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EXACT SOLUTION OF COUPLED PAIRS OF DUAL TRIGONOMETRIC SERIES

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ABSTRACT. Exact solutions are obtained for coupled dual trigonometric series that arise in the study of contact problem of a circular inclusion as well as a set of symmetric curvilinear cracks. The coupled dual series are reduced to coupled integral equations. Simple identities of the kernel functions allow us to decouple these integral equations into two uncoupled singular integral equations. One of these integral equations has a Cauchy type of singularity and can be reduced to the air-foil equation. The other has a logarithmic singularity and is reduced to two Volterra equations.

1. **Introduction.** Dual series equations arise frequently for the solution of mixed boundary value problems in mathematical physics. A comprehensive collection of potential problems with their formal solution is given in Sneddon's monograph [1].

In this paper we derive coupled dual series for an elasticity problem of a circular inclusion, with interface friction, via bipotential Airy-stress functions. Consider a circular insert in a two dimensional infinite medium under uniform tension as shown in Figure 1. Due to lack of bond at interface there will be two distinct sets of regions, namely regions of contact and regions of separations. Mathematically, boundary conditions lead to coupled dual series. Over the regions of contact, stresses and displacements are continuous and over the separated regions stresses vanish. In such a problem not only the contact stresses are unknown but also the regions of receding contact.

In this paper we obtain an exact solution of the coupled dual series. The analysis is formal. The existence and uniqueness of dual and triple series, in a rigorous fashion, have only been recently studied by Kelman [2], [3]. There are, at present, no theorems available for coupled dual series formally solved in the present paper.

Employing the usual notation [4], the boundary conditions for the problem are:

$$u_r = u_r', \; u_\theta = u_\theta', \; \sigma_r = \sigma_r', \; \tau_{r\theta} = \tau'_{r\theta} \text{ for } 0 \leqq \theta < \eta \tag{1}$$

$$\sigma_r = \sigma_r' = 0, \; \tau_{r\theta} = \tau'_{r\theta} = 0 \quad \text{for } \eta < \theta \leqq \pi/2. \tag{2}$$

The primed quantities refer to the insert and the angle η is the unknown angle of receding contact. The general solution of two dimen-

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sional equations of elasticity is well known [4]. The stresses are given in terms of a biharmonic function φ as follows:

$$\sigma_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}, \qquad \sigma_\theta = \frac{\partial^2\varphi}{\partial r^2},$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right).$$

In selecting φ and φ' for the exterior body and the insert we are guided by symmetry, single valuedness of the displacements and regularity condition at infinity, (i.e., $\sigma_y = 1$):

$$\varphi = A_0 \log r + \frac{1}{4}r^2 + \frac{1}{4}r^2\cos 2\theta$$

$$+ \sum_{2,4,\cdots}^{\infty} (A_n r^{-n} + B_n r^{-n+2}) \cos n\theta,$$

$$\varphi' = \frac{1}{2}D_0 r^2 + \sum_{2,4,\cdots}^{\infty} (C_n r^n + D_n r^{n+2}) \cos n\theta,$$

where $A_n - D_n$, $n = 0, 2, \cdots$ are unknown constants. Using $\sigma_r = \sigma_r'$, $\tau_{r\theta} = \tau'_{r\theta}$, (see (1), (2)) we have:

$$D_0 = A_0 + \frac{1}{2}, \quad C_2 = \frac{1}{4} + 3A_2 + 2B_2, \quad D_2 = -2A_2 - B_2$$

$$\left.\begin{aligned} C_n &= (n+1)A_n + nB_n \\ D_n &= -nA_n - (n-1)B_n \end{aligned}\right\} n \geqq 4.$$

On introducing new sets of constants E_n and F_n for convenience,

$$E_0 = 1 + 2A_0, \quad E_2 = \frac{-1}{2} - 6A_2 - 4B_2, \quad F_2 = \frac{1}{2} - 6A_2 - 2B_2,$$

$$\left.\begin{aligned} E_n &= -n(n+1)A_n - (n-1)(n+2)B_n \\ F_n &= -n(n+1)A_n - n(n-1)B_n \end{aligned}\right\} n \geqq 4.$$

Boundary conditions (2) give:

$$\text{(3A)} \qquad \begin{aligned} &\frac{1}{2}E_0 + \sum_{2,4,\cdots}^{\infty} E_n \cos n\theta = 0, \quad \eta < \theta \leqq \pi/2 \\ &\sum_{2,4,\cdots}^{\infty} F_n \sin n\theta = 0, \quad \eta < \theta \leqq \pi/2. \end{aligned}$$

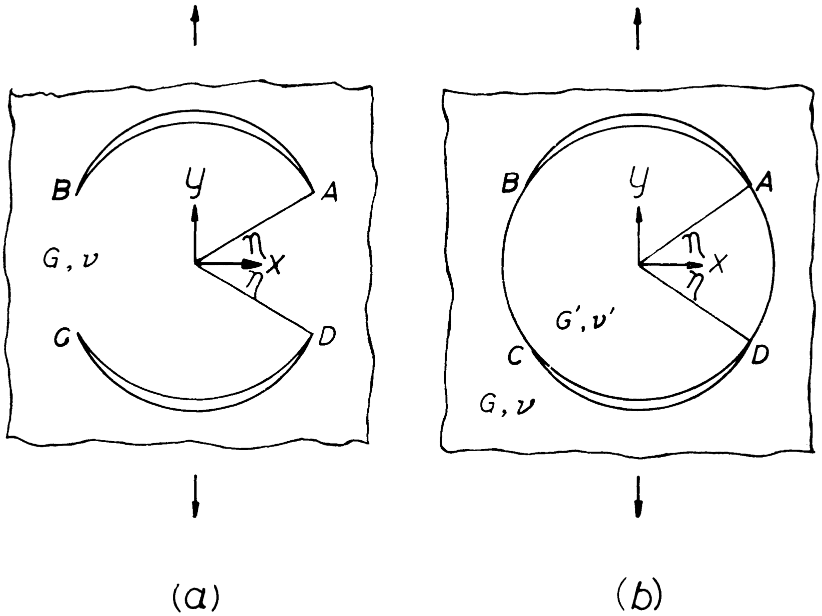

FIGURE 1.

(a) A symmetric set of curvilinear cracks under uniform tension.
(b) Contact problems of an inclusion.

Similarly the displacement conditions of (1) give:

$$\text{(3B)}\quad \begin{aligned} aE_0 + \sum_{2,4,\cdots}^{\infty} \frac{1}{n^2-1}(nE_n - F_n)\cos n\theta &= a - 2a\cos 2\theta,\ 0 \leqq \theta < \eta \\ \sum_{2,4,\cdots}^{\infty} \frac{1}{n^2-1}(nF_n - E_n)\sin n\theta &= 2a\sin 2\theta,\ 0 \leqq \theta < \eta \end{aligned}$$

where a, the elastic parameter, is given by

$$\text{(4)}\qquad a = \frac{1-\nu}{2(1-\nu) + 2(1-\nu')G/G'}\ .$$

(3A–3B) are coupled dual series for the determination of E_n, F_n and the receded contact angle η. For simplicity we have used special relation between the two elastic constants i.e. $G/G' = (1-2\nu)/(1-2\nu')$ in (3B).

In this paper we have given a solution using real variables. The problem can be reduced to a Riemann-Hilbert problem, and the solution obtained via complex variables as is done in [**11**].

2. **Reduction of Dual Series to Integral Equations.** If the insert is smooth the above coupled dual series (3) can be uncoupled. The solution to such a problem was first obtained in [**5**] by complex variables, and in [**6**], by dual series.

Equations (3A) essentially are derived from vanishing of the normal and the shearing stresses in the separated region $\eta < \theta \leqq \pi/2$. Assume $\sigma(\theta)$ and $\tau(\theta)$ to be the unknown normal and shearing stresses over the region $0 \leqq \theta < \eta$, we have:

$$\sigma(\theta) = \frac{1}{2}E_0 + \sum_{2,4,\cdots}^{\infty} E_n \cos n\theta, \qquad 0 \leqq \theta < \eta \tag{5}$$
$$\tau(\theta) = \sum_{2,4,\cdots}^{\infty} F_n \sin n\theta, \qquad 0 \leqq \theta < \eta.$$

Using Fourier inversion formulas, we have from (3A) and (5):

$$E_n = \frac{4}{\pi}\int_0^\eta \sigma(t)\cos nt\,dt \tag{6}$$
$$F_n = \frac{4}{\pi}\int_0^\eta \tau(t)\sin nt\,dt,$$
$$n = 0, 2, 4, \cdots.$$

Substituting from (6) into (3B) and changing the order of integration and summation, a pair of coupled integral equations are obtained:

$$\int_0^\eta \sigma(t)K_1(\theta, t)\,dt - \int_0^\eta \tau(t)K_2(\theta, t)\,dt = \frac{\pi}{4}a(1 - E_0 - 2\cos 2\theta),\ 0 \leqq \theta < \eta \tag{7}$$

$$\int_0^\eta \tau(t)K_4(\theta, t)\,dt - \int_0^\eta \sigma(t)K_3(\theta, t)\,dt = \frac{\pi}{2}a\sin 2\theta, \quad 0 \leqq \theta < \eta, \tag{8}$$

where K_1, K_2, K_3 and K_4 are the kernel functions given by:

$$K_1(\theta, t) = \sum_{2,4,\cdots}^{\infty} \frac{n}{n^2 - 1} \cos n\theta \cos nt \tag{9}$$

$$K_2(\theta, t) = \sum_{2,4,\cdots}^{\infty} \frac{1}{n^2 - 1} \cos n\theta \sin nt \tag{10}$$

$$K_3(\theta, t) = \sum_{2,4,\cdots}^{\infty} \frac{1}{n^2 - 1} \sin n\theta \cos nt \tag{11}$$

$$K_4(\theta, t) = \sum_{2,4,\cdots}^{\infty} \frac{n}{n^2 - 1} \sin n\theta \sin nt \tag{12}$$

These infinite series can be summed exactly. Using, for example, equations (549) and (550) of [7], we get

$$\sum_{0,2,\cdots}^{\infty} \frac{\cos mx}{m+1} = \frac{1}{2} \cos x \log \left| \cot \frac{x}{2} \right| + \frac{\pi}{4} \left| \sin x \right|$$

$$\sum_{0,2,\cdots}^{\infty} \frac{\cos mx}{m-1} = \frac{1}{2} \cos x \log \left| \cot \frac{x}{2} \right| - \frac{\pi}{4} \left| \sin x \right| - 1,$$

These results are used to sum the series in (9). Finally we have

$$K_1(\theta, t) = -\frac{1}{2} + \frac{1}{4} \left(\cos\theta \cos t \log \left| \frac{\cos\theta + \cos t}{\cos\theta - \cos t} \right| + \sin\theta \sin t \log \left| \frac{\sin\theta + \sin t}{\sin\theta - \sin t} \right| \right). \tag{13}$$

Similarly

$$K_2(\theta, t) = \frac{1}{4} \left(\cos\theta \sin t \log \left| \frac{\cos\theta + \cos t}{\cos\theta - \cos t} \right| + \sin\theta \cos t \log \left| \frac{\sin\theta - \sin t}{\sin\theta + \sin t} \right| \right), \tag{14}$$

$$K_3(\theta, t) = \frac{1}{4} \left(\sin\theta \cos t \log \left| \frac{\cos\theta + \cos t}{\cos\theta - \cos t} \right| + \cos\theta \sin t \log \left| \frac{\sin\theta - \sin t}{\sin\theta + \sin t} \right| \right), \tag{15}$$

$$K_4(\theta, t) = \frac{1}{4}\left(\cos\theta \cos t \log \left| \frac{\sin\theta + \sin t}{\sin\theta - \sin t} \right| \right. \tag{16}$$
$$\left. + \sin\theta \sin t \log \left| \frac{\cos\theta + \cos t}{\cos\theta - \cos t} \right| \right).$$

3. **Identities of Kernel Functions and Decoupling.** It can be verified, using (13) through (16) that

$$\frac{\partial K_1(\theta, t)}{\partial\theta} = -K_3(\theta, t) + \frac{\sin 2\theta}{2(\cos 2\theta - \cos 2t)} \tag{17}$$

$$\int_0^\theta K_1(x, t)\,dx = K_3(\theta, t) \tag{18}$$

$$\frac{\partial K_2(\theta, t)}{\partial\theta} = -K_4(\theta, t) \tag{19}$$

$$\int_0^\theta K_2(x, t)\,dx = K_4(\theta, t) - \frac{1}{4}\log\left| \frac{\sin(\theta + t)}{\sin(\theta - t)} \right|. \tag{20}$$

These identities are necessary to decouple the integral equations (7) and (8). Differentiating equation (7) with respect to θ and subtracting (8) and using (17) and (19) we obtain

$$\int_0^\eta \frac{\sigma(t)\,dt}{\cos 2\theta - \cos 2t} = \pi a,\ 0 \leqq \theta < \eta. \tag{21}$$

Integrating equation (7) from 0 to θ and adding (8) and using (18) and (20) we get

$$\int_0^\eta \tau(t)\log\left| \frac{\sin(\theta + t)}{\sin(\theta - t)} \right| dt \tag{22}$$
$$= \pi a(1 - E_0)\theta + \pi a \sin 2\theta, \quad 0 \leqq \theta < \eta.$$

As can be seen, (21) involves a Cauchy kernel and should be understood as the Cauchy principal value, while (22) involves a logarithmic singularity at $\theta = t$.

4. **Solutions of Integral Equations (21) and (22).** Let $\cos 2t = 1 + s\xi$, $\cos 2\theta = 1 + sx$, where s is a constant, selected in such a way, that when $t = 0$ and $t = \eta$ we have $\xi = 0$ and $\xi = 1$, ($s = -2\sin^2\eta$) respectively. Equation (21) is reduced to the airfoil equation:

$$\int_0^1 \frac{H(\xi)}{\xi - x}\,d\xi = 2\pi a, \quad 0 \leqq x \leqq 1 \tag{23}$$

where $H(\xi) \equiv \sigma(t(\xi))/\{\sin 2t(\xi)\}$. The solution of this integral equation, given by Peters [8], is:

$$H(\xi) = \frac{c}{\pi}\frac{1}{\sqrt{\xi(1-\xi)}} - \frac{1}{\pi^2\sqrt{\xi}}\frac{d}{d\xi}\int_\xi^1 \frac{d\lambda}{(\lambda-\xi)^{1/2}}\int_0^\lambda \frac{2\pi a x^{1/2}}{(\lambda-x)^{1/2}}\,dx,$$

and after transforming to the original variables,

$$\sigma(t) = \left(\frac{c}{\pi}+a\right)\frac{2\sin^2\eta\cos t}{(\cos^2 t-\cos^2\eta)^{1/2}} - 4a\cos t(\cos^2 t-\cos^2\eta)^{1/2} \tag{24}$$

where c is a constant which cannot be determined from (21) alone; this can be seen by substituting the solution back into the air-foil equation (23). This constant c has to be determined from the coupled integral equation (7) or (8) as done in the next section. This is due to non-uniqueness of solution of integral equation with a Cauchy kernel.

The integral equation with the logarithmic singularity can be solved by an ingenious device first used by Williams [9] or see [1, p. 155]). We first replace the kernel of (22) by

$$\log\left|\frac{\sin(\theta+t)}{\sin(\theta-t)}\right| = 4\cos\theta\cos t\int_0^{\min(t,\theta)}\frac{\tan v\,dv}{(\cos 2v-\cos 2t)^{1/2}(\cos 2v-\cos 2\theta)^{1/2}}. \tag{25}$$

After changing the order of integration, the Fredholm integral equation (22) is reduced to two Volterra equations of the Abel type:

$$\int_0^\theta \frac{\tan v\,G(v)\,dv}{(\cos 2v-\cos 2\theta)^{1/2}} = \frac{1}{4}\sec\theta\, f(\theta),\quad 0\leqq\theta<\eta \tag{26}$$

$$\int_v^\eta \frac{\cos t\,\tau(t)\,dt}{(\cos 2v-\cos 2t)^{1/2}} = G(v),\quad 0\leqq v\leqq\theta \tag{27}$$

where $f(\theta)$ stands for the right hand side of (22). The solutions of these integral equations are given, for instance, [1, p. 41], by

$$G(v)\tan v = \frac{1}{\pi}\frac{d}{dv}\int_0^v \frac{f(w)\sin w\,dw}{(\cos 2w-\cos 2v)^{1/2}} \tag{28}$$

$$\tau(t)\cos t = -\frac{2}{\pi}\frac{d}{dt}\int_t^\eta \frac{G(w)\sin 2w\,dw}{(\cos 2t-\cos 2w)^{1/2}} \tag{29}$$

With $f(w) = w$ and $f(w) = \sin 2w$ in (28) we obtain $G(v) = 1/2\sqrt{2}$ and $G(v) = (1/\sqrt{2})\cos^2 v$ respectively. Substituting these values for G in (29), we have the corresponding solutions of (29):

$$\tau_1(t) = \sqrt{2}\,\sin t\,(\cos 2t - \cos 2\eta)^{-1/2},$$

$$\tau_2(t) = \frac{2\sqrt{2}}{\pi}\sin t(\cos 2t - \cos 2\eta)^{1/2} + \frac{\sqrt{2}}{\pi}(1 + \cos 2\eta)\sin t(\cos 2t - \cos 2\eta)^{-1/2}.$$

The solution of (22) for τ is simply a superposition of τ_1 and τ_2:

$$\tau(t) = \pi a(1 - E_0)\tau_1(t) + \pi a\,\tau_2(t)$$

where E_0 is found by substituting from (24) into (6),

$$E_0 = \frac{4}{\pi}\int_0^\eta \sigma(t)\,dt = \frac{4}{\pi}c\sin^2\eta.$$

Thus the final solution of the integral equation (22) is

$$\tau(t) = \frac{a\sin t(\pi - 4c\sin^2\eta + 2\cos^2\eta)}{(\cos^2 t - \cos^2\eta)^{1/2}} + 4a\sin t(\cos^2 t - \cos^2\eta)^{1/2}. \tag{30}$$

Substituting into equations (6) from (24) and (30), we find Fourier coefficients for $m = n/2 = 1, 2, \cdots$

$$E_{2m} = \left(\frac{c}{\pi} + a\right)2\sin^2\eta(P_m + P_{m-1}) - a\left[\cos 2\eta\left(\frac{1}{m-1}P_{m-1} - \frac{1}{m+1}P_m\right) + \left(\frac{1}{m+1}P_{m-1} - \frac{1}{m-1}P_m\right)\right] \tag{31}$$

$$F_{2m} = (\pi - 4c\sin^2\eta + 2\cos^2\eta)a(P_{m-1} - P_m) + a\left[\cos 2\eta\left(\frac{1}{m-1}P_{m-1} + \frac{1}{m+1}P_m\right) - \left(\frac{1}{m+1}P_{m-1} + \frac{1}{m-1}P_m\right)\right] \tag{32}$$

where $P_m = P_m(\cos 2\eta)$, $P_{m-1} = P_{m-1}(\cos 2\eta)$ are Legendre polynomials. Thus we have obtained the solution for the coupled dual series (3).

There are yet two parameters, c and contact angle η, to be determined. However if η were the prescribed angle and if the insert and the exterior medium are both of the same material (i.e., $G = G'$ $\nu = \nu'$, $a = 1/4$) then physically the problem reduces to a curvilinear crack-problem as shown in Figure 1b. This is also apparent from the nature of the singularity in the normal ($\sigma(t)$) and shear ($\tau(t)$) stress given in equation (24) and (30) respectively. In the field of fracture mechanics the coefficients of such singularities are called K_I mode and K_{II} mode stress intensity factors. For such a problem we have

$$K_I = \left(\frac{c}{\pi} + a\right)\cdot\sqrt{2\cos\eta}\,\sin^2\eta$$

$$K_{II} = a\sin\eta\,(2\cos\eta)^{-1/2}(\pi - 4c\sin^2\eta + 2\cos^2\eta).$$

5. **Determination of Constant c and the Contact Angle η.** Using equations (7), (9), (10) and letting $\theta = 0$, for simplicity of the computations we have

$$\begin{aligned}&\int_0^\eta \sigma(t) \sum_{2,4,\cdots}^{\infty} \frac{n}{n^2-1}\cos nt\,dt\\ (33)\qquad &\qquad - \int_0^\eta \tau(t)\sum_{2,4,\cdots}^{\infty}\frac{1}{n^2-1}\sin nt\,dt\\ &= -\frac{\pi a}{4}(1+E_0).\end{aligned}$$

Substituting from (24) and (30) into (33) and carryig out the integrations and summations, (see Appendix [11]) we obtain

$$(34)\quad c = \frac{\pi a}{2\sin^2\eta}\;\frac{(\pi-1)\,x\,2\sin^2\eta\;K(\cos\eta) - (\pi-2)E(\cos\eta)}{2a(\pi-1) - K(\cos\eta) - (2a\pi-1)E(\cos\eta}$$

where K and E are complete elliptic functions of the first and second kind.

The physical process of contact of two unbonded surfaces require that the normal contact stress vary from a finite negative value (i.e., compression) over the contact region to a value of zero over the separated region. Hence, at $\theta = \eta$, $\sigma(\eta) = 0$. This gives $c = -\pi a$ from (24), and substituting from (34) we have the following equation for the determination of contact angle η. It should be noted that this equation is unaltered either due to change in load at infinity or the radius of the insert.

$$
\begin{aligned}
&(\pi - 1)(1 + 4a \sin^2\eta) - 4a \sin^2\eta \; K(\cos \eta) \\
&- \{\pi - 2 + 2(2\pi a - 1)\sin^2\eta\} aE(\cos \eta) = 0.
\end{aligned}
\tag{35}
$$

This equation has a real root. For various values of a, the roots of (35) are given in Figure 2.

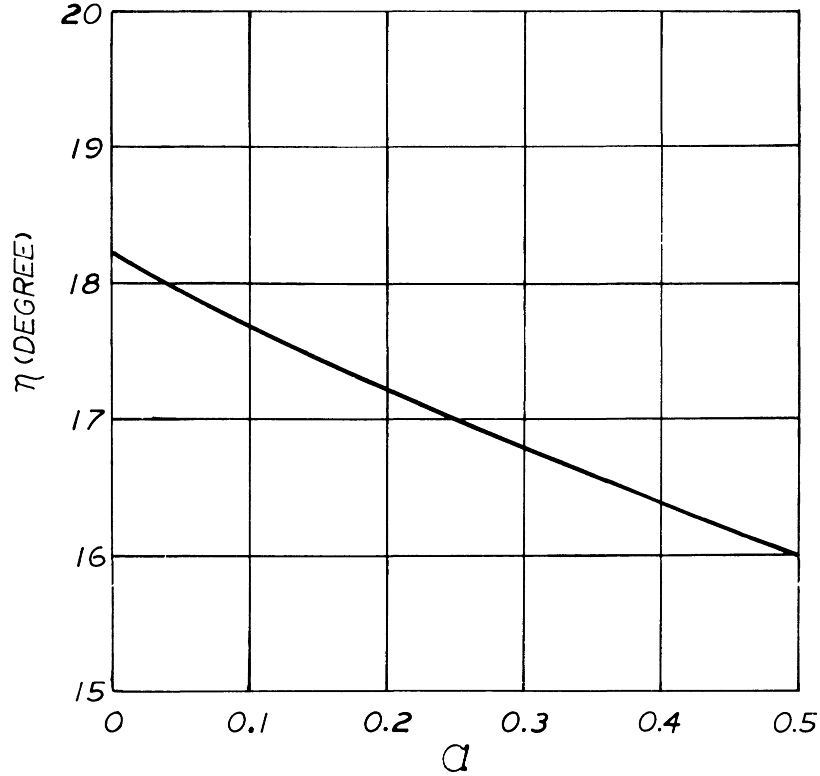

FIGURE 2.

The contact angle η for various values of a.

It is seen that even though the normal stress vanishes at $\theta = \eta$ the shearing stress is singular. This is due to the assumption that no slip takes place. Approximate solution of such slip problems have been attempted in [10].

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