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GENERALIZED BAZILEVIČ FUNCTIONS

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ABSTRACT. This paper defines the generalized Bazilevič functions via the differential equation

$$1 + z\frac{f''(z)}{f'(z)} + (\alpha + i\beta - 1)z\frac{f'(z)}{f(z)} = \alpha z\frac{g'(z)}{g(z)} + z\frac{h'(z)}{h(z)} + i\beta$$

where $\alpha + i\beta \in \mathbf{C} - \{\text{negative integers}\}$ and $g(z)$, $h(z)$ are restricted to various function classes. The geometry of the solutions, their representation, the relation of their univalence to the domain of analyticity, and the motivation for considering the planar projection of the various representations of a generalized Bazilevič function are considered. The extremal problems $\max|a_2|$, $\max|a_3|$ are solved. An explicit bound on the radius of Bazilevičness for S is obtained. A bounded univalent non-Bazilevič function which is a generalized Bazilevič function is constructed. Thomas' result that bounded $B(\alpha, 0)$ functions satisfy $a_n = O(n^{-1})$ is generalized to classes of nonunivalent functions. The paper closes with a conjecture on the analytic structure of a bounded univalent funtion whose coefficients satisfy $a_n = O(n^{-1})$.

0. **Introduction.** Loewner's method [13] introduced in 1923 and developed by Kufarev in 1943 [8] was to imbed an image domain in a continuously increasing family of domains and then describe this family by a differential equation. Using the Loewner-Kufarev differential equation Bazilevič [2] in 1955 was able to prove that the class of functions

$$f(z) = \left\{ \int_0^z g^{\alpha}(\zeta)h(\zeta)\zeta^{i\beta-1}\,d\zeta \right\}^{1/(\alpha+i\beta)} \tag{1}$$

is analytic and univalent in $|z| < 1$ where $g(z)$ is a starlike univalent function, $g(0) = 0$, $h(z)$ is analytic and satisfies $\mathrm{Re}(e^{i\lambda}h(z)) > 0$ in $|z| < 1$ for some real λ, $\alpha > 0$, and β is a real number. Particular choices of α, β, g and h yield the convex, starlike, close-to-convex and spiral-like functions.

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Zamorski [21], found in 1962 the general form of the solution to any extremal problem for the ordinary Bazilevič functions and proved that $|a_n| \leqq n$ for $\alpha = 1, 1/2, 1/3, \cdots, \beta = 0$. In 1965 Pommerenke [15] proved the univalence of (1) using subordination chains. In 1968 Thomas [20] asked if it was possible to give a geometric characterization of the Bazilevič functions when $\beta = 0$. He then proved that the Taylor coefficients of bounded Bazilevič functions with $\beta = 0$ satisfy $a_n = 0(1/n)$ which extended Clunie and Pommerenke's result [5] for bounded close-to-convex functions.

In 1971 Sheil-Small [19] gave an intrinsic characterization for the ordinary Bazilevič functions along the lines of Kaplan's characterization of the close-to-convex functions [6]. He proved each ordinary Bazilevič function $f(z)$ is a solution of the differential equation

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} + (\alpha - 1 + i\beta) \frac{zf'(z)}{f(z)} \\ = \alpha z \frac{g'(z)}{g(z)} + \frac{zh'(z)}{h(z)} + i\beta \end{aligned} \tag{2}$$

(g and h as before) and extended the class to include the case $\alpha = 0$. He gave the first example (albeit non-constructive) of a non-Bazilevič univalent function. Finally he showed the set of $\alpha + i\beta$ which could be used to represent a fixed Bazilevič function in (2) is a closed convex set in the right half plane.

In 1972 Prokhorov [17] characterized Bazilevič functions of type (α, 0) in terms of the geometry of the complement of $f(D)$ as Lewandowski [9, 10] had done for the close-to-convex functions. In 1974 Avhadiev and Aksent'ev [1] completed this program by characterizing Bazilevič functions of type (α, β) in terms of the complement of $f(D)$. By increasing the restrictions on the function $h(z)$ which appears in (1) they were able to let α be a real number $> -1/2$.

In this paper we reverse Loewner's procedure. We define the generalized Bazilevič functions by means of a differential equation. We obtain a generalization by increasing the classes from which g and h are drawn in (2) and by letting $\alpha + i\beta$ be any point in the entire complex plane with the negative integers deleted (section 1). In section 2 we discuss equation (2) at the exceptional points. In section 3 we discuss the geometry of the solutions.

Each solution of (2) is associated with a quadruple (α, β, g, h). In section 4 we examine various properties of the projection onto the complex plane of the first two coordinates of all quadruples which represent a fixed Bazilevič function. Since it is very difficult to explicitly determine to which (α, β) classes a specific Bazilevič function $f(z)$ can belong, theorems 4.2 and 4.3 provide useful simple analytic devices that greatly reduce this difficult problem. Theorem 4.4 shows that the geometry of the projected set can give explicit analytic information about $f(z)$. Much of the work of this section is devoted to an exploration of the following problem: given an arbitrary closed convex set A in the right half plane is there a Bazilevič function whose representation projection is exactly A?

Section 5 solves several extremal problems and provides the machinery to give an upper bound for the radius of Bazilevičness for the univalent functions. We also provide an explicit example of a univalent non-Bazilevič function which is a generalized Bazilevič function. In section 6 we show $a_n = 0(1/n)$ for a class of bounded generalized Bazilevič functions and thereby extend Thomas, Clunie and Pommerenke's results to classes of non-univalent locally univalent functions. We close the paper with a conjecture concerning the analytic structure of bounded univalent functions whose coefficients satisfy $a_n = 0(n^{-1})$.

1. **The Bazilevič Differential Equation.** We begin with the following known result.

LEMMA 1.1. *Let $H(z)$ be an analytic non-vanishing function in $|z| < R$ with $H(0) = 1$. Then there is a unique function $f(z) = z + a_2z^2 + \cdots$, locally univalent and analytic in $|z| < R$, vanishing only at $z = 0$, satisfying the differential equation*

$$zf'(z)/f(z) = H(z), \quad |z| < R. \tag{1.1}$$

PROOF. The function $f(z) = z \exp \int_0^z (H(w) - 1)/w \, dw$ has the desired properties. Suppose $f_1(z)$ and $f_2(z)$ both satisfy (1.1). Then $zf_1'(z)/f_1(z) = zf_2'(z)/f_2(z)$ is equivalent to $z[f_2(z)]^2[f_1(z)/f_2(z)]' \equiv 0$, which, by the identity theorem and the normalization of the solutions, forces $f_1(z) \equiv f_2(z)$.

It will be useful to have an equivalent form for the Bazilevič differential equation.

LEMMA 1.2. *Let $g(z) = b_1z + b_2z^2 + \cdots$, $b_1 \neq 0$, be an analytic function in $|z| < 1$ vanishing only at $z = 0$. Let $h(z) = c_0 + c_1z + \cdots$, $c_0 \neq 0$, be an analytic non-vanishing function in $|z| < 1$. Let $f(z) = z + a_2z^2 + \cdots$ be an analytic locally univalent function in $|z| < R$, $0 < R \leqq 1$, vanishing only at $z = 0$. Then in $|z| < R$ the following two differential equations are equivalent:*

$$
\begin{aligned}
&1 + z\frac{f''(z)}{f'(z)} + (\alpha + i\beta - 1)z\,\frac{f'(z)}{f(z)} \\
(1.2)\qquad &\qquad = \alpha z\frac{g'(z)}{g(z)} + z\,\frac{h'(z)}{h(z)} + i\beta,
\end{aligned}
$$

$$
(1.3)\qquad z\frac{f'(z)}{f(z)} = \left(\frac{g(z)}{z}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\frac{h(z)}{b_1{}^{\alpha}c_0}\,.
$$

PROOF. We may rewrite (1.2) as

$$
\begin{aligned}
&z\frac{f''(z)}{f'(z)} - (\alpha + i\beta - 1)z\left(\frac{f(z)}{z}\right)\frac{f(z) - zf'(z)}{(f(z))^2} \\
&\qquad = \alpha z\left(\frac{z}{g(z)}\right)\left(\frac{zg'(z) - g(z)}{z^2}\right) + z\,\frac{h'(z)}{h(z)},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&z\Big[\log f'(z) - (\alpha + i\beta - 1)\log\frac{z}{f(z)} \\
&\qquad - \alpha\log\frac{g(z)}{z} - \log h(z)\Big]' = 0,
\end{aligned}
$$

which, by the identity theorem and the normalizations of $f(z)$, $g(z)$, and $h(z)$, is equivalent to

$$
z\frac{f'(z)}{f(z)} = \left(\frac{g(z)}{z}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{\alpha+i\beta}\frac{h(z)}{b_1{}^{\alpha}c_0}\,.
$$

We now establish the existence and uniqueness of solutions of the Bazilevič differential equation (1.2) when the parameter $\alpha + i\beta$ is restricted to an appropriately punctured plane.

THEOREM 1.3. *Let* $g(z) = b_1z + b_2z^2 + \cdots$, $b_1 \neq 0$, *be an analytic function in* $|z| < 1$ *vanishing only at* $z = 0$. *Let* $h(z) = c_0 + c_1z + \cdots$, $c_0 \neq 0$, *be an analytic non-vanishing function in* $|z| < 1$. *Then for any complex number* $\alpha + i\beta$ *not equal to a negative integer* $-1, -2, -3, \cdots$, *there exists a real number* R, $0 < R \leqq 1$, *and a unique function* $f(z) = z + a_2z^2 + \cdots$ *locally univalent and analytic in* $|z| < R$, *vanishing only at* $z = 0$, *which in* $|z| < R$ *satisfies the differential equation*

$$
\begin{aligned}
&1 + z\frac{f''(z)}{f'(z)} + (\alpha + i\beta - 1)z\,\frac{f'(z)}{f(z)} \\
(1.2)\qquad &\qquad = \alpha z\,\frac{g'(z)}{g(z)} + z\frac{h'(z)}{h(z)} + i\beta.
\end{aligned}
$$

PROOF. If $\alpha + i\beta \equiv 0$, then by Lemma 1.2 we need only solve the differential equation $zf'(z)/f(z) = h(z)/c_0$ which can be done by Lemma 1.1.

If $\alpha + i\beta$ is any complex number except $0, -1, -2, \cdots$, then

$$F(z) = (g(z)/z)^\alpha h(z) b_1^{-\alpha} c_0^{-1} = 1 + \sum_{n=1}^{\infty} d_n z^n$$

$$G(z) = \frac{1}{\alpha + i\beta} + \sum_{n=1}^{\infty} \frac{d_n}{n + \alpha + i\beta} z^n$$

are well defined analytic functions in $|z| < 1$ which satisfy

$$(1.4) \qquad (\alpha + i\beta)G(z) + zG'(z) = F(z).$$

Since $G(0) = (\alpha + i\beta)^{-1} \neq 0$, the number R, defined as $\sup\{r < 1: G(z) \neq 0 \text{ in } |z| \leqq r\}$, will satisfy $0 < R \leqq 1$. Define

$$(1.5) \qquad H(z) = F(z)/(\alpha + i\beta)G(z).$$

Clearly $H(z)$ is a non-vanishing analytic function in $|z| < R$ satisfying $H(0) = 1$. Therefore Lemma 1.1 guarantees a unique function $f(z)$, locally univalent, analytic, vanishing only at $z = 0$, in $|z| < R$, satisfying $zf'(z)/f(z) = H(z)$. Using the fact that $zf'(z)/f(z) = H(z)$, (1.5), (1.4) and the definition of $F(z)$, it is easy to check that $f(z)$ satisfies (1.2) in $|z| < R$.

To show uniqueness we suppose that $f_1(z)$ and $f_2(z)$ are two distinct solutions to (1.2). Then by Lemma 1.2 both functions would satisfy

$$z\frac{f_j'(z)}{f_j(z)} = \left(\frac{g(z)}{z}\right)^\alpha \left(\frac{z}{f_j(z)}\right)^{\alpha+i\beta} \frac{h(z)}{b_1^\alpha c_0}.$$

Hence in $|z| < R$

$$(1.6) \qquad \left(\frac{f_1(z)}{z}\right)^{\alpha+i\beta-1} f_1'(z) = \left(\frac{f_2(z)}{z}\right)^{\alpha+i\beta-1} f_2'(z).$$

If we set

$$L(z) \equiv \frac{1}{\alpha + i\beta}\left[\left(\frac{f_1(z)}{z}\right)^{\alpha+i\beta} - \left(\frac{f_2(z)}{z}\right)^{\alpha+i\beta}\right],$$

then equation (1.6) becomes $-zL'(z) = (\alpha + i\beta)L(z)$. If $f_1(z)$ is not identically equal to $f_2(z)$ then the normalizations of $f_1(z)$ and $f_2(z)$ force $L(z)$ to have the expansion

$$L(z) = A_n z^n + A_{n+1} z^{n+1} + \cdots,$$

$A_n \neq 0$, for some integer $n \geqq 1$. On the other hand equating coefficients of $-zL'(z) = (\alpha + i\beta)L(z)$ forces $-nA_n = (\alpha + i\beta)A_n$. Since $A_n \neq 0$, this implies $\alpha + i\beta = -n$ which is a contradiction to $\alpha + i\beta$ lying in the plane punctured at the negative integers. This concludes the proof of the theorem.

A function $f(z) = z + a_2 z^2 + \cdots$ analytic, locally univalent, vanishing only at 0 in $|z| < R$ is said to be a generalized Bazilevič function if it satisfies equation (1.2) where $g(z) = b_1 z + b_2 z^2 + \cdots$, $b_1 \neq 0$, is a function analytic in $|z| < 1$ vanishing only at $z = 0$ and $h(z) = c_0 + c_1 z + \cdots$, $c_0 \neq 0$, is an analytic non-vanishing function in $|z| < 1$. Each generalized Bazilevič function is therefore associated with a quadruple (α, β, g, h). Since for any $b \neq 0$ and $c \neq 0$ the quadruple $(\alpha, \beta, g/b, h/c)$ defines the same generalized Bazilevič function as the quadruple (α, β, g, h), we can and will suppose for the rest of the paper that both $g(z)$ and $h(z)$ are normalized by $g(z) = z + b_2 z^2 + \cdots$, $h(z) = 1 + c_1 z + \cdots$.

DEFINITION. Let $\mathscr{G}$ be a class of normalized functions $g(z) = z + b_2 z^2 + \cdots$ analytic in $|z| < 1$ and vanishing only at $z = 0$. Let $\mathscr{H}$ be a class of normalized functions $h(z) = 1 + c_1 z + \cdots$ analytic, and non-vanishing in $|z| < 1$. A function $f(z) = z + a_2 z^2 + \cdots$ analytic, locally univalent and vanishing only at $z = 0$ in $|z| < R$, is a *generalized Bazilevič function of type* $(\alpha, \beta, \mathscr{G}\mathscr{H})$ if and only if $f(z)$ satisfies the differential equation

$$1 + z\frac{f''(z)}{f'(z)} + (\alpha + i\beta - 1)z\,\frac{f'(z)}{f(z)} = \alpha z\,\frac{g'(z)}{g(z)} + z\,\frac{h'(z)}{h(z)} + i\beta$$

in $|z| < R$ where $g(z)$ belongs to $\mathscr{G}$ and $h(z)$ belongs to $\mathscr{H}$.

In 1955 Bazilevič [2] with the help of the Löwner-Kufarev differential equation gave an explicit representation of a class of univalent analytic functions in $|z| < 1$. In 1972 Sheil-Small [19] gave an intrinsic characterization of Bazilevič's functions along the lines of Kaplan's characterization [6] of the close-to-convex functions and proved that Bazilevič's functions corresponded to solutions of (1.2) for $\alpha > 0$, $\mathscr{G}$ the starlike functions, and $\mathscr{H}$ the functions satisfying $\operatorname{Re} e^{i\lambda} h(z) > 0$ for some real number λ. Such functions are obviously a very special subclass of the generalized Bazilevič functions. They will be referred to as the ordinary Bazilevič functions.

2. **The Bazilevič differential equation at the negative integers.** The restriction of theorem 1.2 to C punctured at the negative integers is not accidental. There need be no solutions of (1.2) at $\alpha + i\beta = -n$, $n = 1, 2, \cdots$, and when there are solutions they are never unique.

We already know that if $f(z)$ is a Bazilevič function with representative (α, β, g, h), $\alpha + i\beta \neq -n$, $n = 0, 1, 2, \cdots$, then $f(z)$ can be written as $f(z) = z((\alpha + i\beta)G(z))^{1/(\alpha+i\beta)}$ where $G(z)$ is analytic in $|z| < 1$ and satisfies $(\alpha + i\beta)G(z) + zG'(z) = (g(z)/z)^{\alpha}h(z)$. We need a converse.

Lemma 2.1. *Let $(\alpha + i\beta) \neq 0$. Let $H(z) = (\alpha + i\beta)^{-1} + \sum_{n=1}^{\infty} c_n z^n$ be an analytic function in $|z| < 1$. Suppose $f(z)$ is a generalized Bazilevič function with representative (α, β, g, h). If $f(z)$ can be represented as*

$$f(z) = z((\alpha + i\beta)H(z))^{1/(\alpha+i\beta)} \tag{2.1}$$

in some neighborhood of the origin, then $H(z)$ is a solution of $(\alpha + i\beta)H(z) + zH'(z) = (g(z)/z)^{\alpha}h(z)$ throughout $|z| < 1$.

Proof. Because of (2.1) we may write $H(z)$ as $(\alpha + i\beta)^{-1}(f(z)/z)^{\alpha+i\beta}$ in some neighborhood of the origin. A computation yields

$$(\alpha + i\beta)H(z) + zH'(z) = \left(\frac{f(z)}{z}\right)^{\alpha+i\beta-1} f'(z) \tag{2.2}$$

in some neighborhood of the origin. But by Lemma 1.2 we also have

$$(f(z)/z)^{\alpha+i\beta-1}f'(z) = (g(z)/z)^{\alpha}h(z) \tag{2.3}$$

in some neighborhood of the origin. Therefore by (2.2) and (2.3)

$$(\alpha + i\beta)H(z) + zH'(z) = (g(z)/z)^{\alpha}h(z) \tag{2.4}$$

in some neighborhood of the origin. Both sides of (2.4) are analytic throughout $|z| < 1$. By the identity principle the differential equation (2.4) actually holds throughout $|z| < 1$.

Thus questions about the existence and uniqueness for generalized Bazilevič functions are really questions about the existence and uniqueness of solutions $G(z) = 1/(\alpha + i\beta) + \sum_{n=1}^{\infty} c_n z^n$ to

$$(\alpha + i\beta)G(z) + zG'(z) = F(z) \tag{2.5}$$

where $F(z)$ is an arbitrary analytic non-vanishing function in $|z| < 1$.

To see that the differential equation (2.5) may fail to have an analytic solution at each exceptional point $\alpha + i\beta = -n$, $n = 1, 2, \cdots$, it suffices to let $F(z) = 1 + cz^n$, $0 < |c| \leqq 1$. Letting $G(z) = -n^{-1} + \sum_{k=1}^{\infty} c_k z^k$ and equating coefficients of $-nG(z) + zG'(z) = 1 + cz^n$ yields

$$(k-n)\cdot c_k = \begin{Bmatrix} 0 & \text{if } k \neq n \\ c & \text{if } k = n \end{Bmatrix},$$

which is clearly absurd for $k = n$. One may also turn directly to (1.2) with $g(z) = z$ and $h(z) = 1 + cz^n$ and after elaborate computations verify that there can be no analytic solutions.

The equation $-nG(z) + zG'(z) = F(z)$ can also have non-unique solutions. For example each of the functions $G(z) = -n^{-1} + cz^n$, *c any complex number*, is a solution of $-nG(z) + zG'(z) = 1$. Thus for $(-n, 0, z, 1)$ the Bazilevič differential equation (1.2) has the family of solutions $f(z) = z(1 - ncz^n)^{-1/n}$ analytic in $|z| < |nc|^{-1/n}$. In particular $f(z) = z/(1 - cz)$, $|c| < 1$, is a family of non-unique analytic univalent in $|z| < 1$ functions corresponding to $(-1, 0, z, 1)$.

There are simple necessary and sufficient conditions for the existence of solutions at the points $\alpha + i\beta = -n$, $n = 1, 2, \cdots$. Let $F(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$ and $G(z) = (\alpha + i\beta)^{-1} + \sum_{k=1}^{\infty} c_k z^k$. Upon comparing coefficients of $(\alpha + i\beta)G(z) + zG'(z) = F(z)$ we see that $G(z)$ will be analytic in the same disc as $F(z)$ if and only if we can solve $(\alpha + i\beta + k)c_k = d_k$ for all $k = 1, 2, \cdots$. Thus for the generalized Bazilevič differential equation (1.2) to have a solution at $\alpha + i\beta = -n$, $n = 1, 2, \cdots$ it is necessary and sufficient that d_n, the nth coefficient of $(g(z)/z)^{-n}h(z)$, vanish. There are no solutions for $(-n, 0, z/(1 + z), 1)$ and $(-n, 0, z/(1 - z)^2, 1)$ since d_n is 1 and $(-1)^n\binom{2n}{n}$, respectively. There always is a solution for $(-n, 0, z/(1 + z), 1/(1 + z))$ since d_n is 0. There are also solutions for $(-n, 0, z/(1 - z)^2, (1 + z)/(1 - z))$, since $F(z)$ is $(1 - z)^{2(n-1)}(1 - z^2)$ which has d_n equal to zero.

Although there never is a unique solution at $\alpha + i\beta = -n$, there is a limited family of solutions. If the nth coefficient of $F(z)$ is zero, then it is trivial to check that the functions

$$G(z) = bz^n - \frac{1}{n} + \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{d_k}{k-n} z^k,$$

where b is any complex number, are the only solutions of $-nG(z) + zG'(z) = F(z)$. Consequently we have proved

THEOREM 2.2. *Let $\alpha + i\beta = -n$, $n = 1, 2, \cdots$. Then the generalized Bazilevič differential equation* (1.2) *has a normalized locally univalent analytic solution vanishing only at* 0 *in some neighborhood of zero if and only if the* nth *coefficient of* $(g(z)/z)^{-n}h(z)$ *vanishes. Furthermore, any solution is of the form*

$$f(z) = z\left(1 - \sum_{\substack{k=0\\k\neq n}}^{\infty} \frac{nd_k}{k-n} z^k - nbz^n \right)^{-1/n}$$

where $(g(z)/z)^{-n}h(z) = 1 + \sum_{k=1}^{\infty} d_k z^k$.

3. **Properties of R.** Theorem 1.3 guarantees the existence of a solution to differential equation (1.2) specified by a quadruple (α, β, g, h), $\alpha + i\beta \neq -1, -2, -3, \cdots$. The solution is analytic in $|z| < R$ where R, the modulus of the first zero of an auxillary function $G(z)$, depends on the quadruple (α, β, g, h). A generalized Bazilevič function can have radically different quadruple representations. For example, the Koebe function $z/(1-z)^2$ has both a $(-1/2, 0, z, 1+z)$ as well as a $(1, 0, z/(1-z)^2, (1+z)/(1-z))$ representation. It would be nice to know that R is independent of the quadruple (α, β, g, h) and, if R is less than one, that this reflects the geometry of $f(z)$ itself.

THEOREM 3.1. *Let $f(z)$ be a generalized Bazilevič function with quadruple representations $(\alpha_1, \beta_1, g_1, h_1)$ and $(\alpha_2, \beta_2, g_2, h_2)$, $\alpha_j + i\beta_j \neq -1, -2, -3, \cdots$. Let $R_j = \sup\{r < 1 : G_j(z) \neq 0$ in $|z| < r\}$ where $G_j(z)$ is the function associated with the quadruple $(\alpha_j, \beta_j, g_j, h_j)$, $j = 1, 2$. Then $R_1 = R_2$. If R is less than 1, then*:

$$\alpha > 0 \text{ implies } \inf_{|z|<R} |f(z)/z| = 0, \tag{3.1}$$

$$\alpha = 0 \text{ implies } 0 < A \leqq |f(z)/z| \leqq B < \infty \text{ in } |z| < R, \tag{3.2}$$

$$\alpha < 0 \text{ implies } \sup_{|z|<R} |f(z)/z| = \infty. \tag{3.3}$$

If R is less than 1 and $\alpha + i\beta \neq 1/n$, $n = 1, 2, \cdots$, then R is the exact radius of analyticity of $f(z)$. If R is less than 1 and $\alpha + i\beta = 1/n$, $n = 1, 2, \cdots$, then $f(z)$ has a zero of order n on $|z| = R$. Finally, if R is less than 1, then $f(z)$ grows like $(R - |z|)^{\alpha/(\alpha^2+\beta^2)}$, that is:

$$|f(z)| = O(R - |z|)^{\alpha/(\alpha^2+\beta^2)}. \tag{3.4}$$

PROOF. We first prove $R_1 = R_2$ when $(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) \neq 0$. By Lemma 1.1 the solution of $zf'(z)/f(z) = 1 + zG_j'(z)/[(\alpha_j + i\beta_j)G_j(z)]$ is

$$f(z) = z[(\alpha_j + i\beta_j)G_j(z)]^{1/(\alpha_j+i\beta_j)}, \quad |z| < R_j, \tag{3.5}$$

Thus in $|z| < \min(R_1, R_2)$ we have

$$[(\alpha_1 + i\beta_1)G_1(z)]^{1/(\alpha_1+i\beta_1)} = [(\alpha_2 + i\beta_2)G_2(z)]^{1/(\alpha_2+i\beta_2)}. \tag{3.6}$$

Although G_1 and G_2 may be different functions, since they are both analytic in $|z| < 1$ equation (3.6) guarantees that their zeros (if any) must coincide. Hence R_1 must equal R_2.

We next prove that $R_1 = R_2$ if one quadruple is $(0, 0, g_1, h_1)$ and the other quadruple $(\alpha_2, \beta_2, g_2, h_2)$ satisfies $\alpha_2 + i\beta_2 \neq 0$. The proof of Theorem 1.3 showed that any $f(z)$ defined by $(0, 0, g_1, h_1)$ has R_1 equal to one, that is, $f(z)/z$ is analytic and non-vanishing in $|z| < 1$. But as in the first part of this proof, Lemma 1.1 guarantees that $f(z)$ is equal to $z[(\alpha_2 + i\beta_2)G_2(z)]^{1/(\alpha_2+i\beta_2)}$ in $|z| < R_2$. Thus $G_2(z) = (\alpha_2 + i\beta_2)^{-1}(f(z)/z)^{\alpha_2+i\beta_2}$ in $|z| < R_2$. Both sides of the equation are actually analytic functions throughout $|z| < 1$. Therefore by analytic continuation $G_2(z)$ is equal to the non-vanishing function $(\alpha_2 + i\beta_2)^{-1}(f(z)/z)^{\alpha_2+i\beta_2}$ in $|z| < 1$. Consequently $R_2 = 1 = R_1$ in this case also.

Relations (3.1), (3.2) and (3.3) follow immediately from (3.5) and the fact that $G(z)$ has a zero on $|z| = R < 1$.

We now prove if R is less than 1 and $\alpha + i\beta \neq 1/n$, $n = 1, 2, \cdots$, then T, the radius of analyticity of $f(z)$, is equal to R. We do this by assuming T is greater than R and getting a contradiction for $\alpha < 0$, $\alpha > 0$, and $\alpha = 0$.

We first show this is absurd for $\alpha < 0$. Analyticity in $|z| < T$, $T > R$, implies $\max\{f(z) : |z| \leqq R\} < \infty$ which contradicts (3.3).

We next show this is absurd for $\alpha > 0$. Relation (3.1) implies that if $f(z)$ is analytic in $|z| < T$, $T > R$, then f must have a zero, say z_0, on $|z| = R$. Let the order of this zero be m. Then

$$\lim_{z\to z_0} \frac{f(z)}{(z - z_0)^m} = \lim_{z\to z_0} z \left[(\alpha + i\beta) \frac{G(z)}{(z - z_0)^{m(\alpha+i\beta)}} \right]^{1/(\alpha+i\beta)}$$

as z approaches z_0 radially. The left hand side is finite and non-zero. Since $\alpha + i\beta \neq 1/n$, the right hand side cannot approach a finite non-zero limit. [Note that (1.4) implies that all zeros of $G(z)$ are simple].

Finally we show this is absurd for $\alpha = 0$. Since R is less than 1 the function $G(z)$ has a zero, say z_0, on $|z| = R$. Consequently

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} z[(\alpha + i\beta)G(z)]^{1/(\alpha+i\beta)}$$

as $z \to z_0$ radially. The left hand side approaches a definite limit since $f(z)$ is supposed to be analytic in $|z| < T$, $T > R$. But the right hand side has no limiting value as $z \to z_0$ radially.

If R is less than one and $\alpha + i\beta = 1/n$, $n = 1, 2, \cdots$ then clearly $f(z)$ has a zero of order n on $|z| = R$ and is analytic in a slightly larger disc. Finally, (3.5) and the fact that $G(z)$ has a simple zero on $|z| = R$ shows that (3.4) holds. This completes the proof of the theorem.

Theorem 3.1 has important implications for the univalence of generalized Bazilevič functions.

COROLLARY 3.2. *Let $f(z)$ be univalent within its radius of analyticity. If $f(z)$ has an $\alpha + i\beta$ representation where either $\alpha > 0$ or $(\alpha + (1/4))^2 + \beta^2 < (1/4)^2$, then $f(z)$ is univalent in $|z| < 1$.*

PROOF. If $\alpha > 0$ and $R < 1$, then by (3.1) we would have $\inf\{|f(z)/z| : |z| < R\} = 0$ which is absurd for univalent functions. If $(\alpha + (1/4)^2 + \beta^2 < (1/4)^2$ and $R < 1$, then $f(Rz)/R$ would be a normalized univalent function in $|z| < 1$ which would grow like

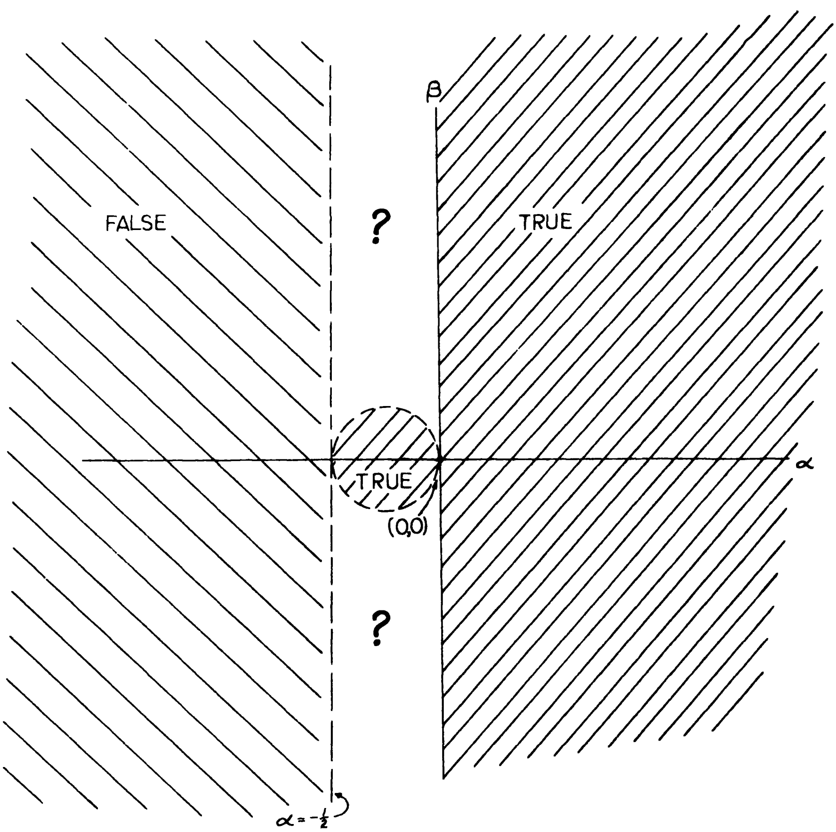

FIGURE 1

$(1 - |z|)^{\alpha/(\alpha^2+\beta^2)}$ which is faster than the maximum possible growth $(1 - |z|)^{-2}$ for normalized univalent functions [15].

Sheil-Small used the first half of this corollary in his proof that ordinary Bazilevič functions are univalent in $|z| < 1$.

It would be of interest to determine the exact region of values $\alpha + i\beta$ in $\mathbf{C}$ where univalence of a Bazilevič function within its radius of analyticity implies that the radius of analyticity is 1 (see Figure 1). For all $\alpha < -1/2$ the phenomena fails. In fact, if $\alpha + i\beta = -1$, the function $f(z) = z/(1 - az)$, with $(-1, 0, z, 1)$ representation, is univalent within the radius of analyticity but for $|a| > 1$ its radius of analyticity is strictly less than one. If $\alpha < -1/2$ and $\alpha + i\beta \neq -1$, then choosing any complex number c, $0 < |c| < 1$, such that $d = (\alpha + i\beta)c/(1 + \alpha + i\beta)$ satisfies $|d| > 1$, we can create the generalized Bazilevič function $f(z) = z(1 + dz)^{1/(\alpha+i\beta)}$ with $(\alpha, \beta, z, 1 + cz)$ representation. Since $zf'(z)/f(z) = (1 + cz)/(1 + dz)$, we see that $f(z)$ is spiral-like, hence univalent, within its radius of analyticity which is strictly less than one.

There are Bazilevič functions with $R < 1$ and $\alpha + i\beta = 1/n$, $n = 1, 2, \cdots$. For example, $f(z) = (e^{4\pi iz} - 1)/4\pi i$ is a generalized Bazilevič function with representation $(1, 0, z, e^{4\pi iz})$ and $R = 1/2$.

4. **The geometry of $\Pi(f, A, B)$.** If $f(z)$ is a generalized Bazilevič function then it is associated with a collection of quadruples (α, β, g, h) which satisfy (1.2). In order to obtain information about f it is necessary to restrict the classes A and B to which the functions g and h are allowed to belong. We let (f, A, B) denote the collection of all quadruples which represent a fixed but arbitrary function f when g is restricted to A and h is restricted to B. We let $\Pi(f, A, B)$ denote the set of all points $(\alpha + i\beta)$ for which f has an (α, β, g, h) representation, $g \in A$, $h \in B$, i.e., $\Pi(f, A, B)$ is the projection into $\mathbf{C}$ of the first two elements of the set of quadruples representing f. In this section we will investigate the relation between the geometry of the point set $\Pi(f, A, B)$ and the geometry of the mapping f.

We first show that $\Pi(f, A, B)$ is either a single point or empty if $f(z)$ is a generalized Bazilevič function with $R < 1$. It clearly suffices to show $\Pi(f, \mathscr{G}, \mathscr{H})$ is a single point where

$$\mathscr{G} = \{g(z) = z + \cdots : g(z)/z \text{ analytic and } \neq 0 \text{ in } |z| < 1\},$$

$$\mathscr{H} = \{h(z) = 1 + \cdots : h(z) \text{ analytic and } \neq 0 \text{ in } |z| < 1\}.$$

Note that $\Pi(f, \mathscr{G}, \mathscr{H})$ is always convex. For if $(\alpha_j, \beta_j, g_j, h_j)$, $j = 1, 2$, are two representatives for $f(z)$ then the quadruple $(\lambda\alpha_1 + (1 - \lambda)\alpha_2, \lambda\beta_1 + (1 - \lambda)\beta_2, g_3, h_3)$ also satisfies (1, 2) where

$$g_3(z) = z\left(\frac{g_1(z)}{z} \right)^{\frac{\lambda\alpha_1}{\lambda\alpha_1+(1-\lambda)\alpha_2}} \left(\frac{g_2(z)}{z} \right)^{\frac{(1-\lambda)\alpha_2}{\lambda\alpha_1+(1-\lambda)\alpha_2}},$$

$$h_3(z) = (h_1(z))^{\lambda}(h_2(z))^{1-\lambda},$$

and $g_3 \in \mathscr{G}$, $h_3 \in \mathscr{H}$.

Let (α, β, g, h) represent an f with $R < 1$. Theorem 3.1 asserts that the growth of $f(z)$ is exactly $(R - |z|)^{\alpha/(\alpha^2+\beta^2)}$. We consider the two cases $\alpha \neq 0$, $\alpha = 0$. First suppose $\alpha \neq 0$. Since the growth of a function is independent of its representation, $\alpha/(\alpha^2 + \beta^2)$ must be a nonzero constant, say k. The locus of (α, β) such that $\alpha/(\alpha^2 + \beta^2) = k$ is a circle. Therefore, $\Pi(f, \mathscr{G}, \mathscr{H})$ is a convex point set lying on a circle and thus must be a single point. On the other hand, if $\alpha = 0$, then β cannot equal 0 since a function with a (0, 0) representation always has $R = 1$. It therefore suffices to show that β (which is $\neq 0$) is uniquely determined by the geometry of f in order to conclude that $\Pi(f, \mathscr{G}, \mathscr{H})$ is a single point. This is immediate since from (3.5) and (3.2)

$$\lim_{z\to z_0} \frac{\arg f(z)/z}{\log \left| \dfrac{1}{(z - z_0)} \right|} = \frac{1}{\beta}$$

where z_0 is a zero of $G(z)$ on $|z| = R$.

Thus whenever we investigate the relation between the geometry of f and the geometry of $\Pi(f, A, B)$ we will be assuming that f is defined and analytic throughout the unit disc.

In order to obtain nontrivial relations between f and $\Pi(f, A, B)$ it is necessary to choose proper subclasses of $\mathscr{G}$ and $\mathscr{H}$. For if $f(z)$ is a generalized Bazilevič function in $|z| < 1$, then $\Pi(f, \mathscr{G}, \mathscr{H})$ is simply $\mathbf{C}$ itself since $(\alpha, \beta, f(z), f'(z)(z/f(z))^{1-i\beta})$ always represents f in $\mathscr{G}$ and $\mathscr{H}$.

So that the reader can have a concrete picture in his mind we have phrased the following results in terms of the ordinary Bazilevič functions. Since the results also apply to classes which do not yet appear in the literature we point out the proper context of our results immediately after the statement of each theorem. Recall that f is an ordinary Bazilevič function, denoted $(f, \mathscr{S}^*, \mathscr{P})$, if f has an (α, β, g, h) representation where α is positive, β is real, g is in $\mathscr{S}^*$ (the normalized starlike functions), and h is in $\mathscr{P}$ (the normalized analytic functions for which for some real λ, $\operatorname{Re} e^{i\lambda}h(z) > 0$ throughout $|z| < 1$).

Sheil-Small showed $\Pi(f, \mathscr{S}^*, \mathscr{P})$ is a closed convex point set. We will find further properties of $\Pi(f, \mathscr{S}^*, \mathscr{P})$ in order to investigate the following general problem: find necessary and sufficient conditions on a point set E so that there will exist a generalized Bazilevič function f and classes A and B for which $\Pi(f, A, B) = E$.

Our first theorem gives a very important restriction on $\Pi(f, A, B)$.

THEOREM 4.1. *If $f(z) \not\equiv z$, then the intersection of $\Pi(f, \mathcal{S}^*, \mathcal{P})$ with any vertical line is a bounded set.* [*The conclusion holds for any $\Pi(f, A, B)$ where A and B are normal families*].

PROOF. If there were a vertical line $x = \alpha$ on which $\Pi(f, \mathcal{S}^*, \mathcal{P})$ were unbounded, then we could find a sequence of quadruples $(\alpha, \beta_k, g_k, h_k)$ representing f for which $|\beta_k| \to \infty$, $g_k \in \mathcal{S}^*$, $h_k \in \mathcal{P}$. Taking the imaginary part of the logarithm of both sides of (1.3) yields

$$\begin{aligned} &\arg \frac{zf'(z)}{f(z)} + \beta_k \log|f(z)/z| + \alpha \arg \frac{f(z)}{z} \\ &\qquad = \alpha \arg \frac{g_k(z)}{z} + \arg h_k(z). \end{aligned} \tag{4.2}$$

Since $f(z) \not\equiv z$, we can find a point z_0 such that $|f(z_0)/z_0| \neq 1$. Although the quantities $g_k(z_0)$ and $h_k(z_o)$ depend on k, nevertheless $|\arg g_k(z_0)/z_0|$ and $|\arg h_k(z_0)|$ are bounded independently of k since $\mathcal{S}^*$ and $\mathcal{P}$ are normal families. As $|\beta_k| \to \infty$ the left hand side of (4.2) diverges while the right hand side remains bounded which is absurd.

Our next theorem is extremely useful in determining what possible value of α and β can be used to represent a generalized Bazilevič function which is unbounded either in modulus or in argument.

THEOREM 4.2. (a) *If $f(z)$ is unbounded then the intersection of $\Pi(f, \mathcal{S}^*, \mathcal{P})$ and any vertical line is at most a single point.*

(b) *If $\arg f(z)/z$ is unbounded, then the intersection of $\Pi(f, \mathcal{S}^*, \mathcal{P})$ and any horizontal line is at most a single point.*

[*The same conclusions hold for $\Pi(f, A, B)$ for any classes A and B for which for each element g in A and h in B we have*

$$\sup_{|z|<1} |\arg g(z)/z| < \infty, \qquad \sup_{|z|<1} |\arg h(z)| < \infty].$$

PROOF. Suppose $f(z)$ is unbounded and $(\alpha_j, \beta_j, g_j, h_j)$, $j = 1, 2$, $\alpha_1 = \alpha_2$, are two representations for $f(z)$ where

$$\sup_{|z|<1} |\arg g_j(z)/z| < \infty, \qquad \sup_{|z|<1} |\arg h_j(z)| < \infty, \tag{4.4}$$

for $j = 1, 2$, by hypothesis. Using (4.2) twice we obtain

$$\beta_1 - \beta_2 = \frac{\alpha(\arg g_1(z)/g_2(z)) + \arg(h_1(z)/h_2(z))}{\log|f(z)/z|}. \tag{4.5}$$

Choosing z_n so that $|f(z_n)| \to \infty$ and remembering (4.4) we see that β_1 must be equal to β_2. A similar proof holds for the assertion concerning $|\arg f(z)/z| \to \infty$.

It is very difficult to know if a given analytically represented function is capable of being written as a reasonably nice Bazilevič function. The next theorem provides an effective test for unbounded functions. One simply tests whether certain limits exist. If the limits do not exist, then either f cannot be represented as a Bazilevič function or else $\Pi(f, A, B)$ consists of a single point. If the limits do exist, then one knows what possible values can be in $\Pi(f, A, B)$.

THEOREM 4.3. *Let $f(z)$ be an unbounded function and let $\Pi(\mathrm{f}, \mathscr{S}^*, \mathscr{P})$ contain at least two distinct points $(\alpha_1 + i\beta_1)$, $(\alpha_2 + i\beta_2)$. Let z_n be any sequence of points in $|z| < 1$ for which $|f(z_n)| \to \infty$ (including sequences z_n which approach $|z| = 1$ tangentially or which have distinct cluster points). Then both of the following limits must exist and must have the indicated values*:

$$\lim_{n\to\infty} \frac{\arg f(z_n)/z_n}{\log|f(z_n)/z_n|} = \frac{\beta_2 - \beta_1}{\alpha_1 - \alpha_2} \tag{4.6}$$

$$\lim_{n\to\infty} \frac{\arg f'(z_n)}{\log|f(z_n)/z_n|} = \frac{(\alpha_2 - 1)\beta_1 - (\alpha_1 - 1)\beta_2}{\alpha_1 - \alpha_2}. \tag{4.7}$$

[*The theorem remains true for $\Pi(f, A, B)$, A, B satisfying the conditions mentioned after Theorem* 4.2].

PROOF. Theorem 4.2 guarantees that $\alpha_2 \neq \alpha_1$. As in Theorem 4.1 we use (4.2) twice to obtain:

$$\begin{aligned} &(\beta_1 - \beta_2)\log|f(z)/z| + (\alpha_1 - \alpha_2)\arg(f(z)/z) \\ &= \alpha_1 \arg(g_1(z)/z) - \alpha_2 \arg(g_2(z)/z) + \arg h_1(z)/h_2(z). \end{aligned} \tag{4.8}$$

Dividing through by $\log|f(z)/z|$, letting $f(z_n) \to \infty$, and using (4.4) yields (4.6). Relation (4.7) is proved similarly.

The next theorem shows how the structure of $\Pi(f, \mathscr{S}^*, \mathscr{P})$ can tell us about the behavior of $f(z)$.

THEOREM 4.4. *If $\Pi(f, \mathscr{S}^*, \mathscr{P})$ is unbounded, then there exists a complex number u, $|u| \leqq 1$ and a $g(z)$ in $\mathscr{S}^*$ such that $f(z) = z(g(z)/z)^u$.*

[*The conclusion holds if $\mathscr{S}^*$ is replaced by a compact family of $\mathscr{G}$ and $\mathscr{P}$ is replaced by any normal subfamily of $\mathscr{H}$*].

PROOF. Since $\Pi(f, \mathscr{S}^*, \mathscr{P})$ is unbounded, then by passing to a subsequence if necessary we can find $\alpha_k + i\beta_k \to \infty$ and complex numbers u, v, $|u| \leqq 1$, $|v| \leqq 1$, $u + v = 1$, such that

$$\lim_{k\to\infty} \frac{\alpha_k}{\alpha_k + i\beta_k} = u, \quad \lim_{k\to\infty} \frac{i\beta_k}{\alpha_k + i\beta_k} = v. \tag{4.9}$$

Each $\alpha_k + i\beta_k$ comes from a quadruple $(\alpha_k, \beta_k, g_k, h_k)$ representing f where $g_k \in \mathscr{S}^*$ and $h_k \in \mathscr{P}$. Using (1.2) we see the following is true for all u, v, and $g(z)$:

$$\begin{aligned} z\frac{f'(z)}{f(z)} - uz\frac{g'(z)}{g(z)} - v = {}& u\left(z\frac{g_k'(z)}{g_k(z)} - z\frac{g'(z)}{g(z)}\right) \\ &+ z\frac{g_k'(z)}{g_k(z)}\left(\frac{\alpha_k}{\alpha_k + i\beta_k} - u\right) + \left(\frac{i\beta_k}{\alpha_k + i\beta_k} - v\right) \\ &+ \frac{1}{\alpha_k + i\beta_k}\left(z\frac{h_k'(z)}{h_k(z)} + z\frac{f'(z)}{f(x)} - 1 - z\frac{f''(z)}{f'(z)}\right). \end{aligned}$$

Thus

$$\begin{aligned} \left| z\frac{f'(z)}{f(z)} - uz\frac{g'(z)}{g(z)} - v\right| \leqq {}& |u|\left|z\frac{g_k'(z)}{g_k(z)} - z\frac{g'(z)}{g(z)}\right| \\ &+ \left|z\frac{g_k'(z)}{g_k(z)}\right| \left|\frac{\alpha_k}{\alpha_k + i\beta_k} - u\right| \\ &+ \left|\frac{i\beta_k}{\alpha_k + i\beta_k} - v\right| \\ &+ \frac{1}{|\alpha_k + i\beta_k|}\left|z\frac{h_k'(z)}{h_k(z)} + z\frac{f'(z)}{f(z)} - 1 - z\frac{f''(z)}{f''(z)}\right|. \end{aligned} \tag{4.10}$$

The normality of $\mathscr{P}$ and the analyticity of $f(z)$ in $|z| < 1$ guarantee that

$$\left| z\frac{h_k'(z)}{h_k(z)} + z\frac{f'(z)}{f(z)} - 1 - z\frac{f''(z)}{f'(z)}\right|$$

is bounded within, say, $|z| \leqq 1/2$ for all k. By passing to another subsequence and using the compactness of $\mathscr{S}^*$ we may assume $g_k(z)$ converges uniformly on $|z| \leqq 1/2$ to $g(z) \in \mathscr{S}^*$. Let u and v be given by (4.9). The left hand side of (4.10) is independent of k while the right hand side goes to zero in $|z| \leqq 1/2$ as $k \to \infty$. Thus

$$z\frac{f'(z)}{f(z)} \equiv uz\frac{g'(z)}{g(z)} + v \tag{4.12}$$

in $|z| \leqq 1/2$. Since $g(z)/z$ is analytic and nonvanishing throughout $|z| < 1$, (4.12) holds throughout $|z| < 1$. Lemma 1.1 guarantees the solution of (4.12) is analytic throughout $|z| < 1$ and is given by $f(z) = z(g(z)/z)^u$.

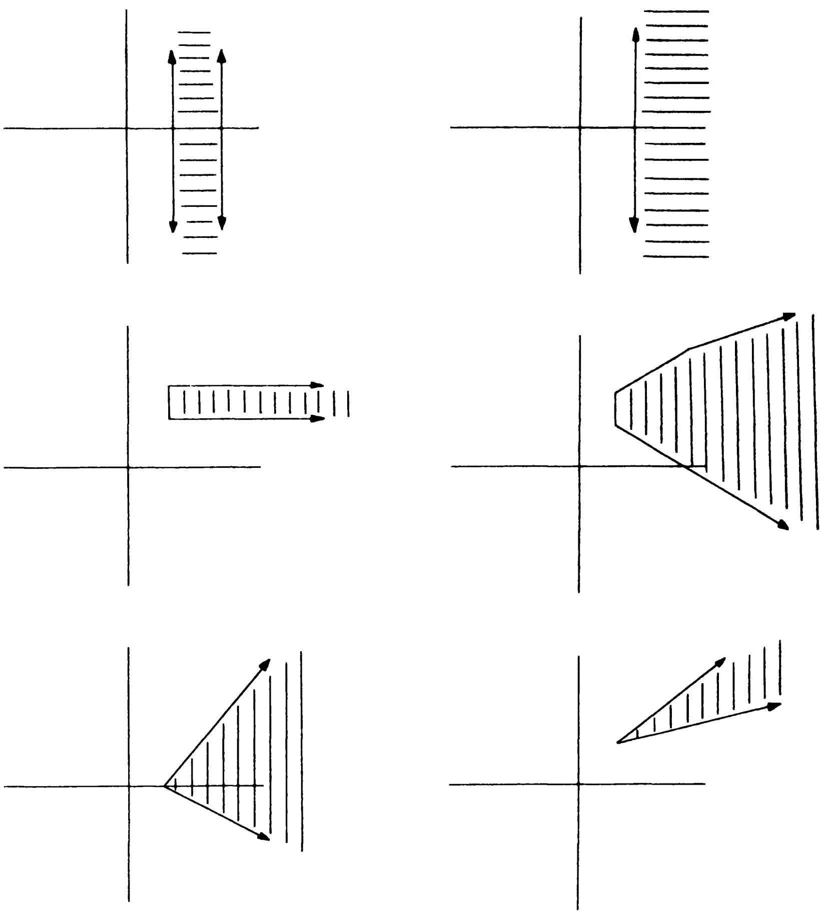

FIGURE 2

If $f(z) \not\equiv z$, then Theorem 4.4 strongly limits the structure of $\Pi(f, \mathcal{S}^*, \mathcal{P})$ in the neighborhood of infinity. In particular, $\Pi(f, \mathcal{S}^*, \mathcal{P})$ (and $\Pi(f, A, B)$ for any A, B satisfying the conditions at the end of Theorem 4.2) cannot contain arbitrarily large points $\alpha_k + i\beta_k$ with $\alpha_k/\beta_k \to 0$.

COROLLARY 4.5. *If $\Pi(f, \mathcal{S}^*, \mathcal{P})$ is unbounded then $f(z)$ is spiral-like. If $\Pi(f, \mathcal{S}^*, \mathcal{P})$ contains a sequence of points $\alpha_k + i\beta_k$, $\beta_k = m\alpha_k + \ell$, $|\alpha_k| \to \infty$, then $f(z)$ is spiral-like of order* arctan m. *If $\Pi(f, \mathcal{S}^*, \mathcal{P})$ is unbounded on any horizontal line, then $f(z)$ is starlike.*

PROOF. This is immediate from (4.9), (4.12) and the definition of spiral-like and starlike.

We now have enough machinery to investigate whether an arbitrary convex set in **C** can be Π of some ordinary Bazilevič function. Theorem 4.1 shows that the convex sets (a) and (b) of Figure 2 can never be a $\Pi(f, \mathcal{S}^*, \mathcal{P})$ set. Corollary 4.5 shows that the convex sets (c), (d), (e) and (f) of Figure 2 can never be a $\Pi(f, \mathcal{S}^*, \mathcal{P})$ set. For if $f(z)$ is starlike it automatically has a representation along the entire real axis (α, 0, $f(z)$, $zf'(z)/f(z)$), while if $f(z)$ is spiral-like of order γ, then it automatically has the representation ($t\cos\gamma$, $t\sin\gamma$, $z(f(z)/z)^{1+i\tan\gamma}$, $zf'(z)/f(z)$) along the entire line through the origin of slope arctan γ.

Given a generalized Bazilevič function f it would be nice to have reasonably general conditions which guarantee that $f(z)$ can be uniformly approximated on compacta by 'nicer' Bazilevič functions which are close to f in the sense of the projection Π.

THEOREM 4.6. *Let $\alpha + i\beta$ be a point in $\Pi(f, \mathcal{S}^*, \mathcal{P})$. Then for each r, $0 < r < 1$, $\Pi(f(rz)/r, \mathcal{S}^*, \mathcal{P})$ contains a neighborhood of the point $\alpha + i\beta$.*

PROOF. Fix r, $0 < r < 1$, and consider $F(z) \equiv f(rz)/r$. By (1.3)

$$z\frac{f'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\beta} = \left(\frac{g(z)}{z}\right)^{\alpha} h(z),$$

where $g(z) \in \mathcal{S}^*$ and $h(z) \in \mathcal{P}$ is such that $|\arg e^{i\lambda}h(z)| < \pi/2$. Let

$$\begin{aligned} G(z) &\equiv g(rz)/r, \\ H(z) &\equiv z\frac{F'(z)}{F(z)}\left(\frac{F(z)}{z}\right)^{\alpha^*+i\beta^*}\left(\frac{z}{G(z)}\right)^{\alpha^*} \\ &= \frac{f(rz)}{rz}^{\alpha^*-\alpha+i(\beta^*-\beta)}\left(\frac{rz}{g(rz)}\right)^{\alpha^*-\alpha} h(rz). \end{aligned}$$

Then

$$\begin{aligned}
&\sup_{|z|<1} |\arg e^{i\lambda} H(z)| \\
&= \sup_{|z|<1} \left| \arg e^{i\lambda}\, h(rz) + (\alpha - \alpha^*) \arg \left(\frac{g(rz)}{rz} \right) \right. \\
&\qquad + (\alpha^* - \alpha) \arg \left(\frac{f(rz)}{\mathrm{rz}} \right) \\
&\qquad \left. + (\beta^* - \beta) \log \left| \frac{f(rz)}{rz} \right| \right| \\
&\leqq \sup_{|z|<1} |\arg e^{i\lambda}\, h(rz)| + |(\alpha^* - \alpha) + i(\beta^* - \beta)| \\
&\qquad \cdot \max_{|z|\leqq 1} \left\{ \left| \arg\left(\frac{g(rz)}{rz} \right) \right|, \right. \\
&\qquad\qquad \left. \left| \arg\left(\frac{f(rz)}{rz} \right) \right|, \left| \log \left| \frac{f(rz)}{rz} \right| \right| \right\}.
\end{aligned}$$

Therefore by choosing $\alpha^* + i\beta^*$ in a sufficiently small neighborhood of $\alpha + i\beta$ we can guarantee that $H(z)$ is in $\mathscr{P}$. The function $G(z)$ is in $\mathscr{S}^*$ since $\mathscr{S}^*$ is contraction invariant. Rewriting $H(z)$ as

$$z \frac{F'(z)}{F(z)} \left(\frac{F(z)}{z} \right)^{\alpha^* + i\beta^*} = \left(\frac{G(z)}{z} \right)^{\alpha^*} H(z),$$

we can use Lemma 1.2 to show that $f(rz)/r$ has an $(\alpha^*, \beta^*, G, H)$ representation as claimed.

We conclude this section with two remarks.

Remark. $\Pi(f, \mathscr{S}^*, \mathscr{P})$ need not be convex in $\mathbf{C}$ although it is always convex in $\alpha \geqq 0$ and $\alpha \leqq 0$ separately. To see this consider $f(z) = z \exp(z/2)$ which is γ-spiral-like for $-\pi/6 \leqq \gamma \leqq \pi/6$. As we indicated before, any γ-spiral-like function automatically has an $\mathscr{S}^*$, $\mathscr{P}$ representation everywhere on the line $\{te^{i\gamma} : -\infty < t < \infty\}$. If $\Pi(f, \mathscr{S}^*, \mathscr{P})$ were convex in $\mathbf{C}$ it would have to contain the convex hull of the points $\{te^{i\gamma} : -\infty < t < \infty,\ -\pi/6 \leqq \gamma \leqq \pi/6\}$. This would be the entire complex plane. Since $f(z) = z \exp(z/2) \not\equiv z$, this is absurd by Theorem 4.2.

Remark. If $f(z)$ is an unbounded γ spiral-like function, then $f(z)$ cannot be γ'-spiral-like, $\gamma' \neq \gamma$. In particular an unbounded starlike function can only be trivially spiral-like. In order to see this it suffices to recall that if a function were both γ and γ' spiral-like, then in the right half plane $\Pi(f, \mathscr{S}^*, \mathscr{P})$ would contain the convex hull of $\{te^{i\gamma}, te^{i\gamma'} : 0 < t < \infty\}$. In particular, the intersection of $\Pi(f, \mathscr{S}^*, \mathscr{P})$ with any vertical line in the right half plane would not be a singleton which is absurd by Theorem 4.2.

5. **Uniqueness of the (α, β) representation of a generalized Bazilevič function.** Each generalized Bazilevič function is given by a quadruple (α, β, g, h). One hopes that the functions g and h reflect the geometry of f itself. Therefore if we fix α and β it is important to determine how much freedom we have in choosing the g and the h which will represent f. For example, $f(z) = (e^{az} - 1)/a$, $0 < |a| \leqq \pi/2$, is an ordinary Bazilevič function with the following two distinct $(1, 0)$ representations in $\mathscr{S}^*, \mathscr{P}$:

$$(1, 0, z, e^{az})$$

$$(1, 0, (e^{az} - 1)/a, aze^{az}/(e^{az} - 1)).$$

This lack of uniqueness is possible for any α, β. For example, if ϵ is sufficiently small, then $f(z) \equiv z$ has the following uncountably many distinct (α, β) representations in $\mathscr{S}^*, \mathscr{P}$: $(\alpha, \beta, z(1 + \epsilon z), (1 + \epsilon z)^{-\alpha})$.

One therefore asks if there are any examples of unique representations. In order to answer this in the affirmative we develop a relation between the coefficients of f and the coefficients of g and h which represent f.

THEOREM 5.1. *Let the generalized Bazilevič function $f(z) = z + a_2z^2 + \cdots$ be represented by the quadruple (α, β, g, h). Then*

$$(1 + \alpha + i\beta)a_2 = \alpha\, b_2 + c_1 \tag{5.1}$$

where $g(z) = z + b_2z^2 + \cdots$ and $h(z) = 1 + c_1z + \cdots$.

PROOF. Multiplying both sides of the fundamental differential equation (1.2) by $f(z)\, f'(z)\, g(z)\, h(z)$ we obtain

$$\begin{aligned} f(z)f'(z)g(z)h(z) &+ z\, f(z)f''(z)g(z)h(z) \\ &+ (\alpha + i\beta - 1)z(f'(z))^2 g\,(z)h(z) \\ &= \alpha z f(z)f'(z)g'(z)h(z) + z\, f(z)f'(z)g(z)h'(z) \\ &+ i\beta f(z)f'(z)g(z)h(z). \end{aligned} \tag{5.2}$$

Equating the coefficients of z^3 for both sides of (5.2) yields

$$(c_1 + b_2 + 2a_2 + a_2) + 2a_2 + (\alpha + i\beta - 1)(c_1 + b_2 + 4a_2)$$
$$= \alpha(c_1 + 2b_2 + 2a_2 + a_2) + c_1 + i\beta(c_1 + b_2 + 2a_2 + a_2)$$

which reduces to $(1 + \alpha + i\beta)a_2 = \alpha b_2 + c_1$. In a similar manner one can obtain, using (5.1),

$$2(2 + \alpha + i\beta)a_3 = \alpha(2b_3 - b_2^2) + 2c_2 - c_1^2 + a_2^2(3 + \alpha + i\beta). \tag{5.3}$$

This allows us to obtain sharp coefficient inequalities for the ordinary Bazilevič functions $B(\alpha, \beta)$, $\alpha \geqq 0$.

THEOREM 5.2. *If f is in $B(\alpha, \beta)$, $\alpha \geqq 0$, then*

$$\left| a_3 - \frac{3 + \alpha + i\beta}{2(2 + \alpha + i\beta)} a_2^2 \right| \leqq \frac{\alpha + 2}{|\alpha + 2 + i\beta|} \tag{5.4}$$

and equality is possible for every (α, β), $\alpha \geqq 0$, $-\infty < \beta < \infty$.

PROOF. Using (5.3) for a_3 yields

$$\begin{aligned} \left| a_3 - \frac{3 + \alpha + i\beta}{2(2 + \alpha + i\beta)} a_2^2 \right| &= \left| \frac{\alpha(b_3 - \frac{1}{2} b_2^2)}{2 + \alpha + i\beta} + \frac{c_2 - \frac{1}{2} c_1^2}{2 + \alpha + i\beta} \right| \\ &\leqq \frac{\alpha|b_3 - \frac{1}{2} b_2^2|}{|2 + \alpha + i\beta|} + \frac{|c_2 - \frac{1}{2} c_1^2|}{|2 + \alpha + i\beta|} \\ &\leqq \frac{\alpha + 2}{|2 + \alpha + i\beta|}, \end{aligned} \tag{5.5}$$

where the last inequality uses the fact that $|c_2 - (1/2)c_1^2| \leqq 2$ for functions of positive real part and $|b_3 - (1/2)b_2^2| \leqq 1$ for starlike functions. By choosing $h(z) = (1 + z^2)/(1 - z^2) = 1 + 2z^2 + \cdots$ and $g(z) = z/(1 - z^2) = z + z^3 + \cdots$ we obtain equality in (5.4) for any $\alpha \geqq 0$ and $-\infty < \beta < \infty$.

THEOREM 5.3. *If $f(z)$ is in $B(\alpha, \beta)$, $\alpha \geqq 0$, then*

$$|a_2| \leqq \frac{2(\alpha + 1)}{|\alpha + 1 + i\beta|}. \tag{5.6}$$

If $f(z)$ is in $B(0, \beta)$, then

$$|a_3| \leqq \frac{2}{|2 + i\beta|} \cdot \max \left\{ 1, \left| \frac{3 + i\beta}{(1 + i\beta)^2} \right| \right\}. \tag{5.7}$$

Both inequalities (5.6) *and* (5.7) *are best possible.*

PROOF. Inequality (5.6) is immediate from (5.1) and the fact that

$|c_2| \leqq 2$ and $|b_2| \leqq 2$. The uniqueness (up to rotations) of functions in $\mathscr{S}^*$ and $\mathscr{P}$ satisfying $|c_1| = 2$, $|b_2| = 2$, shows that equality holds in (5.6) if and only if $f(z)$ is a rotation of the function defined by $(\alpha, \beta, z/(1-z)^2, (1+z)/(1-z))$.

If α is zero, then (5.3) and (5.1) together yield

$$(5.8) \quad |a_3| = \frac{1}{|2+i\beta|}\,|c_2 - \lambda c_1{}^2|,\ \lambda = \frac{1}{2}\left(1 - \frac{3+i\beta}{(1+i\beta)^2}\right).$$

But for functions of positive real part

$$|c_2 - \lambda c_1{}^2| \leqq 2\max\{1, |2\lambda - 1|\}$$

and the inequality is sharp. This immediately proves (5.7) and also proves that (5.7) is sharp.

The conclusions of Theorem 5.2 and 5.3 actually hold for the larger class of functions which have an (α, β, g, h) representation with g spiral-like (instead of just starlike) and h in $\mathscr{P}$. We only needed $|b_3 - (1/2)b_2{}^2| \leqq 1$ and this holds for spiral-like as well as starlike functions [7].

The class of functions which have an (α, β, g, h) representation, g spiral-like, $h \in \mathscr{P}$, contains nonunivalent functions as well as univalent functions which are not ordinary Bazilevič functions. To be specific, let us consider functions with a $(1, 0, g, h)$ representation, g spiral-like, h in $\mathscr{P}$. All analytic functions $f(z)$ for which $zf'(z)$ is spiral-like belong to this class; in fact, they have the representation $(1, 0, zf'(z), 1)$. Robertson [18] showed that there are nonunivalent functions for which $zf'(z)$ is spiral-like. Therefore, there are nonunivalent functions with a $(1, 0, g, h)$ representation, g spiral-like, h in $\mathscr{P}$.

We now prove that there are univalent functions with a $(1, 0, g, h)$ representation which are not in $B(\alpha, \beta)$ for any $\alpha \geqq 0$, and $-\infty < \beta < \infty$. The function

$$(5.9) \qquad f(z) = \frac{1 - (1-z)^{\exp(-2i\gamma)}}{\exp(-2i\gamma)}$$

defined by $(1, 0, z(1-z)^{\exp(-2i\gamma)-1}, 1)$ is univalent for $0 < \gamma < \pi/6$ by corollary 2 of [14] since $0 < \cos(\gamma - \pi/2) \leqq 1/2$ and

$$\operatorname{Re} e^{i(\gamma-\pi/2)}\left[1 + z\frac{f''(z)}{f'(z)}\right]$$
$$= \operatorname{Re} e^{i(\gamma-\pi/2)}\left[\frac{1 - ze^{-2i\gamma}}{1-z}\right] > 0$$

in $|z| < 1$.

On the other hand, Sheil-Small [19, p. 142] showed that if C is a positively described arc of the simple closed curve $f(|z| = r)$, $0 < r < 1$, $f(z)$ univalent analytic, then $f(z)$ is an ordinary Bazilevič function of type (α, β), $\alpha \geqq 0$, if and only if for each such arc

$$\Delta_C (\arg dw + (\alpha - 1)\arg w + \beta \log|w|) > -\pi \tag{5.10}$$

where if C joins w_1 to w_2, $\Delta_C u(w) \equiv u(w_2) - u(w_1)$. Since $(1 - z)^{\exp(-2i\gamma)}$ is bounded in $|z| < 1$ and goes to zero as $z \to 1$ in any direction, it is easy to see that $f(z)$ and $\arg f(z)$ are bounded in $|z| < 1$. A direct computation shows $\arg f'(z)$ is unbounded in $|z| < 1$. Consequently for any $\alpha \geqq 0$ and any β, $-\infty < \beta < \infty$, the function (5.9) can be made to violate (5.10). Thus $f(z)$ cannot be an ordinary Bazilevič function. This provides an explict example of a bounded univalent non-Bazilevič function.

One might suspect from Sheil-Small's example of a univalent non-Bazilevič function and from the univalent non-Bazilevič function (5.9) that all such examples must be 'bad' on the boundary. Such is not the case. There are functions univalent analytic on the *closed* unit disc which are quasiconformally extendable to the whole plane, but which are not ordinary Bazilevič functions. It suffices to note that the compactness of the ordinary Bazilevič functions [19, p. 141] implies that if $f(z)$ is (5.9) then only finitely many elements of the collection $\{f(r_n z)/r_n)\}_{n=1}^{\infty}$, $0 < r_n < 1$, $r_n \to 1$, can be ordinary Bazilevič functions. Consequently $f(r_n z)/r_n$ for all n greater than some n_0 are non-Bazilevič functions univalent in the closed unit disc, and trivially quasiconformally extendable.

Theorem 5.2 yields the first quantitative bound for the radius of ordinary Bazilevičness for the class $\mathscr{S}$. Let $R_{\beta z}$ denote the supremun of these $r < 1$ such that for every $f(z)$ in $\mathscr{S}$, $f(rz)/r$ is in some class $B(\alpha, \beta)$, $\alpha \geqq 0$.

THEOREM 5.5. $R_{\beta z} \leqq (1 + 2\exp(-6))^{-1/2} = .99753 \cdots$.

PROOF. An analysis of Szegö and Fekete's proof that for any $f(z)$ in $\mathscr{S}$, $|a_3 - \lambda a_2{}^2| \leqq 1 + 2 \exp(-2\lambda/(1 - \lambda))$, $0 \leqq \lambda < 1$, [16, p. 167] shows that for $\lambda = 3/4$ there is a function $f(z)$ in $\mathscr{S}$ with

$$a_2 = 8e^{-3} \qquad a_3 = 1 + 50e^{-6}. \tag{5.11}$$

If $f(z)$ is this function with a_2 and a_3 given by (5.11) then we will apply Theorem 5.3 to the coefficients of $f(rz)/r = z + A_2 z^2 + A_3 z^3 + \cdots$ where $A_2 = ra_2$, $A_3 = r^2 a_3$. Computing (5.4) for the coefficients of $f(rz)/r$ yields

$$\left| A_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)} A_2^2 \right| = r^2 \left| a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta} a_2^2 \right| . \tag{5.12}$$

But a_2 and a_3 are real and positive so

$$\left| a_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta)} a_2^2 \right| \geqq \left| a_3 - \frac{3}{4} a_2^2 \right| \tag{5.13}$$

for all $\alpha \geqq 0$ and all β, $-\infty < \beta < \infty$. Thus by (5.11), (5.12) and (5.13) the coefficients of $f(rz)/r$ satisfy

$$\left| A_3 - \frac{3+\alpha+i\beta}{2(2+\alpha+i\beta} A_2^2 \right| \geqq r^2(1 + 2\exp - 6).$$

Therefore by (5.4) the univalent function $f(rz)/r$ cannot be in any $B(\alpha, \beta)$ class, $\alpha \geqq 0$, if $r^2(1 + 2\exp(-6))$ is greater than 1. Consequently $R_{\beta_z} \leqq (1 + 2\exp(-6))^{-1/2}$. Better bounds are easily obtained for any specific $B(\alpha, \beta)$ class.

We return to the problem of unique representation of a Bazilevič function. The Koebe function $z/(1-z)^2$ can have an (α, B), $\alpha > 0$, representation in the ordinary Bazilevič function only if $\beta = 0$ and in this case the functions g and h are uniquely determined. In fact let (α, β, g, h) be representation for $z/(1-z)^2$, $g \in \mathscr{S}^*$, $h \in \mathscr{P}$, $\alpha \geqq 0$. Then by Theorem 5.1, $2 = (\alpha b_2 + c_1)/(1 + \alpha + i\beta)$ where b_2 is the second Taylor coefficient of g and c_1 is the first Taylor coefficient of h. If $|b_2|$ or $|c_1|$ were strictly less than 2 we would have

$$2 = \left| \frac{\alpha b_2 + c_1}{1 + \alpha + i\beta} \right| < \frac{2\alpha + 2}{\sigma + 1} = 2$$

which is absurd. Thus $|b_2| = |c_1| = 2$. The same argument forces $\beta = 0$. Repeating the argument a third time forces $\arg b_2 = \arg c_1 = 0$. Thus $b_2 = 2$, $c_1 = 2$ which, as is well known for $\mathscr{S}^*$ and $\mathscr{P}$, forces $g(z) = z/(1-z)^2$ and $h(z) = (1+z)/(1-z)$. One notices that the Koebe function has the same unique representation in the much larger class of functions with an (α, β, g, h) representation, $g \in \mathscr{S}$, $h \in \mathscr{P}$, $\alpha > 0$.

Since the Koebe function still has the representation $(\alpha, 0, z/(1-z)^2, (1+z)/(1-z))$ on the negative real axis, Theorem 4.2 and the convexity of that part of $\Pi(f, \mathscr{S}^*, \mathscr{P})$ which lies in the left plane again imply that there can only be an $(\alpha, 0)$ representation in $\mathscr{S}^*$, $\mathscr{P}$, $\alpha < 0$, for the Koebe function. However g and h are no longer unique for the

Koebe function, $\alpha < 0$. In fact,

$$\begin{array}{ll} (\alpha,\ 0,\ z/(1-z)^{(2\alpha+1)/\alpha},\ 1+z), & \alpha < -1/2 \\ (\alpha,\ 0,\ z/(1-z)^{2(\alpha+1)/\alpha},\ 1-z^2), & \alpha < -1 \\ (-\frac{1}{2},\ 0,\ z/(1+az)^2,\ (1+az)(1+z)), & -\frac{1}{3} \leqq a \leqq 0 \\ (-\frac{1}{2},\ 0,\ z,\ 1+z) & \end{array}$$

are all distinct (α, β, g, h) representations, $g \in \mathscr{S}^*$, $h \in \mathscr{P}$, $\alpha < 0$. In particular we can represent the Koebe function by two bounded functions $g(z) = z$, $h(z) = 1 + z$.

6. **Application to the $a_n = O(1/n)$ problem.** One of the interesting results of univalent function theory relates the growth of the maximum modulus of a function, $M(r, f)$, to the growth of the coefficients of the function $f(z) = \sum a_n z^n$. In 1932 Littlewood and Paley [12] proved for univalent functions that $M(r, f) = O(1-r)^{-\gamma}$ implies $|a_n| = O(n^{\gamma-1})$, $\gamma > 1/2$. This phenomonon breaks down for $\gamma < 1/2$. In fact Littlewood [11] proved there is an odd bounded univalent function for which $|a_n| > n^{.17-1}$ for infinitely many n. Thus to prove that $|a_n| = O(n^{-1})$ for a class of bounded univalent functions requires additional assumptions about the geometry of elements of the class. Clunie and Keogh [4] in 1960 proved that $a_n = O(n^{-1})$ for bounded starlike functions while Clunie and Pommerenke [5] in 1966 proved $a_n = O(n^{-1})$ for bounded close-to-convex functions. Thomas [20] in 1968 proved $a_n = O(n^{-1})$ for bounded $B(\alpha, 0)$, $\alpha > 0$, functions (for which $B(1, 0)$ yields Clunie and Pommerenke's close-to-convex result).

In the other direction Clunie [3] in 1968 answered in the negative a question of Pommerenke concerning Littlewood's $|a_n| > n^{.17-1}$ counterexample. Clunie exhibited a bounded univalent function with $|\arg f'(z)| < \pi$ for which $a_n \neq O(n^{-1})$. Thus, Clunie remarked, being 'similar' to a close-to-convex function is not sufficient to guarantee $a_n = O(n^{-1})$. This raised the question as to whether one can characterize the bounded univalent functions for which $a_n = O(n^{-1})$. After stating and proving the next theorem, which extends Thomas' result we make two conjectures about the geometry of a function and $a_n = O(n^{-1})$ behavior.

We recall that a starlike, close-to-convex, or ordinary Bazilevič function of type $(\alpha, 0)$, $\alpha > 0$, has the representation

$$\left(\frac{z}{f(z)}\right)^{\alpha-1} f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha} h(z),$$

where $g(z)$ is starlike and $\text{Re}\, e^{i\lambda}h(z)$ is of positive real part. But for the starlike function $g(z)$ in the representation of $f(z)$ we always have $\int_0^{2\pi} |d_\theta[arg\, g(z)\}| = \int_0^{2\pi} d_\theta\{\arg g(z)\} = 2\pi$. It appears that the crucial element in Thomas' proof that bounded univalent ordinary Bazilevič functions have $a_n = O(n^{-1})$ is that f have a representation in a $\mathcal{G}$, $\mathcal{H}$ class where $\sup \int_0^{2\pi} |d_\theta\{\arg g(z)\}|$ is finite rather than necessarily 2π. A careful examination of Thomas' proof also indicates that his heavy dependence on the univalence of f (for example Lemma 1 and Lemma 2, p. 356 [20]) is not necessary. The appropriate setting for Thomas' proof is the context of generalized Bazilevič functions which have an α, 0 representation, α real,

$$\left(\frac{z}{f(z)}\right)^{\alpha-1} f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha} h(z),$$

where $g(z)$ satisfies $\sup \int_0^{2\pi} |d_\theta\{\arg g(z)\}| < \infty$ and $h(z)$ satisfies $\text{Re}\, e^{i\lambda}h(z) > 0$ for some λ.

THEOREM 6.1. *Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be a generalized Bazilevič function with representation $(\alpha, 0, g, h)$ where $\sup \int_0^{2\pi} |d_\theta \{\arg g(z)\}| \leqq A < \infty$, $\text{Re}\, e^{i\lambda}h(z) > 0$, and α is real. If $0 < m \leqq |f(z)/z| \leqq M < \infty$ in $|z| < 1$ then $n|a_n| \leqq K(\alpha, M, m, A)$, where $K(\alpha, M, m, A)$ is a constant independent of f and depends only on α, M, m and A.*

PROOF. Let $(\alpha, 0, g, h)$ *be a representation for f.* By (1.3) $f(z)$ satisfies

$$z\frac{f'(z)}{f(z)} = \left(\frac{g(z)}{z}\right)^{\alpha}\left(\frac{z}{f(z)}\right)^{\alpha} h(z)$$

where for some λ, $h_1(z) = e^{-i\lambda}h(z)$ is of positive real part. Thus

$$\begin{aligned} e^{-i\lambda}zf'(z) &= 2(f(z))^{1-\alpha}(g(z))^{\alpha} Re\, h_1(z) \\ &\quad - (f(z))^{1-\alpha}(g(z))^{\alpha}\overline{h_1(z)}. \end{aligned} \tag{6.1}$$

Since na_n is equal to $(2\pi r^n)^{-1} \int_0^{2\pi} zf'(z)e^{-in\theta}\, d\theta$, (6.1) yields

$$\begin{aligned} n|a_n| &\leqq (\pi r^n)^{-1} \int_0^{2\pi} |f(z)^{1-\alpha}g(z)^{\alpha}|\, \text{Re}\, h_1(z)\, d\theta \\ &\quad + (2\pi r^n)^{-1} \left| \int_0^{2\pi} f(z)^{1-\alpha}g(z)^{\alpha}\overline{h_1(z)}e^{-in\theta}\, d\theta \right| \\ &= I_1(r) + I_2(r). \end{aligned}$$

From the hypothesis on $f(z)$ we have $|f(z)|^{1-\alpha} \leqq K_1(r)$ where $K_1(r)$ is $(Mr)^{1-\alpha}$ if $-\infty \leqq \alpha \leqq 1$ and is $(mr)^{1-\alpha}$ if $1 \leqq \alpha$. Thus

$$I_1(r) \leqq K_1(r)(\pi r^n)^{-1} \int_0^{2\pi} |g(z)|^\alpha \operatorname{Re} h_1(z)\, d\theta.$$

Using (6.1) and the fact that $\operatorname{Re} h_1(z) \geqq 0$ in $|z| < 1$ we find

$$\begin{aligned} I_1(r) &\leqq K_1(r)(\pi r^n)^{-1} \operatorname{Re} \int_0^{2\pi} |g(z)|^\alpha h_1(z)\, d\theta \\ &= K_1(r)(\pi r^n)^{-1} \operatorname{Re} \int_0^{2\pi} e^{-i\lambda} z f'(z) f(z)^{\alpha-1} \exp(-i\alpha \arg g(z))\, d\theta. \end{aligned}$$

After integrating this by parts we obtain

$$I_1 \leqq K_1(r)(\pi r^n)^{-1} \operatorname{Re} \int_0^{2\pi} f(z)^\alpha \exp(-i\alpha \arg g(z))\, d_\theta\{\arg g(z)\}.$$

Taking absolute values in this integral and using the fact that $\sup \int_0^{2\pi} |d_\theta\{\arg g(z)\}| \leqq A$ together with the fact that $|f(z)|^\alpha \leqq (M|z|)^\alpha$ if $\alpha \geqq 0$ and $|f(z)|^\alpha \leqq (m|z|)^\alpha$ if $\alpha \leqq 0$, we obtain

$$I_1(r) \leqq 2AK_2(\pi r^n)^{-1} \tag{6.2}$$

where

$$K_2 = \left\{ \begin{array}{ll} M^{1+|\alpha|}/m^{|\alpha|} & \alpha \leqq 0 \\ M & 0 \leqq \alpha \leqq 1 \\ M^\alpha m^{1-\alpha} & 1 \leqq \alpha \end{array} \right\}. \tag{6.3}$$

We now bound $I_2(r)$. On taking complex conjugates and using (6.1) again, we obtain

$$\begin{aligned} I_2(r) &= (2\pi r^n)^{-1} \left| \int_0^{2\pi} \overline{f(z)}^{1-\alpha} \overline{g(z)}^\alpha h_1(z) e^{in\theta}\, d\theta \right| \\ &= (2\pi r^n)^{-1} \left| \int_0^{2\pi} \overline{f(z)}^{1-\alpha} z f'(z) f(z)^{\alpha-1} \exp(-2i\alpha \arg g(z)) e^{in\theta}\, d\theta \right| \\ &= (2\pi r^n)^{-1} \left| \int_0^{2\pi} z f'(z) \exp(2i(\alpha - 1) \arg f(z)) \exp(-2i\alpha \arg g(z)) e^{in\theta}\, d\theta \right| \\ &= (2\pi r^{2n})^{-1} \left| \int_0^{2\pi} z^{n+1} f'(z) \exp(2i(\alpha - 1) \arg f(z)) \exp(-2i\alpha \arg g(z))\, d\theta \right| \end{aligned} \tag{6.4}$$

If we define $f_n(z)$ as

$$f_n(z) = \int_0^z \zeta^n f'(\zeta)\, d\zeta, \tag{6.5}$$

then integrating (6.4) by parts yields

$$I_2(r) = (2\pi r^{2n})^{-1}\left| \int_0^{2\pi} f_n(z)\, d_\theta\{\exp[2i(\alpha - 1)\arg f(z)]\exp(2i\alpha \arg g(z))\}\right|$$

$$\leqq \frac{|\alpha - 1|}{\pi r^{2n}}\left| \int_0^{2\pi} f_n(z)\exp[2i(\alpha - 1)\arg f(z)]\exp(-2i\alpha \arg g(z)\mathrm{Re}\, z\, \frac{f'(z)}{f(z)}\, d\theta\right|$$

$$+ \frac{|\alpha|}{\pi r^{2n}}\left| \int_0^{2\pi} f_n(z)\exp[2i(\alpha - 1)\arg f(z)]\exp(-2i\alpha \arg g(z)\, d_\theta\{\arg g(z)\}\right|$$

$$\equiv J_1(r) + J_2(r).$$

We now estimate $J_1(r)$ and $J_2(r)$. Upon integrating (6.5) by parts we obtain

$$|f_n(z)| \leqq 2Mr^n. \tag{6.6}$$

Since $\sup \int_0^{2\pi} |d_\theta\, \{\arg g(z)\}| \leqq A$ we obtain by (6.6)

$$J_2(r) \leqq 4MA|\alpha|r^{-n} \tag{6.7}$$

Applying the Schwarz inequality to

$$J_1(r) \leqq \frac{|\alpha - 1|}{\pi r^{2n}}\left| \int_0^{2\pi} f_n(z)\exp[2i(\alpha - 1)\arg f(z)]\right.$$

$$\left. \cdot \exp(-2i\alpha \arg g(z))\mathrm{Re}\, z\, \frac{f'(z)}{f(z)}\, d\theta\right|$$

yields

$$J_1^{\,2}(r) \leqq \frac{(\alpha - 1)^2}{\pi^2 r^{4n}} \int_0^{2\pi} |f_n(z)|^2\, d\theta \int_0^{2\pi} \left|\mathrm{Re}\, z\, \frac{f'(z)}{f(z)}\right|^2 d\theta. \tag{6.8}$$

Noting that

$$|\mathrm{Re}\, zf'(z)/f(z)|^2 \leqq m^{-2}|f'(z)|^2 \tag{6.9}$$

yields

$$\begin{aligned}\int_0^{2\pi} |\mathrm{Re}\, zf'(z)|^2\, d\theta &\leqq m^{-2} \int_0^{2\pi} |f'(z)|^2\, d\theta \\ &= 2\pi m^{-2} \sum_{k=0}^{\infty} k^2|a_k|^2 r^{2k-2} \\ &\leqq 2\pi m^{-2} r^{-2}\ (\sup_k kr^k) \sum_{k=0}^{\infty} k|a_k|^2 r^k \\ &\leqq 2\pi m^{-2} r^{-2} e^{-1}(1 - r)^{-1} \sum_{k=0}^{\infty} k|a_k|^2 r^k.\end{aligned} \tag{6.10}$$

On the other hand $f_n(z) = \Sigma_{k=0}^{\infty} (k/(n+k))^2 a_k z^{n+k}$ so

$$\int_0^{2\pi} |f_n(z)|^2 \, d\theta = 2\pi \sum_{k=0}^{\infty} (k/(n+k))^2 |a_k|^2 r^{2n+2k}$$

$$\text{(6.11)} \qquad \leqq 2\pi r^{2n} \left(\sup_k \frac{kr^k}{(n+k)^2} \right) \sum_{k=0}^{\infty} k|a_k|^2 r^k$$

$$\leqq 2\pi r^{2n} e^{-1}(1-r)^{-1} n^{-2} \sum_{k=0}^{\infty} k|a_k|^2 r^k.$$

Thus

$$J_1(r) \leqq 2|\alpha - 1| M^2 m^{-1} r^{-n} e^{-1}(1-r)^{-1} n^{-1},$$

since $\Sigma\, k|a_k|^2 r^k$ is less than or equal to $M^2 r(1-r)^{-2}$ by the Cauchy estimate $|a_n| \leqq M$. Putting all these estimates together we have

$$\text{(6.12} \qquad \begin{aligned} n|a_n| \leqq{} & 2AK_2(\pi r^n)^{-1} + 4MA|\alpha| r^{-n} \\ & + 2|\alpha - 1| M^2 m^{-1} r^{-n} e^{-1}(1-r)^{-3} n^{-1}. \end{aligned}$$

Choosing r so that $(1-r)^3 n \to \infty$ as $n \to \infty$ (for example, $r = 1 - n^{-1/4}$) we see that

$$\limsup_{n \to \infty} \; n|a_n| \leqq 2AK_2 e^3 \pi^{-1} + 4MA|\alpha| e^3 = K(A, \alpha, M, m)$$

as claimed. Note in particular that this proof shows $J_1(r)$ does not contribute anything to the determination of $\limsup n|a_n|$.

COROLLARY 6.2. *If $f(z)$ is a bounded univalent function which has an $(\alpha, 0, g, h)$ representation,* $\sup \int_0^{2\pi} |d_\theta \{\arg g(z)\}| < \infty$, $h \in \mathscr{P}$, *α real, then* $a_n = O(n^{-1})$.

Corollary 6.2 generalizes Thomas' result not only to a larger class of functions but also allows α to be negative as well.

We close the paper with the conjecture that these generalized Bazilevič functions are the natural setting for the phenomenon relating boundedness of univalent $f(z)$ to $a_n = O(n^{-1})$.

CONJECTURE A. If $f(z)$ is a bounded univalent function with $a_n = O(n^{-1})$, then f has an $(\alpha, 0, g, h)$ representation with $\sup \int_0^{2\pi} |d_\theta \{\arg g(z)\}| < \infty$, $h \in \mathscr{P}$.

CONJECTURE B. If $f(z)$ is a locally univalent function, $0 < m \leqq |f(z)/z| \leqq M < \infty$ in $|z| < 1$, and $a_n = O(n^{-1})$, then f has an $(\alpha, 0, g, h)$ representation with $\sup \int_0^{2\pi} |d_\theta \{\arg g(z)\}| < \infty$, $h \in \mathscr{P}$.

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