ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 3, Summer 1979

IDEMPOTENT ARTINIAN RINGS AND PROJECTIVE COVERS

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ABSTRACT. It is shown that the idempotent, left artinian rings form a collection of (non-abelian) categories in which every object has a unique projective cover. It is also shown that, over a LEFT artinian ring with identity, the projective cover of every finite RIGHT (unitary) module is again finite.

1. **Introduction.** Charles Hopkins [2, 7.1] noted that every left artinian ring A is a sum (not necessarily direct) $A = R + N$ of an *idempotent* one R (i.e., $R = R^2$) and a nilpotent one N. Furthermore, by taking the Peirce decomposition of the idempotent summand R with respect to a suitable idempotent element e, he noted that R could be built by starting with the left artinian ring $U = eRe$ *with identity*. (The details of this building process will be reviewed in §2.)

Our interest in the subject comes from the fact that, by taking a more categorical point of view than was possible in the 1930's, one can continue Hopkins' program, obtaining some interesting structure in an area singularly lacking in modern results : idempotent artinian rings.

Specifically, we show that the class $\mathscr{I}(U)$ of all idempotent left artinian rings R which can be built from any *fixed* U by Hopkins' procedure forms a category in which every element has a unique projective cover. In the process, we show how to construct many examples of rings in $\mathscr{I}(U)$, not a difficult task using module theory.

In a subsequent paper [6], Bittman will determine for which rings U and V the categories $\mathscr{I}(U)$ and $\mathscr{I}(V)$ are equivalent.

Let R be any left artinian ring. Then $R/\text{rad } R$, being semisimple artinian, always has an identity [4]; and since rad R is a nil ideal, that identity can be lifted to an idempotent e of R. Such an idempotent of R is called a *principal idempotent*, and the ring $U = eRe$ is easily seen to be a left artinian ring with identity. We call U a *unitary ring* of R.

If e' is another principal idempotent of R, then the ring $U' = e'Re'$ is isomorphic to U (see [5, 3.1]). Thus the isomorphism class of U is an invariant of R, and we regard U as the first step in building R from rings with identity. We formalize this in:

BASIC DEFINITION. Let U be a left artinian ring with identity. We denote, by $\mathscr{I}(U)$, the category whose objects are all idempotent, left artinian rings with unitary ring $\cong U$. The *morphisms* of $\mathscr{I}(U)$ are defined

Received by the editors on June 13, 1977, and in revised form on September 13, 1977.

to be all ring homorphisms $R \rightarrow S$, between objects of $\mathscr{S}(U)$, which take some unitary ring of R isomorphically onto some unitary ring of S.

We will prove, in §3, that a morphism $R \rightarrow S$ of $\mathscr{S}(U)$ must take *every* unitary ring of R isomorphically onto one S. Thus composition of maps $R \rightarrow S$ and $S \rightarrow T$ is possible, and $\mathscr{S}(U)$ is indeed a category. Moreover, every ring homomorphism: R *onto* S, with R and S in $\mathscr{S}(U)$, turns out to be a morphism of $\mathscr{S}(U)$ (§3). The corresponding result for one-to-one maps is false: For example, some fields can be mapped isomorphically onto proper subfields of themselves.

We will call a ring P in $\mathscr{S}(U)$ *projective* if, whenever f and g are morphisms of $\mathscr{S}(U)$, with f *onto*, there exists a morphism h of $\mathscr{S}(U)$ such that the diagram below commutes.

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ R \arrow[r, two heads, "f"] & S \end{tikzcd}$$

(onto)

A *projective cover* of R [R in $\mathscr{S}(U)$] is a morphism $\varphi : \hat{R}$ onto R, with $\hat{R}$ projective, which is a *minimal epimorphism* ("epimorphism" meaning "onto") in the sense that no subring $X \subset \hat{R}$ (proper inclusion), X in $\mathscr{S}(U)$, satisfies $\varphi(X) = R$.

The main results of this paper (Theorems 4.1 and 4.5) state: *Every ring R in $\mathscr{S}(U)$ has a projective cover $\varphi : \hat{R} \twoheadrightarrow R$. If $f : P \twoheadrightarrow R$ is another projective cover of R, then there is a ring isomorphism $\theta : \hat{R} \rightarrowtail\!\!\!\!\!\twoheadrightarrow P$ such that $\varphi = f\theta$.* Moreover, φ is *absolutely minimal* in the sense that no subring $X \subset \hat{R}$ satisfies $\varphi(X) = R$, *regardless* of whether or not $X \in \mathscr{S}(U)$.

The reason underlying this striking analogy with module theory is that the additive group $(R, +)$ is a direct sum $A \oplus U \oplus B \oplus N$ where A is a *finite* (unitary) right U-module, B is a left U-module of finite (composition) length, and N is a homomorphic image of the group $A \otimes_U B$. This (simple, and rather old) observation is discussed in §2. It provides us with an easy source of examples of rings in $\mathscr{S}(U)$ and with the motivation for the proof of the main theorems: R turns out to be projective in $\mathscr{S}(U)$ if and only if A and B are projective U-modules and $N = A \otimes_U B$ (see §4).

In making the transition from module theory over U to ring theory in $\mathscr{S}(U)$ we require the following fact, proved in the Appendix: *Over a* *left artinian ring U with identity, the projective cover of every finite* RIGHT (*unitary*) *U-module is again finite* (but the result is false for projective covers of finite LEFT U-modules).

Since the motivation for the main results of this paper is categorical, we remark that *being onto or $1 - 1$ are categorical properties of morphisms in $\mathscr{I}(U)$*, that is, they are preserved by category equivalences $\mathscr{I}(U) \to \mathscr{I}(V)$. The proofs of these facts will appear in [**6**].

2. **The PPD and Examples.** Let R be *any ring* which has a principal idempotent e (= an idempotent which becomes the identity element in $R/\text{rad } R$). The *Principal Peirce Decomposition* of R with respect to $e = 1_U$ is the decomposition

$$\text{(PPD)} \qquad (R, +) = \underbrace{(1 - e)Re}_{=A} \oplus \underbrace{eRe}_{=U} \oplus \underbrace{eR(1 - e)}_{=B} \oplus \underbrace{(1 - e)R(1 - e)}_{=N}.$$

Here $1 - e$ is used symbolically only; that is, $(1 - e)x$ means $x - ex$.

Note that A and B are, respectively, right and left *unitary* U-modules, N is a ring (nilpotent when R is artinian, because e becomes the identity modulo rad R), $AB \subseteq N$, and $BA \subseteq U$.

THEOREM 2.1. *Keep the above notation. Then*

(1) *R is left artinian and left noetherian $\Leftrightarrow$ A and N are finite sets and ${}_UU$ and ${}_UB$ are modules with composition series.*

(2) $BA \subseteq \text{rad } U$.

(3) *If R is left artinian, $R = R^2 \Leftrightarrow AB = N$.*

THEOREM 2.2. *Every idempotent left artinian ring is also left noetherian.*

These facts, due mainly to Hopkins and Szele, are assembled in [**5**, §2]. Note that the PPD displayed above, together with Hopkins' observation (3), show that *every idempotent left artinian ring R is built from a ring U with identity and two unitary U-modules : A and B.* We will elaborate on this in 2.5–2.7. First we obtain two applications of 2.1 which will be needed later.

PROPOSITION 2.3. *Let U be a unitary ring of a left artinian, left noetherian ring R. Then any ring S with $U \subseteq S \subseteq R$ is left artinian and left noetherian, and U is a unitary ring of S.*

REMARK. The significance of this proposition is: *If $f : R' \to R$ is a morphism in $\mathscr{I}(U)$, then the subring $f(R')$ of R is an object of $\mathscr{I}(U)$.*

PROOF. Form the PPD of R with respect to $e = 1_U$. Then note that (For details see [**5**, Proposition 2.0])

$$(1) \qquad \text{rad } R = A + \text{rad } U + B + N.$$

We show now: e is a *principal idempotent of* S. Let $(S, +) = A' \oplus U' \oplus B' \oplus N'$ be the Peirce decomposition of S with respect to e (so $A' = (1 - e)Se \subseteq A$, etc.). By hypothesis, $S \supseteq U$, so $U' = eSe \supseteq eUe = U = eRe \supseteq eSe = U'$, so $U = U'$. Thus

(2) $(S, +) = A' \oplus U \oplus B' \oplus N'$ with $A' \subseteq A$, $B' \subseteq B$, and $N' \subseteq N$.

Thus, to see that $e = 1_U$ becomes the identity in $S/\text{rad } S$, it suffices to show that the ideal $I = A' + \text{rad } U + B' + N'$ is contained in rad S. But, by (1), I is contained in the radical of the left artinian ring R and is therefore nilpotent; so $I \subseteq \text{rad } S$ and e is a principal idempotent of S.

Now $U = eSe$ is a unitary ring of S, and so (2) is a PPD of S. To see that S is left artinian and left noetherian, we must show that the terms of (2) have finiteness conditions required by Theorem 2.1. But since R is left artinian and left noetherian, Theorem 2.1 shows that A and N are finite and ${}_UB$ has a composition series. These conditions are clearly inherited by A', B' and N' in (2).

From equation (1) above, we also see that

COROLLARY 2.4. *Let U be a unitary ring of the left artinian left noetherian ring R. Then $R/\text{rad } R \cong U/\text{rad } U$.*

CONSTRUCTION LEMMA 2.5. *Let the following data be given.*

(1) *A left artinian ring U with identity.*

(2) *A finite right U-module A, and a left U-module B of finite composition length (both unitary).*

(3) *A $U - U$ bimodule homomorphism $\rho : {}_U(B \otimes_Z A)_U \to \text{rad } U$.*

Then there is exactly one way of making the group

(4) $$(R, +) = A \oplus U \oplus B \oplus N \qquad (N = A \otimes_U B)$$

into an idempotent left artinian ring whose multiplication extends the ring and module multiplications given in (1) *and* (2), *such that* (4) *is the* PPD *of R with respect to $e = 1_U$, and such that $a \cdot b = a \otimes b$ and $b \cdot a = \rho(b \otimes a)$.*

PROOF. Uniqueness: The requirement that (4) be a PPD forces all "non-matching" products to be zero, that is,

(5) $$O = A^2 = B^2 = UA = BU = AN = NB = UN = NU.$$

Of the remaining 8 types of products, three are given by (1) and (2) and two at the end of the lemma. The remaining three types are:

NA. Recall that $N = A \otimes B = A \cdot B$. The desired product is then

$$(a \cdot b) \cdot a_1 = a \cdot (b \cdot a_1) = a \cdot \rho(b \otimes a).$$

BN is similarly determined. N^2 is determined by

$$(a \cdot b) \cdot (a_1 \cdot b_1) = a \cdot (b \cdot a_1) \cdot b_1 = a \cdot \rho(b \otimes a_1) \cdot b_1.$$

Conversely, a long but straightforward computation shows that the multiplication described above is well-defined and associative, and thus makes $(R, +)$ into a ring. Theorem 2.1 shows that the ring R is idempotent and left artinian, once we verify that $N = A \otimes_U B$ is finite. But ${}_UB$ is finitely generated and *unitary*, say $B = \Sigma_{i=1}^{s} Ub_i$. Since A is finite, so therefore is $N = \Sigma A \otimes b_i$.

EXAMPLES 2.6. The purpose of these examples is to show that $\mathscr{I}(U)$ contains a large number of easily constructible, non-isomorphic rings. Some comments are added to help make the reader comfortable with this unfamiliar class of rings.

NOTATION. Let U be your favorite finite ring (with identity). The examples will be more interesting if rad $U \neq 0$. Let A and B be, respectively, any right and left ideals of U. We now have items (1) and (2) in the Construction Lemma. (If U is infinite, there is the additional complication of choosing A to be finite.) Finally, let p be any element of U such that $BpA \subseteq$ rad U, for example, any element of rad U. Then we have the bimodule homomorphism $\rho(b \otimes a) = bpa$ required by (3).

The Construction Lemma and its proof now describe in detail how to make the additive direct sum

$$(4) \qquad (R, +) = A \oplus U \oplus B \oplus N \qquad (N = A \otimes_U B)$$

into the PPD, with respect to $e = 1_U$, of a ring in $\mathscr{I}(U)$. Given elements $a \in A$ and $b \in B$ it will be important to distinguish their products ab and ba in U from their products $a \cdot b = a \otimes b \in N$ and $b \cdot a = bpa \in$ rad U where, on the left side of the last two products, A and B are considered as terms of (4).

To indicate the number of non-isomorphic rings thus constructed, we note:

> Changing the isomorphism class of any of the additive groups A, B, or BpA changes the isomorphism class of the ring R.

We postpone the proof to item (v) below, and now give some special cases of this construction.

(i) Let A and B equal U itself, hence also $(N, +) = A \otimes_U B = U$. Theorem 4.6 then states that the ring R is projective in $\mathscr{I}(U)$, regardless of choice of p in rad U.

CAUTION. Although $(N, +) = U$ in (i), the *ring* N is *not* isomorphic to the ring U, because N is nilpotent and U has an identity. The product in N is given by $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1(b_1pa_2) \otimes b_2$.

(ii) If we take A or $B \subseteq \operatorname{rad} U$, then it is easy to check (using 2.1) that the subring

$$R' = \begin{bmatrix} U & B \\ A & AB \end{bmatrix}$$

of 2 by 2 matrices over U belongs to $\mathscr{I}(U)$ and

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a principal idempotent of R'.

If we build the ring R in (4) using $p = 1$, then (by Lemma 4.4) there will be a "natural" ring homomorphism of R onto R' which is a morphism in $\mathscr{I}(U)$.

(iii) $N \neq A \otimes_U B$. One of the main theorems (4.6) to be proved states that an R in $\mathscr{I}(U)$ is projective in $\mathscr{I}(U) \Leftrightarrow A_U$ and ${}_UB$ are projective modules and $N = A \otimes_U B$ [more precisely, the multiplication map $A \otimes_U B \to A \cdot B$ (the "N" term of the PPD) is an isomorphism]. So we need an example showing that A_U and ${}_UB$ can be projective without $N = A \otimes_U B$ holding.

Take A_U and ${}_UB$ projective such that $AB \neq 0$ (product in U), hence $A \otimes_U B \neq 0$. Also take $p = 0$. Then the term N in (4) annhihilates R on the left and on the right, so any subgroup N' of $(N, +)$ is an ideal of R. Hence we can form the ring $R' = R/N'$ in $\mathscr{I}(U)$. It has a PPD

$$(R', +) = A \oplus U \oplus B \oplus (N/N').$$

Thus any $N' \neq 0$ will furnish the desired example.

(iv) *Every* COMMUTATIVE *idempotent artinian ring has an identity.* In fact, examination of any PPD will show that commutativity forces $A = 0$ and $B = 0$, hence $N = A \cdot B = 0$, so $R = U$. Thus all nontrivial rings considered in this paper will be strictly noncommutative.

(v) Changing the isomorphism class of any of the additive groups A, B, or BpA changes the isomorphism class of the ring R. First note that $(BpA, +) \cong (B \cdot A, +)$. By analogy with (4), let (4)″ be the PPD of an idempotent left artinian ring constructed from ingredients A'', U'', B'' and p''; and write the multiplication in R'' in the form $x \cdot\cdot\, y$ to distinguish it from the multiplication $x \cdot y$ in R. Suppose there is an isomorphism φ of R'' onto R. Then φ takes the identity e'' of U'' to a principal idempotent of R. It is proved in [5, 3.1] that any two principal idempotents of R are conjugate under some automorphism of R.

Therefore, we can suppose that $\varphi(e'') = e(= 1_U)$. Then φ provides the following additive isomorphisms

$$A = (1 - e) \cdot R \cdot e \cong (1 - e'') \cdot \cdot R'' \cdot \cdot e'' = A''.$$

B is handled similarly, and

$$B \cdot A = (1 - e) \cdot R \cdot e \cdot R \cdot (1 - e)$$

$$\cong (1 - e'') \cdot \cdot R'' \cdot \cdot e'' \cdot \cdot R'' \cdot \cdot (1 - e'') = B'' \cdot \cdot A''.$$

The first sentence of this proof now shows that the additive groups BpA and $B''p''A''$ are isomorphic.

REMARKS 2.7. To see what portion of all rings in $\mathscr{I}(U)$ can be constructed by Lemma 2.5, let R in $\mathscr{I}(U)$ be given, and take a PPD $(R, +) = A \oplus U \oplus B \oplus N$. Then let

$$(R', +) = A \oplus U \oplus B \oplus N' \qquad (N' = A \otimes_U B)$$

be made into a ring by the Construction Lemma, with $\rho(b \otimes a)$ defined to be the product ba in R. Then the map obtained by extending linearly

$$a + u + b + (a_1 \otimes b_1) \longrightarrow a + u + b + a_1 b_1$$

is easily seen to be a ring homomorphism of R' onto R (see 4.4).

3. **U-Preserving Homomorphisms.** PROPOSITION 3.1. *Let $f : R \longrightarrow S$ be a ring homomorphism, where R is left artinian and f is $1 - 1$ on some unitary subring of R. Then* $\ker f \subseteq \operatorname{rad} R$.

PROOF. Let $f(r) = 0$, and suppose f is $1 - 1$ on eRe. Then $f(ere) = 0$ and hence $ere = 0$. But $r - ere$ becomes zero modulo rad R because e becomes the identity there. Hence $r = r - ere \in \operatorname{rad} R$.

THEOREM 3.2. *Let $f : R_1 \longrightarrow R_2$ be a homomorphism of left artinian, left noetherian rings whose unitary ring is $\cong U$. Suppose either*

(1) *f is onto, or*

(2) *f takes some unitary ring of R_1 isomorphically onto one of R_2.*

Then f takes every *unitary ring of R_1 isomorphically onto one of R_2.*

PROOF. Suppose first that f is *onto*. Let e_1 be any principal idempotent of R_1, and $e_2 = f(e_1)$. Since f is onto, $f(\operatorname{rad} R_1) \subseteq \operatorname{rad} R_2$, so f induces a ring homomorphism $\bar{f} : \overline{R}_1 \twoheadrightarrow \overline{R}_2$ where $\overline{R}_i = R_i/\operatorname{rad} R_i$. Since $\overline{R}_1$ and $\overline{R}_2$ are both isomorphic to the semisimple artinian ring $U/\operatorname{rad} U$ (by 2.4) we conclude that $\bar{f}$ is an isomorphism. It follows that *e_2 is a principal idempotent of R_2.*

We show next: *f is $1 - 1$ on e_1Re_1.* By uniqueness of unitary rings,

$$U \cong e_1R_1e_1 \xrightarrow{f} e_2R_2e_2 \cong U.$$

But U has finite composition length on the left (by 2.1), and this forces f to be $1 - 1$ on $e_1R_1e_1$.

Thus $f(e_1R_1e_1) = e_2R_2e_2$, a unitary ring of R_2; so the theorem is proved when f is onto.

Next suppose (2) holds. We first do the case: f is *the inclusion map*: $R_1 \subseteq R_2$. In particular, R_1 has a unitary ring—call it U—which, by (2), is also a unitary ring of R_2. What we want to prove is: *every unitary ring V of R_1 is a unitary ring of R_2.* Since U is a unitary ring of both R_1 and R_2, 1_U is a principal idempotent of these rings and $1_UR_11_U = 1_UR_21_U$ ($= U$). Thus it suffices to find an automorphism of R_2 which takes 1_U to 1_V and takes R_1 to itself.

Let *ring* $(\mathbf{Z} + R_1)$ be the ring with identity whose additive group is $\mathbf{Z} \oplus R_1$, its multiplication being $(z_1 + r_1)(z_2 + r_2) = (z_1z_2) + (z_1r_2 + r_1z_2 + r_1r_2)$. It was shown in [5, 3.1] that, given any two principal idempotents 1_U and 1_V of R_1, there is a unit $(1 + r)$ in ring $(\mathbf{Z} + R_1)$ such that $(1 + r)^{-1}1_U(1 + r) = 1_V$. Since R_1 is a 2-sided ideal of ring $(\mathbf{Z} + R_1)$, conjugation by $1 + r$ takes R_1 to itself. Moreover, $1 + r$ remains a unit in the larger ring, ring $(\mathbf{Z} + R_2)$; so conjugation by $1 + r$ is the desired automorphism of R_2.

General $f : R_1 \to R_2$. Here we are given unitary rings U_1 and V_1 of R_1 such that f takes U_1 isomorphically onto the unitary ring $f(U_1) = U_2$ of R_2. We want to prove that f is $1 - 1$ on V_1, and $V_2 = f(V_1)$ is a unitary ring of R_2. Consider the factorization of f :

$$R_1 \xrightarrow{\quad f \quad} f(R_1) = S \xrightarrow{\text{inclusion}} R_2.$$

Since $f : R_1 \to S$ is onto, S is left artinian and noetherian; and by 2.3 S has the unitary ring $U_2 \cong U_1$. So by the "onto" case of the theorem, f is $1 - 1$ on V_1 and V_2 is a unitary ring of S. Finally by the "inclusion" case of the theorem, V_2 is also a unitary ring of R_2.

REMARKS 3.3. For readers interested in extending these results, we note that Theorem 3.2 holds when R_1 and R_2 are only left artinian (and *not necessarily* noetherian). Similarly 2.3 and 2.4 hold if "noetherian" is deleted from both the hypothesis and conclusion.

For the proof of this more general version, one merely uses the more complicated finiteness conditions in [5, 2.1] in place of Theorem 2.1 of the present paper. In this connection, see also Remark 4.7.

4. **Projective and Protective Covers in $\mathscr{I}(U)$.** THEOREM 4.1 (UNIQUENESS OF THE PROJECTIVE COVER). *Let $f_i : P_i \twoheadrightarrow R (i = 1, 2)$ be projective covers of R in $\mathscr{I}(U)$. Then there is a ring isomorphism $\theta : P_1$ onto P_2 such that $f_1 = f_2\theta$.*

$$\begin{tikzcd} P_1 \arrow[rr, dashed, tail, two heads, "\theta"] \arrow[dr, two heads, "f_1"'] & & P_2 \arrow[dl, two heads, "f_2"] \\ & R & \end{tikzcd} \qquad \begin{tikzcd} P_1 \arrow[dr, two heads, "f_1"'] & & P_2 \arrow[ll, dashed, two heads, "\varphi"'] \arrow[dl, two heads, "f_2"] \\ & R & \end{tikzcd}$$

PROOF. Since P_1 is projective in $\mathscr{I}(U)$ and f_2 is onto, there is a morphism θ in $\mathscr{I}(U)$ such that $f_1 = f_2\theta$. Recall, by 2.2, that rings in $\mathscr{I}(U)$ are left noetherian.

By 2.3, $X = \theta(P_1)$ is a subring in $\mathscr{I}(U)$ of P_2; and $f_2(X) = R$. Minimality of f_2 shows $\theta(P_1) = P_2$. Since P_2 is projective and θ is onto, there is a morphism $\varphi : P_2 \rightarrow P_1$ such that $\theta\varphi =$ identity on P_2. Thus φ is $1 - 1$. Note that the diagram containing φ, f_1, and f_2 commutes. Minimality of f_1 now shows, as with θ, that φ is onto, hence an isomorphism. Therefore $\theta = \varphi^{-1}$ is an isomorphism, too.

We now prepare for the proof that projective covers exist.

"SMALLNESS" LEMMA 4.2. *Let $(R, +) = A \oplus U \oplus B \oplus N$ be the* PPD *of an idempotent left artinian ring R with respect to $e = 1_U$, and let K be a 2-sided ideal of R. Then ${}_RK$ is small in ${}_RR$ if and only if*

(1) ${}_U(K \cap U)$ is small in ${}_UU$; and

(2) ${}_U(K \cap B)$ is small in ${}_UB$.

A similar pair of statements, involving U and A, equivalent to K_R being small in R_R.

PROOF. Note first that since K is 2-sided,

$$(3) \qquad (K, +) = \underbrace{(1-e)Ke}_{=K\cap A} \oplus \underbrace{eKe}_{=K\cap U} \oplus \underbrace{eK(1-e)}_{=K\cap B} \oplus \underbrace{(1-e)K(1-e)}_{=K\cap N}$$

Now suppose ${}_RK$ is small in ${}_RR$. To prove (2), we suppose $B = B' + (K \cap B)$ where B' is a U-submodule of B. Let

$$(4) \quad (X, +) = A + U + (B' + JB) + N \qquad (\text{where } J = \text{rad } U).$$

Next, note that X is a left ideal of R: It suffices to multiple each term on the right of (4) by each of A, U, B, and N. We remark that one of these products requires the hypothesis $R = R^2$:

$$BN = B(AB) = BA(B) \subseteq JB \qquad (\text{Use } 2.1).$$

Finally, note that $X + K = R$ because of (3). Hence smallnes of ${}_RK$ in R shows that $X = R$, so $B' + JB = B$. We can now use either nilpotence of J, or finite generation of ${}_UB$ (see 2.1) together with Nakayama's Lemma, to conclude $B' = B$ as desired.

To prove (1) we use a similar computation, with $U' + (K \cap U) = U$ and

$$X = A + (U' + J) + B + N.$$

Now suppose (1) and (2) hold, and let X be a left ideal of R such that

$$R = K + X.$$

Left multiply by e.

$$eK + eX = eR = U + B.$$

Also, by (3), $eK = (K \cap U) + (K \cap B)$. So, by (1) and (2), eK is small in the left U-module $U \oplus B$. Hence $eX = U + B$; in particular ${}_RX$ contains both U and B.

Hence X contains $AU = A$ and also $AB = N$; so $X = R$.

PROPOSITION 4.3. *The following assertions about an epimorphism* $f : R \twoheadrightarrow S$ *in* $\mathscr{I}(U)$ *are equivalent.*

(1) *f is ABSOLUTELY MINIMAL, that is, if X is a subring of R and $X \neq R$, then $f(X) \neq S$ whether or not $X \in \mathscr{I}(U)$.*

(2) ker *f is a small left ideal of R, and also a small right ideal.*

PROOF. (1) $\Rightarrow$ (2). We show that ker f is a small *left* ideal of R by verifying conditions (1) and (2) of the Smallness Lemma. A symmetric argument then shows ker f is small on the right, too.

So suppose ${}_UB' \subseteq B$ and $B' + (\ker f \cap B) = B$. Let

$$R' = A + U + (B' + JB) + N \qquad (J = \operatorname{rad} U).$$

Then R' is a subring of R because $B'N \subseteq BAB \subseteq JB$; and $f(R') = f(R)$ because $f(B') = f(B)$. So, by (1), $R' = R$; therefore $B' + JB = B$. Nilpotent of J or finite generation of ${}_UB$ now implies $B' = B$, so (2) of the Smallness Lemma holds.

For (1) of the Smallness Lemma recall that, by 3.2, f is one-to-one on U; so $(\ker f) \cap U = 0$. By the Smallness Lemma, ker f is a small left ideal of R.

(2) $\Rightarrow$ (1). Suppose R' is a subring of R such that $f(R') = S$. Let $\bar{e}$ be a principal idempotent of S. Since, by 3.1, ker $f \subseteq$ rad R, a nilpotent ideal, $\bar{e}$ can be lifted to an idempotent e in R'.

Of course, e is also an idempotent of R; and the choice of $\bar{e}$, together with the fact that ker $f \subseteq$ rad R, shows that e is a principal idempotent of R.

We claim $eR'e = eRe$. By hypothesis $f(eR'e) = f(eRe)$. But since f is a morphism in $\mathscr{I}(U)$, f is one-to-one on the unitary ring eRe (by 2.2 and 3.2), proving the claim.

Now we form a PPD of R and apply the Smallness Lemma to it. Let $U = eRe$, and form the PPD $R = A + U + B + N$ with respect to $e = 1_U$. By hypothesis, ker f is a small left ideal of R, so by (2) of the Smallness Lemma, (ker f) $\cap$ B is a small U-submobile of B. But $f(R') = f(R)$ shows $f(eR'(1 - e)) = f(B)$, so smallness of (ker f) $\cap$ B shows $eR'(1 - e) = B$.

Similarly, the right-handed version of the Smallness Lemma shows $(1 - e)R'e = A$. Finally, idempotence of R shows.

$$\begin{aligned}(1 - e)R(1 - e) &= N = AB \\ &= (1 - e)R'eeR'(1 - e) \subseteq (1 - e)R'(1 - e)\end{aligned}$$

and since the opposite inclusion is trivial, we are done.

Map-Building Lemma 4.4 *Let* $(R_i, +) = A_i \oplus U \oplus B_i \oplus N_i$ $(i = 1, 2)$ *be* PPD's, *with each* $R_i \in \mathscr{I}(U)$, *and let* $\varphi : R_1 \rightarrow R_2$ *be an additive homomorphism which takes* $A_1 \rightarrow A_2$, $U \rightarrow U$, $B_1 \rightarrow B_2$, $N_1 \rightarrow N_2$. *Suppose further that*

(1) φ *is the identity on* U,

(2) $\varphi : A_1 \rightarrow A_2$ *and* $\varphi : B_1 \rightarrow B_2$ *are respectively homomorphisms of right and left U-modules, and*

(3) $\varphi(a_1b_1) = (\varphi a_1)(\varphi b_1)$ *and* $\varphi(b_1a_1) = (\varphi b_1)(\varphi a_1)$ *for* $a_1 \in A_1$ *and* $b_1 \in B_1$.

Then φ *is a ring homomorphism.*

Proof. It suffices to check that $\varphi(xy) = (\varphi x)(\varphi y)$ for x and y in each of A_1, U, B_1, N_1. Of these 16 types of products, 5 behave properly because of (1), (2), and (3), and 8 are always zero (see (5) of 2.5). The remaining 3 types: NA, BN, and NN are easily checked by using $N = AB$ (see 2.1).

Second Main Theorem 4.5. *Every* R *in* $\mathscr{I}(U)$ *has a projective cover* $\varphi : \hat{R} \twoheadrightarrow R$, *and any such* φ *is absolutely minimal.*

Proof. Choose a principal idempotent e of R and let

$$(R, +) = A \oplus U \oplus B \oplus N \tag{1}$$

be the PPD of R with respect to e. We build φ separately for each term in (1).

Over a left artinian ring with identity, every left *and* every right unitary module has a projective cover [1, Theorem P]. Let $\varphi : \hat{A} \twoheadrightarrow A$ and $\varphi : \hat{B} \twoheadrightarrow B$ be projective covers of, respectively, the right and left U-modules A and B. Set

$$(\hat{R}, +) = \hat{A} \oplus U \oplus \hat{B} \oplus (\hat{A} \otimes_U \hat{B}) \tag{2}$$

and extend φ to an additive map: $\hat{R} \twoheadrightarrow R$ by $\varphi =$ identity on U, and $\varphi(\hat{a} \otimes \hat{b}) = (\varphi\hat{a})(\varphi\hat{b}) \in AB = N$.

We make $\hat{R}$ into a ring in $\mathscr{I}(U)$ by means of the Construction Lemma 2.5, then verify that φ is the required projective cover of the *ring* R.

The most sensitive item needed in the hypotheses of the Construction Lemma is that $\hat{A}$ is finite. This follows from the fact: *over a LEFT artinian ring* ($= U$) *with identity, the projective cover of every finite RIGHT module is finite.* This is proved in the Appendix, Theorem A3. Since $\hat{B} \twoheadrightarrow B$ is a projective cover of a finitely generated U-module, $\hat{B}$ is finitely generated; so since U is left artinian, ${}_U\hat{B}$ has a composition series.

Finally define the U-U bilinear map $\rho : \hat{B} \otimes_Z \hat{A} \to \operatorname{rad} U$ by $\rho(\hat{b} \oplus \hat{a}) = (\varphi\hat{b})(\varphi\hat{a})$.

Using the Construction Lemma, we can now make $\hat{R}$ into a ring in $\mathscr{I}(U)$; and by the Map Building Lemma, φ will be a ring homomorphism as soon as we verify that $\varphi(\hat{a}\hat{b}) = (\varphi\hat{a})(\varphi\hat{b})$ and $\varphi(\hat{b}\hat{a}) = (\varphi\hat{b})(\varphi\hat{a})$. The first of these is clear since $\hat{a}\hat{b} = \hat{a} \otimes \hat{b}$. For the second of these, note first that $\hat{b}\hat{a} \in U$ where φ equals the identity. Then

$$\begin{aligned} \hat{b}\hat{a} &= \rho(\hat{b} \otimes \hat{a}) && \text{(definition of } \hat{b}\hat{a} \text{ in 2.5)} \\ &= (\varphi\hat{b})(\varphi\hat{a}) && \text{(definition of } \rho). \end{aligned}$$

By construction, φ is onto. To check that φ is absolutely minimal we have to verify (by 4.3) that $\ker \varphi$ is small in $\hat{R}$, both as a left ideal and as a right ideal; and by the Smallness Lemma (4.2) these facts follow from the facts that $\varphi : \hat{B} \twoheadrightarrow B$ and $\varphi : \hat{A} \twoheadrightarrow A$ are projective covers of ${}_UB$ and A_U and φ equals the identity on U.

It remains only to check that $\hat{R}$ is a projective object of $\mathscr{I}(U)$, and this follows the next theorem.

THEOREM 4.6. *Let* $(R, +) = A \oplus U \oplus B \oplus N$ *be a* PPD *of* $R \in \mathscr{I}(U)$. *Then* R *is a projective object of* $\mathscr{I}(U)$ *if and only if*

(1) A_U *and* ${}_UB$ *are projective modules, and*

(2) *The multiplication map* $\mu : A \otimes_U B \twoheadrightarrow AB = N$ *is* $1 - 1$.

PROOF. Suppose (1) and (2) hold, and let morphisms f and g in $\mathscr{I}(U)$

be given, with f onto. We seek φ in $\mathscr{I}(U)$ such that the following diagram commutes.

$$\begin{tikzcd} & R \arrow[dl, dashed, "\varphi"'] \arrow[d, "g"] \\ X_2 \arrow[r, two heads, "f"] & X_1 \end{tikzcd}$$

Since g is a morphism in $\mathscr{I}(U)$, g takes the unitary ring U of R isomorphically onto a unitary ring of X_1. We identify U and $g(U)$, thus making g the identity on U. Since $\ker f \subseteq \operatorname{rad} X_2$ (by 3.1) $e = 1_U$ can be lifted to an idempotent e' of X_2; and since the homomorphism $X_1/\operatorname{rad} X_1 \twoheadrightarrow X_2/\operatorname{rad} X_2$ induced by f is an isomorphism (see the beginning of the proof of 3.2), e' is a principal idempotent of X_2. Therefore f carries the unitary ring $e'X_2e'$ of X_2 isomorphically onto the unitary ring $eX_1e = U$ of X_1. We identify $e'X_2e'$ and U, so that f becomes the identity on U.

We can now take the PPD of each X_i with respect to $e = 1_U$:

$$X_i = A_i \oplus U \oplus B_i \oplus N_i \qquad (i = 1, 2) \tag{3}$$

and f takes $A_2 = (1 - e)X_2e$ onto A_1, B_2 onto B_1, and N_2 onto N_1. Also, since f equals the identity on U and is a ring homomorphism, $f : A_2 \twoheadrightarrow A_1$ and $f : B_2 \twoheadrightarrow B_1$ are maps of right and left U-modules, respectively. Similar "compatibility" remarks apply to the way g takes the original PPD of R to that of X_1.

We can now use projectivity of A_U and ${}_UB$ to find U-linear maps $\varphi : A \longrightarrow A_2$ and $\varphi : B \longrightarrow B_2$ such that $f\varphi = g$ on A and on B. To extend φ to an additive map: $R \longrightarrow X_2$, set $\varphi =$ the identity on U. Then identify N with $A \otimes_U B$ (hypothesis (2)) and set $\varphi(a \otimes b) = (\varphi a)\cdot(\varphi b) \in A_2B_2 = N_2$.

To see that φ is a ring homomorphism, we invoke the Map-Building Lemma 4.4. The only hypothesis of that Lemma remaining to be verified is that $\varphi(ba) = (\varphi b)(\varphi a)$. But since both sides belong to U where f, g, and φ equal the identity, and since f and g are ring homomorphisms, we get

$$\begin{aligned} \varphi(ba) &= f\varphi(ba) = g(ba) = (gb)(ga) \\ &= (f\varphi b)(f\varphi a) = f[(\varphi b)(\varphi a)] = (\varphi b)(\varphi a) \end{aligned}$$

and this completes the proof that R is projective in $\mathscr{I}(U)$.

Conversely, suppose that R is projective in $\mathscr{I}(U)$. Choose a PPD of R and construct the projective cover $\varphi : \hat{R} \twoheadrightarrow R$ used in the proof of Theorem 4.5. Since R is projective, the identity map $R \longrightarrow R$ is also a projective cover. Hence, by Uniqueness of the Projective Cover (4.1), φ is an isomorphism; and this establishes the assertions of (1) and (2), all at once!

REMARKS 4.7. In proving that R was projective provided it satisfies (1) and (2) of 4.6, we never used idempotence of X_1 and X_2. In fact, use of [5, 2.1] in place of Theorem 2.1 of the present paper shows: *if $R \in \mathscr{I}(U)$ satisifies* (1) *and* (2) *of* 4.6, *then R is a projective object in the category of all artinian rings with unitary ring $\cong U$.* We don't know if R being projective in this larger category forces R to be in $\mathscr{I}(U)$.

Appendix. Finite-Infinite Decompositions. We will say that a division ring Δ is *associated* with a left artinain ring U if Δ is isomorphic to one of the division rings which occur in the decomposition of $\overline{U} = U/\operatorname{rad} U$ into a direct sum of full matrix rings over division rings.

All rings considered in this appendix will have identity elements, and all modules will be unitary.

FINITE-INFINITE DECOMPOSITION LEMMA A1. *Every left artinain ring U (with identity) has orthogonal idempotents i and f such that $i + f = 1$, and an "upper triangular" decomposition*

$$(U, +) = \underbrace{iUi}_{=I} \oplus \underbrace{iUf}_{=M} \oplus \underbrace{fUf}_{=F} \qquad \text{with } fUi = 0 \tag{1}$$

in which

(2) *F is a finite ring, and*

(3) *${}_II$ and ${}_IM$ have finite (composition) length, and the ring I has no finite associated division rings (hence no finite left or right nonzero modules).*

If $M \neq 0$, then U is neither right artinian nor right noetherian.

PROOF. If i and f are any orthogonal idempotents such that $i + f = 1$, then $(U, +) = I \oplus M \oplus F \oplus fUi$. Furthermore: ${}_II$, ${}_IM$, ${}_FF$ and ${}_F(fUi)$ all have finite composition length: For example, to see that ${}_IM$ has finite composition length, merely note that if M' is an I-submodule of M, then the group $M' \oplus (fUi)M'$ is a left ideal of U.

The semisimple artinian ring $\overline{U} = U/\text{rad } U$ can be written $\overline{U} = \overline{I} \oplus \overline{F}$ where $\overline{I}$ is a direct sum of matrix rings over infinite division rings and $\overline{F}$ is a finite ring. Let the corressponding decomposition of the identity element of $\overline{U}$ be $\overline{1} = \overline{i} + \overline{f}$ and lift $\overline{i}$ and $\overline{f}$ to orthognal idempotents i and f of U. Then $1 = i + f$.

To establish (2), recall that ${}_FF$ has finite composition length. So it will suffice to show that all the composition factors of ${}_FF$ are finite.

But these are all simple modules over the finite ring $\overline{F} \cong F/\text{rad } F$, hence they are finite.

Similarly, *no* division ring associated with I is finite. If I had a finite nonzero left or right module, then it would have a finite simple left or right module. Thus $I/\text{rad } I$ would have a finite simple module, contrary to the fact that the division rings associated with I are all infinite. Thus (3) is proved.

Since ${}_F(fUi)$ has finite length and F is finite, fUi is finite. And since fUi is a right I-module, its finiteness forces $fUi = 0$, proving (1).

The supplementary statement is true since F is finite, M is infinite (because it is a left I-module) and every right F-submodule of M is a right ideal of U.

EXAMPLE A2. Given I, ${}_IM_F$, and F satisfying (2) and (3) of the lemma, the ring $U = \left[\begin{smallmatrix} I & M \\ 0 & F \end{smallmatrix}\right]$ will produce the decomposition (1). For a specific example, let I be any infinite field of characteristic $\neq 0$, F the prime field of I, and $M = I$.

THEOREM A3. *Over a LEFT artinian ring U, the projective cover of every finite RIGHT module (exists and) is finite.*

PROOF. Existence is part of Bass's Theorem P [1]. Note also that if $\varphi : P \twoheadrightarrow A$ is a projective cover of a finitely generated module A (over any ring) then minimality of the epimorphism φ shows that P is finitely generated, too.

Now let $\varphi : P \twoheadrightarrow A$ be a projective cover of a finite right U-module A, and choose i and f according to the lemma above. Then Ai is a finite right I-module, so (3) of the lemma shows $0 = Ai = A(1 - f)$. Hence $A = Af$. But then, $\varphi(Pf) = Af = A$; and since Pf is a U-submodule of P (see (1) of the lemma), minimality of $\varphi : P \twoheadrightarrow A$ shows $Pf = P$.

Finally, since P is a finitely generated right U-module, $Pf = P$ is a finitely generated module over the finite ring F, and hence is finite.

We now show that projective covers of finite LEFT U-modules need not be finite.

PROPOSITION A4. *Let U be a left artinian ring such that, in the notation of Lemma A1, $M \neq 0$. Then U has a finite left module B whose projective cover is infinite.* (Such rings U exist by Example A2.)

PROOF. The natural homomorphism $\varphi : Uf \twoheadrightarrow B = Uf/Jf$, where $J = \text{rad } U$, is a projective cover of ${}_UB$. Note that $iUf \subseteq J$ because iUf is a left ideal of U whose square is zero. Hence $iUf = iJf$. Now finiteness of B follows from:

$$(B, +) = \frac{iUf}{iJf} \oplus \frac{fUf}{fJf} = 0 \oplus \overline{F} = \text{finite}.$$

On the other hand, $(Uf, +) = M \oplus F$. Since M is a nonzero left I-module, M must be infinite. Hence Uf is infinite, too.

Finally, we mention that the above machinery contains a proof of the following (probably known) result.

COROLLARY A5. *Let U be an indecomposable left and right artinian ring. Then the division rings associated with U are either all finite or else all infinite.*

PROOF. (In the notation of Lemma A1:) U being *right* artinian forces $M = 0$. But then indecomposability of U (as a ring) forces $U = I$ or $U = F$.

REMARK A6. At the suggestion of the referees, we note that the results of this appendix can be extended by letting U be a left artinian algebra over a commutative ring K: Replace the phrase "is finite" by "has finite length as a K-module" whenever it appears in a theorem or its proof. Thus, in Lemma A1, statement (2) becomes, "F is a ring of finite K-length," and the last assertion of (3) becomes, "I has no associated division algebras of finite K-length (hence no nonzero left or right modules of finite K-length)."

The following version of A5 then might be of interest:

Let U be an indecomposable left and right artinian algebra over a field K. Then the division algebras associated with U are either all finite dimensional or all infinite dimensional.

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