannon at the University of Wisconsin, Madison.

PROOF. We assume P is connected. If P has more than one component we repeat the following argument for each component of P. We assume that a neighborhood U of P is given and let N be a regular neighborhood of P which is contained in U. Let B be a regular neighborhood of a maximal tree of P which is contained in the interior of N. The set B is a PL 3-cell. We let $\sigma_1, \cdots, \sigma_k$ denote the 1-simplexes of P. Each σ_i is contained in an open set U_i of N which is homeomorphic to Euclidean 3-space. We choose a PL 3-cell B_i in each U_i such that (B_i, σ_i) is an unknotted cell pair and $B_i \cap B_j = \sigma_i \cap \sigma_j$ for $i \neq j$. We let S_i^2 denote the boundary of B_i. We now let G_i be a regular neighborhood of $\mathrm{Bd}\, \sigma_i$ in S_i^2 which is contained in B. We then denote by N_i a regular neighborhood of $\mathrm{Cl}(S_i^2 - G_i)$ which is contained in U_i. Let C_i be an Antoine's necklace [**2**, **12**] in $U_i - N_i$ which links N_i; i.e., the inclusion from N_i into $U_i - C_i$ induces a monomorphism on fundamental groups.

Since B is cellular, there exists a map [**5**] $P : \mathrm{M} \to M$ such that the restriction to $M - N$ is the inclusion and the only nondegenerate inverse set is B. Let C be a Cantor set in N which contains the compact 0-dimensional set $P(\bigcup C_i \cup B)$. The Cantor set C is the desired Cantor set.

Let γ be a loop in $M - U$ which is inessential in $M - C$. Thus, there is a map $f : D \to M - C$ where D is a disk and $f \mid \mathrm{Bd}\, D = \gamma$. The map P^{-1} of f shows that γ bounds in the complement of $\bigcup C_i \cup B$. Hence, there is a map $F : D \to M - (\bigcup C_i \cup B)$ such that $F \mid \mathrm{Bd}\, D = \gamma$ and F is in general position with respect to the 2-spheres S_i^2. The set $F^{-1}(\bigcup S_i^2)$ is a collection of disjoint simple closed curves in D. The map F restricted to each of these simple closed curves is a trivial loop in some $S_i^2 - G_i$. If this were not the case, a simple argument would show that the inclusion from N_i into $U_i - C_i$ does not induce a monomorphism on fundamental groups. Let $J_1, \cdots, J_m$ be the outermost simple closed curves. Redefine F on the interior of the disks bounded by the J_i by sending the interior of each disk to some $S_j^2 - G_j$ and we see that γ is inessential in $M - P$.

On the other hand, if γ is a loop of $M - U$ which is inessential in $M - P$, γ is inessential in $M - N$ and hence in $M - C$.

I am indebted to Ric Ancel for pointing out the following result. Ancel as shown [1] the surprising result that the following lemma is false in all dimensions greater than three.

LEMMA 2.2. *Let K be a compactum in the interior of a 3-manifold M. If U is a neighborhood of K, then there exists a smaller neighborhood V of K such that a loop in $M - U$ which is inessential in $M - K$ is inessential in $M - V$.*

PROOF. We may assume that U is a compact PL 3-manifold with boundary. For each component C_i of Bd U, we let M_i be the component of $M - K$ which contains C_i. The subgroup of $\pi_1(M_i)$ which is the image of $\pi_1(C_i)$ under inclusion is finitely generated and hence [10] finitely presented. Therefore, the kernel of the homomorphism on fundamental groups induced by the inclusion of C_i into M_i is the normal closure of a finite number of elements which we designate as H_i. We let V be a neighborhood of K contained in U such that, for each i, any element of H_i is inessential in $M - V$. By general position and cut and paste the set V satisfies the conclusion of the lemma.

We give an outline of the reason Lemma 2.2 fails in higher dimensions. Let G be a finitely generated group with a countable presentation but which fails to have a finite presentation. Let K be a locally finite 2-complex with carrier P such that $\pi_1(P) = G$.

Now P can be embedded in $E^n (n \geqq 5)$ as a closed polyhedron. If the complex K is chosen carefully, P can also be embedded as a closed polyhedron in E^4 [1]. Let W be an open neighborhood of the embedded P which strong deformation retracts onto P. Hence, $\pi_1(W) = G$ and W is an example of an n-manifold $(n \geqq 4)$ in which Scott's theorem is false. We consider $E^n \subset \Sigma^n$ (the 1-point compactification of E^n). We set $X = \Sigma^n - W$. Let U be any neighborhood of X in Σ^n which misses a finite set of generators for $\pi_1(W)$. It is now evident that Lemma 2.2 fails with this choice of X and U in Σ^n; otherwise, all the relators for $\pi_1(W)$ would bound in a compact subset of W which would imply that $\pi_1(W)$ is finitely presented.

THEOREM 2.1. *Suppose K is a compactum in the interior of a 3-manifold M. Then K is approximable by Cantor sets.*

PROOF. Let U be a neighborhood of K. Let V be a neighborhood as given by Lemma 2.2. We further assume that V is a PL 3-manifold with boundary in the interior of U. By Lemma 2.1 there is a Cantor set C in the interior of a small regular neighborhood of the 1-skeleton of the boundary of V which approximates that 1-skeleton with respect to the small regular neighborhood. The Cantor set C is the desired Cantor set.

If γ is a loop in $M - U$ which is inessential in $M - K$, then γ is inessential in $M - V$ and hence in $M - C$. On the other hand if γ is inessential in $M - C$, then γ is inessential in the complement of the 1-skeleton of V and hence in $M - V$ which is contained in $M - K$.

3. **Punctured cells.** Suppose $C, C_1, \cdots, C_k$ is a finite collection of PL n-cells in Euclidean n-space such that $C_i \cap C_j = \Phi$ if $i \neq j$ and

$C_i \subset \operatorname{Int} C$. Then $C - \bigcup \operatorname{Int} C_i$ is a cell with holes and any set homeomorphic to $C - \bigcup \operatorname{Int} C_k$ is called a punctured n-cell. The boundary of a punctured n-cell is the union of a finite number of $(n-1)$-spheres. Two punctured n-cells are homeomorphic if their boundaries have the same number of components. In fact if H_1 and H_2 are punctured n-cells with the same number of boundary components and f is a homeomorphism between a boundary component of H_1 and a boundary component of H_2, then f can be extended to a homeomorphism of H_1 onto H_2. This fact is well known if the H_i have only one boundary component. To prove the general case it is helpful to take handle decompositions of the punctured n-cells H_i in which each decomposition consists of a single 0-handle and k $(n-1)$-handles. The map f is then extended on each handle by using induction on n and the well known case in which there is only one boundary component.

The following lemma will be used only in the case $n = 3$, but we prove it for all dimensions.

LEMMA 3.1. *Suppose M is an n-dimensional manifold. Then M is homeomorphic to the n-sphere S^n minus a tame compact* 0*-dimensional subset if and only if M can be written as the union of punctured n-cells $H_i (i = 1, 2, 3, \cdots)$ where $H_i \subset \operatorname{Int} H_{i+1}$.*

PROOF. If M is homeomorphic to S^n minus a tame compact 0-dimensional subset, then it is easy to show that M is the union of an ascending sequence of punctured n-cells.

We now suppose that M is the union of an ascending sequence of punctured n-cells H_i. By shrinking each H_i slightly we may assume that $\operatorname{Bd} H_i$ is collared in M. We wish to find a sequence H_i' of punctured n-cells such that $H_i \subset H_i' \subset \operatorname{Int} H_{i+1}$ and $\operatorname{Cl}(H'_{i+1} - H_i')$ is a finite collection of punctured n-cells. We define $H_1' = H_1$ and assume that by induction we have defined $H_1', \cdots, H_m'$ where $\operatorname{Bd} H_m'$ is collared in M. Let $f: H_{m+2} \to S^n$ be an embedding such that $\operatorname{Cl}(S^n - f(H_m'))$ is the union of disjoint PL n-cells. In each component U_i of $S^n - f(H_{m+2})$ we choose a PL n-cell B_i. Since $\operatorname{Cl}(U_i)$ is cellular there exists a homeomorphism $g: S^n \to S^n$ which fixes $f(H_{m+1})$ and takes $\operatorname{Cl}(U_i)$ into $\operatorname{Int} B_i$ [5]. We define H'_{m+1} to be the set $(g \circ f)^{-1}(S^n - \bigcup \operatorname{Int} B_i)$.

We now define an embedding $h: M \to S^n$. We suppose inductively that we have defined h on H_i' and that $\operatorname{Cl}(S^n - h(H_i'))$ is the union of a finite number of PL n-cells of diameter less than $1/i$. Since each component of $\operatorname{Cl}(H'_{i+1} - H_i')$ is a punctured n-cell, there is a natural way to extend h to H'_{i+1} such that $\operatorname{Cl}(S^n - h(H'_{i+1}))$ is the disjoint union of PL n-balls of diameter less than $1/(i + 1)$. The map h is an embedding and $S^n - h(M)$ is a compact 0-dimensional set. Since for each $\epsilon > 0$

there exists a finite number of disjoint n-cells of diameter less than ϵ whose interiors cover $S^n - h(M)$, we conclude [9, Theorem 1] that $S^n - h(M)$ is tame.

4. **3-manifolds in which Cantor sets lie in open 3-cells.** To answer Bing's question for $n = 3$ we will need the following theorem.

THEOREM 4.1. *Suppose M is a 3-manifold in which every Cantor set lies in an open 3-cell of M. Then every polyhedral finite graph in M is contained in an open 3-cell of M.*

PROOF. The proof is implicit in [13, Theorem 2.3]. For completeness we give a proof which is similar to the proof of our Lemma 2.1.

Let K be a polyhedral finite graph in M. Since the hypothesis of the theorem implies M is connected, we assume, without loss of generality, that K is connected. We let $\sigma_1, \cdots, \sigma_m$ be the 1-simplexes of K. Each σ_i is contained in an open set U_i of M which is homeomorphic with Euclidean 3-space. There exist PL 2-spheres S_i^2 in each U_i such that $\sigma_i \subset S_i^2$, S_i^2 is the join of $\mathrm{Bd}\,\sigma_i$ and a simple closed curve Σ_i, and $S_i^2 \cap S_j^2 = \sigma_i \cap \sigma_j$ for $i \neq j$. Let B be a regular neighborhood of a maximal tree of K. For each i, let G_i be a regular neighborhood of the end points of σ_i in S_i^2 such that $G_i \subset B$. We let N_i be a regular neighborhood of Σ_i in U_i such that $S_i^2 - G_i \subset N_i$. Let C_i be an Antoines necklace [2, 12] in $U_i - N_i$ which links N_i.

If we identify the cell B to a point, we get M/B which is homeomorphic to M [5]. Let P be the identification map from M to M/B. The compact 0-dimensional set $P(\bigcup C_i \cup B)$ is contained in a Cantor set C in M/B which in turn is contained in an open 3-cell U' in M/B by the hypothesis of the theorem. The set $U = P^{-1}(U')$ is also an open 3-cell [5] in M which contains $\bigcup C_i \cup B$.

We let W be a closed collared 3-cell in U such that $\bigcup C_i \cup B \subset W$. Triangulate each S_i^2 so finely that if a simplex of S_i^2 intersects W then it is contained in U. Let P_i be the union of all the closed simplexes of S_i^2 which miss W. The polyhedron P_i is contained in $S_i^2 - G_i$ and hence in N_i. If P_i separates the end points of γ_i in S_i^2, then P_i contains a simple closed curve γ_i which is homotopic to Σ_i in N_i. Since γ_i is in the complement of W, γ_i is homotopically trivial in $M - W$. Since U_i is simply connected at infinity, γ_i is homotopically trivial in $U_i - C_i$ which is a contradiction to the choice of C_i. Thus, we may conclude that P_i does not separate the end points of σ_i in S_i^2.

Choose a polyhedral arc δ_i connecting the end points of σ_i in $S_i^2 - p_i$. Let K' be the polyhedron formed by taking the union of the δ_i. The polyhedron K' is contained in the open 3-cell U. We take an isotopy of the 2-spheres S_i^2 which takes each δ_i onto σ_i, keeping the end points fixed. By using collars on the S_i^2 we extend this to an isotopy of M which takes K' to K. Hence K is also contained in an open 3-cell.

THEOREM 4.2. *Suppose M is a 3-manifold. Then M is homeomorphic to S^3 or S^3 minus a tame compact 0-dimensional subset if and only if each polyhedral finite graph in M is contained in an open 3-cell of M.*

PROOF. If every finite polyhedral graph of M is contained in an open 3-cell of M and M is compact then M is homeomorphic to S^3. For, using complementary 1-skeleta and Stallings' stretching technique [**11**], M is the union of two open 3-cells and hence [**5**] is homeomorphic to S^3. If M is not compact, we show that M is the union of an ascending sequence of punctured 3-cells. Hence by Lemma 3.1 we can conclude that M is homeomorphic to S^3 minus a tame compact 0-dimensional subset. It will suffice to show that every compact subset of M is contained in the interior of a punctured 3-cell. An arbitrary compact subset of M is contained in a finite polyhedron L. By hypothesis the 1-skeleton of L is contained in an open 3-cell. By the Hauptvermutung for 3-manifolds the 1-skeleton of L lies in the interior of a PL 3-cell B which we assume to be in general position with respect to L. By the techniques used in the proof of [**3**, Lemma 4] it is possible to modify B by cut and paste to obtain a punctured 3-cell H which contains L in its interior.

As a consequence of Theorems 4.1 and 4.2 we get our desired answer to Bing's question for $n = 3$.

THEOREM 4.3. *Suppose M is a 3-manifold. Then M is homeomorphic to S^3 or S^3 minus a tame compact 0-dimensional subset if and only if each Cantor set of M is contained in an open 3-cell of M.*

CONJECTURE 4.1. Suppose M is an n-manifold ($n \geqq 3$). Then M is homeomorphic to S^n or S^n minus a tame compact 0-dimensional subset if and only if each Cantor set of M is contained in an open n-cell of M.

Bing showed [**3**, Lemma 7] that if each polygonal simple closed curve of a 3-manifold lies in a topological 3-cell in the manifold, then each polyhedral finite graph in the manifold lies in an open 3-cell of the manifold. Hence we also have the following corollary which strengthens a theorem [**6**, Theorem 2] of Costich, Doyle and Galewski.

COROLLARY 4.1. *Suppose M is a 3-manifold. Then M is homeomorphic to S^3 or S^3 minus a tame compact 0-dimensional subset if and only if each polygonal simple closed curve of M lies in a topological 3-cell of M.*

COROLLARY 4.2. *A contractible 3-manifold M is topologically E^3 if and only if each Cantor set in M lies in an open 3-cell of M.*

Our final corollary is only a slight modification of a theorem of Bing [3, Theorem 2].

COROLLARY 4.3. *A contractible 3-manifold M is topologically E^3 if and only if each polygonal simple closed curve in M lies in a topological 3-cell of M.*

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