ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 4, Fall 1979

A NOTE ON MODULUS OF APPROXIMATE CONTINUITY ON $R(X)$

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1. Let X be a compact subset of the plane. We denote by $R(X)$ the uniform closure of $R_0(X)$, the set of rational functions having no poles on X. We say that ϕ is an *admissible function* if (a) ϕ is a positive, non-decreasing function defined on $(0, \infty)$ and (b) $\psi(r) = r/\phi(r)$ is also non-decreasing and $\lim_{r\to 0^+} r/\phi(r) = 0$.

Fix $x \in X$. Suppose ϕ is an admissible function and $\phi(0^+) = 0$. We say that the unit ball of $R(X)$ admits ϕ as a *modulus of approximate continuity at* x if

$$|f(y) - f(x)| \leqq \phi(|y - x|) \text{ for all } f \in R(X), \ \|f\| \leqq 1$$

and all y in a subset having full area density at x. Some properties concerning the modulus of approximate continuity have been investigated in [**5**] and [**6**]. It is known, for instance, at a non-peak point x, there exists an admissible function ϕ with $\phi(0^+) = 0$ such that the unit ball of $R(X)$ admits $\epsilon\phi$ as a modulus of approximate continuity at x, for every $\epsilon > 0$.

One can define a fractional order bounded point derivation in terms of representing measure, analytic capacity and modulus of approximate continuity respectively. However, it turns out that the definitions are not equivalent (see [**6**]).

Although the existence of modulus of approximate continuity at a point is in general a weaker condition than some other properties, we will show that it does imply that X has more than full area density at that point (Corollary 3).

Let E be a bounded plane set and denote by $H(E)$ the set of functions holomorphic off a compact subset of E, bounded in modulus by one, which vanish at ∞. The analytic capacity of E is $\gamma(E) = \sup\{|f'(\infty)| : f \in H(E)\}$.

In [**6**], it was conjectured that the convergence of a "generalized Melnikov's series" implies the unit ball of $R(X)$ admits ϕ as a modulus of approximate continuity at a point. We are unable to prove this. Using a well known localization procedure and Melnikov's estimate for Cauchy integrals [**2**], however, we can get a weaker result (Theorem 4). Hayashi [**3**] has obtained a similar result independently when he considered the case of the first order bounded point derivations.

Received by the editors on September 13, 1977, and in revised form on December 19, 1977.

2. For $x, y \in X$, the Gleason distance is

$$||y - x|| = \sup\{|f(y) - f(x)| : f \in R(X), ||f|| \leqq 1\}.$$

It is clear that the unit ball of $R(X)$ admits ϕ as a modulus of approximate continuity at x if and only if $||y - x|| \leqq \phi(|y - x|)$ for all y in a subset having full area density at x.

Let ϕ be an admissible function. We denote by $\Delta(x; r)$ this disk $\{|z - x| < r\}$, $A_n(x)$ the annulus $\{2^{-n-1} \leqq |z - x| \leqq 2^{-n}\}$, $d(z, E)$ the distance from z to a set E and $\operatorname{int} X$ the interior of X.

The following lemma is due to Curtis [1].

LEMMA 1. *If $x, y \in X$, then*

$$||y - x|| \geqq \frac{\gamma(\Delta(x; r)\setminus X)}{r + \gamma(\Delta(x; r)\setminus X)} - \frac{\gamma(\Delta(x; r)\setminus X)}{d(y, \Delta(x; r)\setminus X)}$$

for every $r > 0$ such that y has positive distance to $\Delta(x; r)\setminus X$.

PROOF. It is clear that for every such r and every $g \in H(\Delta(x; r)\setminus X)$ we have

$$\begin{aligned} ||y - x|| &\geqq |g(x)| - |g(y)| \\ &\geqq |g(x)| - \gamma(\Delta(x; r)\setminus X)/d(y, \Delta(x; r)\setminus X). \end{aligned}$$

Taking g to be $(r + f'(\infty))^{-1}[(x - z)\ f(z) + f'(\infty)]$ where f varies in $H(\Delta(x; r)\setminus X)$, we obtain the desired estimate.

THEOREM 2. *If for every $\epsilon > 0$, $||y - x|| \leqq \epsilon\phi\,(|y - x|)$ for all y in a subset having full area density at x, then $\lim_{r\to 0}\gamma(\Delta(x; r)\setminus X)\,/\,(r\phi(r)) = 0$.*

PROOF. Suppose there exists a $C > 0$ and $r_n \downarrow 0$ so that $\gamma(\Delta(x; r_n)\setminus X) > C\, r_n\phi(r_n)$. By Lemma 2, we have, for every $y \in X$,

$$\begin{aligned} ||y - x|| &> Cr_n\phi(r_n)\ [(r_n + \gamma(\Delta(x; r_n)\setminus X))^{-1} \\ &\quad - d(y, \Delta(x; r_n)\setminus X)^{-1}] \\ &= C\,\phi(r_n)\ [(1 + \gamma(\Delta(x; r_n)\setminus X)/r_n)^{-1} - \\ &\quad r_n\, d\,(y, \Delta(x; r_n)\setminus X)^{-1}] \\ &\geqq C\,\phi(r_n)\ [1/2 - r_n\, d\,(y, \Delta(x; r_n)\setminus X)^{-1}] \end{aligned}$$

for every r_n such that $d(y, \Delta(x; r_n)\setminus X) > 0$. Let $E_n = X \cap [\Delta(x; 5r_n)\setminus\Delta(x; 4r_n)]$. The $\cup E_n$ has positive upper area density at x. If $y_n \in E_n$, then $d(y_n, \Delta(x; r_n)\setminus X) > 3r_n$ and $\phi(|y_n - x|) \leqq 5\phi(r_n)$, and thus we obtain $||y_n - x|| > 1/6\, C\,\phi(r_n) \geqq 1/30\, C\,\phi(|y_n - x|)$. Therefore the inequality $||y - x|| \leqq 1/30\ C\ \phi(|y - x|)$ does not hold for all y in a subset having full area density at x.

COROLLARY 3. *If the unit ball of $R(X)$ admits $\epsilon\phi$ as a modulus of approximate continuity at x for every $\epsilon > 0$, then*

$$m(\Delta(x; r)\setminus X) = o(r^2\phi(r)^2)$$

where m is the plane Lebesgue measure.

PROOF. Note that $m(E) \leqq 4\pi\gamma(E)^2$ (e.g., [2], theorem VIII.3.2).

THEOREM 4. *Suppose* $\Sigma 2^n\phi(2^{-n})^{-1}\gamma(A_n(x)\setminus X) < \infty$. *Then* $||y - x|| \leqq K_1\ \phi(|y - x|)$ *if*

$$\sum_{N(y)}^{\infty} 2^n\ \gamma(A_n(y)\setminus X)/\phi(|y - x|) \leqq K_2$$

where $N(y)$ is a positive integer depending only on the Euclidean distance $|y - x|$, K_1 is a constant depending on K_2.

PROOF. Throughout the proof $C_1, C_2, \cdots$ are universal constants, and $\gamma_n(z)$ is $\gamma(A_n(z)\setminus X)$.

Let $f \in R_0(X)$, $||f|| \leqq 1$. We can choose some neighborhood U of X such that $||f||_U \leqq 2||f||_X$ and define $g(z) = [f(z) - f(y)]/(z - y)$ when $z \in U$ and $g \equiv 0$ outside U. By the Cauchy integral formula,

$$g(x) = \frac{1}{2\pi i}\left[\int_{|\zeta - x| = 1}\frac{g\,d\zeta}{\zeta - x} - \sum_{0}^{M}\int_{bA_n(x)}\frac{g\,d\zeta}{\zeta - x}\right]$$

for some large M.

Let $y \in A_k\ (x)$ and lt $\tilde{A}_k(x) = A_{k-1}(x) \cup A_k(x) \cup A_{k+1}(x)$. We have

$$\begin{aligned}
|f(x) - f(y)| \leqq |x - y| \Bigg[& ||g||_{|y-x|=1} \\
& + C_1 \sum_{|n-k|\geqq 2} 2^n\gamma_n(x)||g||_{A_n(x)} \\
& + \frac{1}{2\pi}\left|\int_{b\tilde{A}_k(x)}\frac{g\,d\zeta}{\zeta - x}\right|\Bigg] \\
\leqq \phi(|x - y|)\ ||f|| \Bigg[& C_2\ \psi(|y - x|) \\
& + C_3 \sum_{|n-k|\geqq 2} 2^n\ \phi(2^{-n})^{-1}\gamma_n(x) \\
& + \frac{|x - y|}{2\pi}\left|\int_{b\tilde{A}_k(x)}\frac{g\,d\zeta}{\zeta - x}\right|\Bigg].
\end{aligned}$$

Choose $h \in C_0^\infty\ (\tilde{A}_k(x))$ such that $0 \leqq h \leqq 1$, $h \equiv 1$ on $\Delta(y; \sigma/2\,|x-y|)$, $h \equiv 0$ off $\Delta(y; \sigma|x-y|)$ and $|\text{grad}\,\zeta| \leqq C_4/(\sigma|x-y|)$ where $0 < \sigma < 1/2$. We write

$$G(\zeta) = \frac{1}{\pi} \int \int \frac{g(w) - g(\zeta)}{w - \zeta} \frac{\partial h}{\partial \overline{w}}\, dudv,$$

where $w = u + iv$. Then G is holomorphic wherever g is and off $\Delta(y; \sigma|x-y|)$ and $G - g$ is holomorphic wherever g is and in $\Delta(y; \sigma/2\,|x-y|)$. By Cauchy's theorem we have

$$|x-y| \left| \int_{b\tilde{A}_k(x)} \frac{g d\zeta}{\zeta - x} \right| \leqq |x-y| \left| \int_{b\tilde{A}_k(x)} \frac{(g-G)\, d\zeta}{\zeta - x} \right| + |x-y| \left| \int_{b\Delta(y;\sigma|x-y|)} \frac{G d\zeta}{\zeta - x} \right| = |I_1| + |I_2|.$$

By the maximum modulus principle,

$$|I_1| \leqq C_5\, |x-y| \left(\sum_{n=k-1}^{k+1} 2^n\, \gamma_n(x) \right) ||g - G||_{A_k(x)}$$

$$\leqq C_5\, |x-y| \left(\sum_{n=k-1}^{k+1} 2^n\, \gamma_n(x) \right) ||g - G||_{b\Delta(y;(\sigma/2)|x-y|)}$$

$$\leqq \frac{C_6}{\sigma}\, \phi(|y-x|)\, ||f|| \left(\sum_{n=k-1}^{k+1} 2^n\, \phi(2^{-n})^{-1}\, \gamma_n(x) \right).$$

let $N(y)$ be the integer so that

$$2^{-N-1} \leqq \sigma\, |x-y| < 2^{-N}. \text{ Then } |I_2| \leqq |x-y| \cdot \left| \sum_{N(y)}^{\infty} \int_{bA_n(y)} \frac{G d\zeta}{\zeta - x} \right|$$

$$\leqq C_7 \sum_{n=N(y)}^{\infty} ||G||_{A_n(y)}\, \gamma_n(y)$$

$$\leqq C_8\, \phi(|y-x|)\, ||f|| \left(\sum_{n=N(y)}^{\infty} 2^n \gamma_n(y))/\phi(|y-x| \right).$$

Hence,

$$|f(x) - f(y)| \leqq \phi(|y-x|)\, ||f||\, [C_2\, \psi(|y-x|)$$

$$+ \; C_3 \sum_{|n-k| \geqq 2} 2^n \, \phi(2^{-n})^{-1} \, \gamma_n(x)$$

$$+ \frac{C_6}{\sigma} \sum_{n=k-1}^{k+1} 2^n \, \phi(2^{-n})^{-1} \, \gamma_n(x)$$

$$+ \; C_8 \sum_{m=N(y)}^{\infty} 2^n \, \gamma_n(y) \, / \, \phi(|y - x|)].$$

By hypothesis, the first three terms inside the square brackets are bounded and the last is dominated by a constant multiple of K_2, hence we can find a constant $K_1 > 0$ such that $||y - x|| \leqq K_1 \, \phi \, (|y - x|)$ and the theorem is proved.

We remark that for fixed σ, $0 < \sigma < 1/2$ we can take K_1 small if both $|y - x|$ and K_2 are small. This generalizes some results on non-tangential limits found by O'Farrell ([4], Theorem 1 and Corollary 1). Suppose x satisfies a "cone condition" at x, that is, there exists $r_0 > 0$ and an open interval I such that the sector $\{y : 0 < |y - x| < r_0, \arg(y - x) \in I\}$ is contained in $\operatorname{int} X$. Let J be a closed interval contained in I, and put $C_\delta = \{y : 0 < |y - x| \leqq \delta, \arg(y - x) \in J\}$.

COROLLARY 5. *Suppose* $\Sigma \, 2^n \, \phi(2^{-n})^{-1} \, \gamma(A_n(x) \setminus X) < \infty$. *Then for every* $\epsilon > 0$, $||y - x|| \leqq \epsilon\phi(|y - x|)$ *for all* y *in* C_δ *when* $\delta > 0$ *is sufficiently small.*

PROOF. For a suitable choice of σ, K_2 can be taken zero for all y in C_δ.

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