ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 2, Spring 1980

LINEAR SYMPLECTIC STRUCTURES ON BANACH SPACES

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ABSTRACT. In recent years, both J. Marsden and A. Weinstein have called attention to the practical and theoretical importance of symplectic forms on Banach manifolds. In particular, Weintein's proof of the Darboux theorem in infinite dimensions reduces the local classification of symplectic structures on a Banach manifold to the study of linear (i.e., "constant") symplectic forms on a Banach space.

In previous work, the author has proved a version of the Morse index theorem for partial differential equations, using a topological approach based on symplectic forms in Hilbert space.

In the present paper, we attempt a more systematic study of the linear theory. The work is expository but includes several new results and examples. We prove in §*1*, for example, that the pull-back action of the general linear group on symplectic forms is *stable*; i.e., the orbits are open sets.

We define isotropic and lagrangian subspaces and give their elementary properties. The grassmannian of lagrangian subspaces is given the structure of a Banach subvariety of the full grassmannian of complemented subspaces of a Banach space. We show that under the action of the symplectic group, orbits of lagrangian subspaces are diffeomorphic to Banach homogeneous spaces of the symplectic group.

Finally, in the special case of a Hilbert space, it is proved that the lagrangian grassmannian and, as a consequence, the linear symplectic group are contractible topological spaces. Examples are sketched which show that this result is false if either general Banach spaces or weak symplectic structures are allowed.

1. **Linear Symplectic Structures.** In the following **E** is a Banach space, equipped with a continuous skew-symmetric bilinear form $\omega : \mathbf{E} \times \mathbf{E} \to \mathbf{R}$. Define the "flatted" map $\omega_b : \mathbf{E} \to \mathbf{E}^*$ by setting $\omega_b(e) \cdot f = \omega(e, f)$. If ω_b is an isomorphism, then ω is (strongly) non-degenerate and the pair $(\mathbf{E}, \omega)$ is called a (strong) linear symplectic structure. If it happens that ω_b is merely injective, then $(\mathbf{E}, \omega)$ is a *weak* symplectic structure. Although equivalent in finite dimensions, the two notions differ for the general Banach space case; and, in fact, weak structures play a dominant rôle in J. Marsden's formulation [2] of infinite dimensional mechanics. Other possibilities lie between the extremes of weak or strong; e.g., Tromba [15] has defined an "almost symplectic" struc-

Received by the editors on October 16, 1976.

AMS 1970 *subject classification.* Primary 58 H05, 58 B05; Secondary 22E65.

Key Phrases. Banach space, linear symplectic structure, lagrangian subspaces, symplectic group, fibre bundles.

ture arising from a weaker norm $\| \|_w$ defined on $\mathbf{E}$.

The existence of a linear symplectic structure $(\mathbf{E}, \omega)$ implies not only that $\mathbf{E}$ is isomorphic to $\mathbf{E}^*$ but also that $\mathbf{E}$ is reflexive, since the isomorphism $(\omega_b^{-1})^* \circ \omega_b : \mathbf{E} \to \mathbf{E}^{**}$ is simply the natural injection into the second dual.

If F is any Banach space, the product $F \times F^*$ carries a natural *weak* symplectic form Ω defined by

$$\Omega[(e, \alpha), (f, \beta)] = \langle \alpha, f \rangle - \langle \beta, e \rangle,$$

where $\langle \ , \ \rangle$ denotes the dual pairing of F and F^*. Thus, Ω_b maps each (e, α) to (α, e) in $\mathbf{E}^*$, if e is identified with its image in $\mathbf{E}^{**}$ under the natural injection. It follows that F is reflexive if and only if $(\mathbf{E}, \Omega)$ is a strong symplectic structure. Call a symplectic structure $(\mathbf{E}, \omega)$ *Darboux* if $\mathbf{E}$ is isomorphic to $F \times F^*$, F is reflextive, and $\omega = A^*\Omega$, where the pull-back $A^*\Omega(x, y) = \Omega(Ax, Ay)$ for all $x, y \in \mathbf{E}$, and A is an isomorphism $A : \mathbf{E} \to F \times F^*$. It is not known whether every strong symplectic structure is Darboux. We shall return to Darboux structures in §2. In general, any isomorphism $A : \mathbf{E}_1 \to \mathbf{E}_2$ of Banach spaces with symplectic structures $(\mathbf{E}_1, \omega_1)$ and $(\mathbf{E}_2, \omega_2)$ such that $A^*\omega_2 = \omega_1$ is called *symplectic*.

We now confine our attention to the set $\mathscr{A}_{\mathscr{P}}(\mathbf{E})$ of all strongly symplectic structures on $\mathbf{E}$. Then $\mathscr{A}_{\mathscr{P}}(\mathbf{E})$ is an open subset of $L_a{}^2(\mathbf{E})$, the continuous alternating forms on $\mathbf{E}$, since the isomorphism $\mathbf{E} \to \mathbf{E}^*$ form an open set in the space of continuous linear maps $L(\mathbf{E}, \mathbf{E}^*)$. We assume that $\mathbf{E}$ admits at least one symplectic structure.

If $\mathrm{GL}(\mathbf{E})$ denotes the general linear group of $\mathbf{E}$, there is a natural action $\mathrm{GL}(\mathbf{E}) \times \mathscr{A}_{\mathscr{P}}(\mathbf{E}) \to \mathscr{A}_{\mathscr{P}}(\mathbf{E})$ via the pullback map $(A, \omega) \rightsquigarrow A^*\omega$, defined above. Thus, $\mathscr{A}_{\mathscr{P}}(\mathbf{E})$ partitions into isometric classes determined by $\mathrm{GL}(\mathbf{E})$. For a given form ω, we may define the orbit map $f_\omega : A \rightsquigarrow A^*\omega$. The isotropy subgroup of $\mathrm{GL}(\mathbf{E})$ which fixes ω is called the *symplectic group* (at ω) which we denote by $\mathrm{Sp}(\mathbf{E}, \omega)$. Recall that a group action is *stable* if all the orbits are open sets.

In the following, the term "split-submersion" is applied to a differentiable map $f : M \to N$ if the exact sequence $\operatorname{Ker} T_x f \to T_x M \xrightarrow{T_x f} T_x N$ splits for all $x \in M$, where M and N are Banach manifolds.

THEOREM 1.1. *Suppose the Banach space* $\mathbf{E}$ *admits a linear symplectic structure. Then the following assertions hold*:

(i) $\mathrm{GL}(\mathbf{E})$ *acts stably on* $\mathscr{A}_{\mathscr{P}}(\mathbf{E})$

(ii) *The projection* $\pi : \mathrm{GL}(\mathbf{E}) \to \mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E}, \omega)$ *defines a locally trivial fibre bundle*

(iii) *Each orbit* $\mathrm{Orb}(\omega)$ *is diffeomorphic to a quotient group* $\mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E}, \omega)$.

PROOF. (i) We want to show that for every ω, the image of the orbit map $f_\omega : \mathrm{GL}(\mathbf{E}) \to \mathscr{A}_{\mathscr{P}}(\mathbf{E})$ contains a neighborhood of ω. Note $\mathrm{GL}(\mathbf{E})$ and $\mathscr{A}_{\mathscr{P}}(\mathbf{E})$ are Banach manifolds modeled, respectively, on $L(\mathbf{E})$, the space of continuous linear maps on $\mathbf{E}$, and on $L_a{}^2(\mathbf{E})$ defined above. If the differential $D_I f_\omega : L(\mathbf{E}) \to L_a{}^2(\mathbf{E})$ is split-surjective, then we may conclude by the implicit function theorem for Banach manifolds (Lang [10]) that the orbit is open. Thus, we need to show the sequence

$$\operatorname{Ker} D_I f_\omega \to L(\mathbf{E}) \xrightarrow{D_I f_\omega} L_a{}^2(\mathbf{E})$$

is exact and splits.

Since f_ω is quadratic, its differential is easily computed as $D_I f_\omega(A)(x, y) = \omega(Ax, y) + \omega(x, Ay)$, for every $A \in L(\mathbf{E})$ and $x, y \in \mathbf{E}$. To show surjectivity, if $\alpha \in L_a{}^2(\mathbf{E})$, define $B \in L(\mathbf{E})$ such that $B = \omega_b^{-1} \circ \alpha_b$. Then $\alpha(x, y) = \omega(Bx, y) = \omega(x, By)$, and setting $A = (1/2)B$, we obtain $D_I f_\omega(A) = \alpha$. Note that $A \in \operatorname{Ker} D_I f_\omega$ if and only if the induced bilinear form $\beta(x, y) = \omega(Ax, y)$ is symmetric. The space of bilinear forms splits as $L^2(\mathbf{E}) = L_a{}^2(\mathbf{E}) \oplus L_s{}^2(\mathbf{E})$; i.e., into alternating and symmetric parts, since $\mathbf{E}$ admits a non-degenerate form. The correspondence $G : A \rightsquigarrow \omega(Ax, y)$, for $x, y \in \mathbf{E}$ defines an isomorphism of $L(\mathbf{E})$ onto $L^2(\mathbf{E})$. Therefore, $L(\mathbf{E}) = G^{-1}(L_a{}^2(\mathbf{E})) \oplus G^{-1}(L_s{}^2(\mathbf{E}))$, and in particular, $\operatorname{Ker} D_I f_\omega = G^{-1}(L_s{}^2(\mathbf{E}))$ which establishes the desired splitting.

(ii) From the proof of (i), we know that $\mathrm{Sp}(\mathbf{E}, \omega) = f_\omega^{-1}(\omega)$ is a closed subgroup of $\mathrm{GL}(\mathbf{E})$ whose tangent space may be identified with $\operatorname{Ker} D_I f_\omega$, whose elements are called infinitesimally symplectic and which is formally the Lie algebra "$sp(\mathbf{E}, \omega)$" corresponding to the Banach Lie group $\mathrm{Sp}(\mathbf{E}), \omega)$.

Now let π, f_ω be as above and suppose $F : \mathrm{Orb}(\omega) \to \mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E}, \omega)$ is the bijective correspondence $\alpha \rightsquigarrow \pi A$, for $\alpha = A^*\omega$. A standard result for (Banach) Lie groups (cf. de la Harpe [6]) is that the coset space admits a smooth manifold structure which is uniquely determined by the requirement that π be a smooth submersion. Moreover, the tangent space $T_{\pi I}[\mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E}, \omega)]$ is isomorphic to the quotient space $L(\mathbf{E})/\operatorname{Ker} D_I f_\omega$, which is, in turn, isomorphic to $G^{-1}[L_a{}^2(\mathbf{E})]$ as in the proof of (i). Since $\operatorname{Ker} D_I \pi$ equals $\operatorname{Ker} D_I f\omega$, π is a split-submersion and by the implicit function theorem must be locally trivial. That is, there is a neighborhood V_1 of πI in $\mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E}, \omega)$, a neighborhood U of I in $\mathrm{GL}(\mathbf{E})$, and a diffeomorphism $h : V_1 \times V_2 \to U$ such that $\pi \circ h : V_1 \times V_2 \to V_1$ is the projection map. In particular, we may take V_2 to be the fibre $\mathrm{Sp}(\mathbf{E}, \omega)$. Since we are dealing with a smooth group $\mathrm{GL}(\mathbf{E})$, the same argument holds in a neighborhood of any element of $\mathrm{GL}(\mathbf{E})$.

(iii) By (i) and (ii), F is a local diffeomorphism, and since it is bijective, it must be a global diffeomorphism.

2. **The Lagrangian Grassmannian.** Weinstein [**16**] has treated lagrangian subspaces of a symplectic Banach space. We shall build on his work, below, in order to characterize certain subsets of the grassmannian of lagrangian subspaces as specific Banach homogeneous spaces.

A subspace F of a Banach space $\mathbf{E}$ is *isotropic*, with respect to a symplectic structure $(\mathbf{E}, \omega)$, if for all $x, y \in F$, $\omega(x, y) = 0$. Isotropic subspaces are always closed. By Zorn's lemma, *maximal* (by inclusion) isotropic subspaces always exist. Let F^ω denote the ω-annihilator of a subspace F, i.e., $F^\omega = \{e \in \mathbf{E} | \omega(e, f) = 0, \text{ for all } f \in F\}$. Then, evidently, a subspace F in $\mathbf{E}$ is maximal isotropic if and only if $F = F^\omega$. An isotropic subspace F is called *lagrangian* if it is complemented in $\mathbf{E}$ by an isotropic subspace, i.e., $\mathbf{E} = F \oplus G$ and G is isotropic. Using the non-degeneracy of the form ω, it is not hard to show that lagrangian subspaces are always maximal isotropic, even when ω is only a *weak* form. Weinstein [**16**] has shown that in Hilbert space maximal isotropic subspaces coincide with lagrangian subspaces if ω is strongly non-degenerate.

The set of all lagrangian subspaces is called the *lagrangian grassmannian*, which we write as $\Lambda(\mathbf{E}, \omega)$. The terminology is due to Arnol'd (cf. [**1**]) in finite dimensions. The next theorem, due to Weinstein [**16**], relates lagrangian subspaces to Darboux structures.

THEOREM 2.1. *If* $(\mathbf{E}, \omega)$ *is a symplectic structure and* L *is a lagrangian subspace of* $\mathbf{E}$, *then there exists a symplectic isomorphism* $A : \mathbf{E} \to L \times L^*$, *where* $L \times L^*$ *is equipped with the canonical symplectic structure defined in* §1.

COROLLARY. $\Lambda(\mathbf{E}, \omega) \neq \Phi$ *if and only if the structure* $(\mathbf{E}, \omega)$ *is Darboux.*

PROOF (of theorem). We shall sketch Weinstein's proof. Since L is lagrangian, we may set $\mathbf{E} = L \oplus M$, for a lagrangian subspace M. Define a map $A : L \oplus M \to L \times L^*$ such that $A = \mathrm{id}_L \times \pi_{L^*} \circ (\omega_b | M)$, where π_{L^*} is the projection of $\mathbf{E}^*$ onto L^*, i.e., the restriction map. Then A is the desired symplectic isomorphism.

The proof of the corollary is immediate. Note that Theorem 2.1 implies that a given Darboux symplectic form is determined (up to isometry) by its set of isomorphically distinct lagrangian subspaces. The name "Darboux" comes from the fact that Theorem 2.1 yields "Darboux coordinates."

We now observe that a given symplectic structure may have several distinct Darboux decompositions in the sense that $\mathbf{E} \simeq L \times L^* \simeq M \times M^*$, but L and M are non-isomorphic lagrangian subspaces (the symbol $\simeq$ denotes a Banach space isomorphism).

EXAMPLE. Suppose ℓ_p is the p-summable sequence space, with dual space $\ell_p^* = \ell_q$. We may equip $\mathbf{E}_p = \ell_p \times \ell_q$ with the standard symplectic form Ω_π. Evidently, $\mathbf{E}_p \simeq \mathbf{E}_q$ although ℓ_p and ℓ_q are not isomorphic unless $p = q = 2$. The symplectic structure $(\mathbf{E}_p, \Omega_p)$ admits at least three isomorphism classes of lagrangian subspaces: ℓ_p, ℓ_q, or $\ell_p \times \ell_q$. By the latter, we mean a lagrangian subspace L which is isomorphic to $\mathbf{E}_p$. To see that such a subspace exists, put $L = \ell_p' \times \ell_q''$ where ℓ_p' is derived from ℓ_p by setting every even entry equal to zero; while ℓ_q'' consists of sequences whose odd entries are zero. Then L is lagrangian and is complemented by $\ell_p'' \times \ell_q'$. There is a plausible conjecture associated with this example.

CONJECTURE. For any symplectic structure $(\mathbf{E}_p, \omega)$ on $\mathbf{E}_p$, each lagrangian subspace is isomorphic to ℓ_p, ℓ_q, or $\ell_p \times \ell_q$.

REMARK. The conjecture is rendered likely by the fact that the spaces ℓ_p, $1 \leqq p \leqq \infty$, are all *prime* Banach spaces, i.e., any complemented subspace is isomorphic to ℓ_p or to a finite dimensional space. A related notion is that of *primary* Banach spaces. $\mathbf{E}$ is primary if whenever $\mathbf{E} = F \oplus G$, $\mathbf{E} \simeq F$ or $\mathbf{E} \simeq G$. The function spaces $L_p[0, 1]$, $1 < p < \infty$ are primary, and we believe the conjecture to hold for the symplectic space $L_p \times L_q$. For a discussion of prime and primary spaces consult [11].

In the author's thesis [14], the following is proved.

PROPOSITION 2.2. *$\Lambda(\mathbf{E}, \omega)$ is a closed (non-singular) analytic sub-variety of the full grassmannian on $\mathbf{E}$, $\mathscr{G}(\mathbf{E})$, i.e., the totality of closed complemented subspaces of $\mathbf{E}$.*

For a treatment of analytic Banach varieties, consult Douady [4]. Charts for $\Lambda(\mathbf{E}, \omega)$ consist of pairs $(\Gamma_{F,G}, U_G)$, such that for each lagrangian subspace G, $U_G = \{F \in \Lambda(\mathbf{E}, \omega) \mid \mathbf{E} = F \oplus G\}$ and the map $\Gamma_{F,G} : U_G \to Q(F)$ is defined by the rule $F' \rightsquigarrow \omega(Ax, y)$. To obtain the symmetric bilinear form $\omega(Ax, y)$, we may express F' as the graph $\{(x, Ax) \mid x \in F\}$ for some continuous linear map $A : F \to G$. The form $\omega(Ax, y)$ is always symmetric if F and G are isotropic subspaces. Thus, $\Lambda(\mathbf{E}, \omega)$ is locally modeled on the Banach space $Q(F)$ of symmetric bilinear forms on F.

The symplectic group $\mathrm{Sp}(\mathbf{E}, \omega)$ acts on the variety $\Lambda(\mathbf{E}, \omega)$ by sending a subspace to its image under a symplectic automorphism. To verify this fact, note that symplectic maps preserve isotropic subspaces and for A invertible, $\mathbf{E} = F \oplus G$ if and only if $AF \oplus AG = \mathbf{E}$. Denote the orbit of $L \in \Lambda(\mathbf{E}, \omega)$ under $\mathrm{Sp}(\mathbf{E}, \omega)$ as $\Lambda_L(\mathbf{E}, \omega)$. Now assume that the form ω is fixed and Darboux; i.e., lagrangian subspaces exist. Hereafter, we shall omit all inessential references to ω.

THEOREM 2.3. *With the induced structure from* $\Lambda(\mathbf{E})$, *the orbit* $\Lambda_L(\mathbf{E})$ *is a smooth* (*even analytic*) *manifold modeled on* $Q(L)$.

PROOF. It is enough to know that for every lagrangian subspace $L' \in \Lambda_L(\mathbf{E})$ such that $\mathbf{E} = L' \oplus M$, i.e., $L' \in U_M$, it follows that U_M is contained in $\Lambda_L(\mathbf{E})$. That suffices since by Proposition 1, charts of $\Lambda(\mathbf{E})$ are compatible, the overlap maps are non-singular, and every chart maps into an isomorphic copy of the Banach space $Q(L)$.

We shall show that $\Lambda_L(\mathbf{E})$ contains all lagrangian subspaces which are isomorphic to L, and hence must also contain the chart domain U_M. Suppose there is an isomorphism $B : L \to L'$. Then $C = (B^*)^{-1} : L^* \to (L')^*$ is an isomorphism, and we may conclude that the map $A = B \times C : L \times L^* \to L' \times (L')^*$ is an isomorphism. A is symplectic with respect to the canonical structures on its domain and range. This follows from the fact that for any pair $(x, y) \in L \times L^*$

$$\langle A(\phi, 0), A(0, x)\rangle = \langle C\phi, Bx\rangle = \langle \phi, x\rangle,$$

where $\langle\ ,\ \rangle$ denotes the dual pairing of $\mathbf{E}$ and $\mathbf{E}^*$. However, by Theorem 1, there are symplectic maps $A_1 : \mathbf{E} \to L \times L^*$ and $A_2 : \mathbf{E} \to L' \times (L')^*$. Therefore, since $A_1 \mid L = \mathrm{id}_L$ and $A_2 \mid L' = \mathrm{id}_{L'}$, the map $S = A_2^{-1}AA_1 : \mathbf{E} \to \mathbf{E}$ is a symplectic automorphism such that $S \mid L = B$.

Now let $\alpha_L : \mathrm{Sp}(\mathbf{E}) \to \Lambda_L(\mathbf{E})$ denotes the orbit map $A \rightsquigarrow A(L)$. Suppose G_L denotes the closed subgroup of $\mathrm{Sp}(\mathbf{E})$ which stabilizes L. If $\pi : \mathrm{Sp}(\mathbf{E}) \to \mathrm{Sp}(\mathbf{E})/G_L$ is the natural projection, then there is a bijection $\Phi : \Lambda_L(\mathbf{E}) \to \mathrm{Sp}(\mathbf{E})/G_L$ such that $\Phi \circ \alpha_L = \pi$. Again, $\mathrm{Sp}(\mathbf{E})/G_L$ may be given the structure arising from the assertion that π is a (split) submersion. We now give the analog of Theorem 1.1 for the above group action.

THEOREM 2.4. *Let* α_L, π, *and* Φ *be as above*; *then*
(i) α_L *is a split-submersion onto* $\Lambda_L(\mathbf{E})$
(ii) Φ *is a diffeomorphism.*

COROLLARY. *The map* π (*or equivalently* α_L) *defines a smooth fibre bundle structure.*

PROOF (of theorem). (i) Assume $\mathbf{E} = L \oplus m$; then it is clear that every symplectic map A lying in some neighborhood of the identity on $\mathbf{E}$ is such that $\mathbf{E} = A(L) \oplus M$. That is, $A(L)$ lies in the chart U_M. Hence, in a chart we may represent $A(L)$ as the graph of a linear map $B : L \to M$, such that the form $\omega(Bx, y)$ for $x, y \in L$, is symmetric. If π_L and π_M are the coordinate projections on $\mathbf{E}$ it is clear that

$$\begin{aligned} A \mid L &= \pi_L A \mid L \oplus B\pi_L A \mid L \\ &= \pi_L A \mid L \oplus B\pi_M A \mid L. \end{aligned}$$

Thus, if $j : L \longrightarrow \mathbf{E}$ denotes inclusion, we know

$$B\pi_L A j = \pi_M A j, \text{ or } B = (\pi_M A j)(\pi_L A j)^{-1}.$$

Now, as a manifold, the symplectic group is modeled on $sp(\mathbf{E})$, the space of infinitesimally symplectic operators. The exponential map is a chart which sends a neighborhood of zero in $sp(\mathbf{E})$ onto a neighborhood of the identity in $\mathrm{Sp}(\mathbf{E})$. We may assume that the latter neighborhood is mapped into U_M by α_L. Thus, locally α_L is given by the correspondence $a \rightsquigarrow B(a) = (\pi_M \exp(a)j)(\pi_L \exp(a)j)^{-1}$. The differential at zero is easily computed (leibniz' rule) to be $D_0B(a) = \pi_M a j$. Hence, the tangent map $T_I\alpha_L : sp(\mathbf{E}) \longrightarrow Q(L)$ has the form $a \rightsquigarrow \omega(\pi_M ax, y)$. But since L and M are lagrangian and ω is symplectic, we know that $\omega(\pi_M axy,) = \omega(ax, y)$ for all x, y in L.

Because $a \in sp(\mathbf{E})$ if and only if the form $\omega(ax, y)$ is symmetric on $\mathbf{E}$, the map $T_I\alpha_L$ is surjective. By Theorem 2.1, we may identify $\mathbf{E}$ with $L \times L^*$, and, consequently, elements of $sp(\mathbf{E})$ may be written in the block form

$$\begin{bmatrix} a & b \\ c & -a^* \end{bmatrix},$$

where $a : L \longrightarrow L$ is continuous linear, and the maps $b : L^* \longrightarrow L$ and $c : L \longrightarrow L^*$ determine symmetric bilinear forms on, respectively, L^* and L. (Recall that L is reflexive.)

Thus, $sp(\mathbf{E})$, which we identify with $T_I\mathrm{Sp}(\mathbf{E})$, splits canonically as

$$\begin{bmatrix} a & b \\ c & -a^* \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & -a^* \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}.$$

The kernel of $T_I\alpha_L$ consists precisely of operators of the type of the first summand on the right, and $Q(L)$ is isomorphic to the space of operators $c : L \longrightarrow L^*$ which are symmetric.

(ii) We may identify the tangent space $T_{\pi I}[\mathrm{Sp}(\mathbf{E})/G_L]$ with the space $Q(L)$, since T_IG_L is isomorphic to the kernel of $T_I\alpha_L$. With this identification, the differential of π is, like that of α_L, given by $T_I\pi(a) = \omega(ax, y)$. Hence, in suitably chosen coordinates the map Φ is a local diffeomorphism. Since Φ is bijective, it must be a global diffeomorphism.

The corollary is immediate from the implicit function theorem for Banach manifolds (see the proof of Theorem 1 in §1).

By Theorem 2.4, we may regard $\Lambda_L(\mathbf{E})$ as a Banach homogeneous space of the symplectic group; i.e., $\Lambda_L(\mathbf{E}) = \mathrm{Sp}(\mathbf{E})/G_L$. The characterization may be made completely explicit by representing the isotropy group in a particularly simple form given in the next proposition. We omit the proof, as it is straight forward and parallels the finite dimensional case (e.g., [5]). A complete treatment is given in [**14**].

PROPOSITION 2.5. *Suppose $L \in \Lambda(\mathbf{E})$ and $\mathbf{E}$ is identified with $L \times L^*$. Elements of G_L may be expressed in block form as*

$$\begin{bmatrix} A & AB \\ 0 & (A^*)^{-1} \end{bmatrix}$$

where $A : L \to L$ is invertible, and $B : L^* \to L$ determines a symmetric bilinear form on L^*.

COROLLARY. *The group G_L is diffeomorphic (and isomorphic) to the product* $\mathrm{GL}(L) \times Q(L^*)$, *where* $\mathrm{GL}(L)$ *is the linear group of L and $Q(L^*)$ is the totality of symmetric forms on L^*.*

In the next section, we shall apply this corollary to obtain homotopy results for the symplectic group.

3. **The Topology of the Symplectic Group.** In this section, our goal is to relate the topologies of the symplectic group and the lagrangian grassmannian. For the special case in which the underlying space $\mathbf{E}$ is a Hilbert space, we prove that both $\mathrm{Sp}(\mathbf{E})$ and $\Lambda(\mathbf{E})$ are contractible topological spaces.

We first require a strong lemma from homotopy theory in infinite dimensions.

LEMMA 3.1 (R. Palais [13]). *Let X and Y denote metrizable manifolds; i.e., paracompact manifolds modeled on locally convex topological vector spaces which are metrizable. Then the following hold:*

(i) *If $f : X \to Y$ is a continuous map inducing isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n \geqq 0$, then f is a homotopy equivalence.*

(ii) $\pi_n(X) = 0$ *for all n if and only if X is contractible.*

COROLLARY. 1. *Suppose K is a closed subgroup of a metrizable Banach group G, such that the projection $\pi : G \to G/K$ defines a (locally trivial) fibre bundle. Then*

(i) *if K is contractible, π is a homotopy equivalence.*

(ii) *if any two among the triple $(K, G, G/K)$ are contractible manifolds, the same holds for the third.*

PROOF (of corollary). Since $G \to G/K$ is a fibering, we have the following infinite exact homotopy sequence:

$$\cdots \to \pi_{n+1}K \to \pi_{n+1}G \to \pi_{n+1}\, G/K \to \pi_n K \to \pi_n G \to \cdots$$

valid for $n \geqq 0$ (cf. Hu [7]).

For a contractible space every homotopy group vanishes. The corollary now follows from Lemma 3.1.

COROLLARY 2. *Suppose a reflexive Babach space L has a contractible general linear group. Then if* E *is the symplectic space* $L \times L^*$, *the orbit map* $\alpha_L : \mathrm{Sp}(\mathrm{E}) \to \Lambda_L(\mathrm{E})$ *is a homotopy equivalence.*

PROOF. From §2, the isotropy subgroup G_L is isomorphic to $\mathrm{GL}(L) \times Q(L^*)$, and, therefore, is contractible. Since $\Lambda_L(\mathrm{E})$ is diffeomorphic to the homogeneous space $\mathrm{Sp}(\mathrm{E})/G_L$, it follows from Corollary 1 that α_L is a homotopy equivalence.

REMARKS. The linear group is contractible for an astonishing variety of Banach function spaces (cf. Mityagin [12]) including the Sobolev and Hölder spaces.

If the underlying Banach space E is not only symplectic but "Hilbertable," we shall see that some pleasant consequences arise.

A *complex-symplectic structurre* on a Banach space E is a pair (J, ω) such that $\omega \in \mathscr{A}_\theta(\mathrm{E})$ and $J : \mathrm{E} \to \mathrm{E}$ is symplectic; i.e., $J^*\omega = \omega$, with the property that $J^2 = -I$. From the definition it follows that the form $g(x, y) = -\omega(Jx, y)$ is non-degenerate and symmetric. If for some choice of (J, ω), $g(x, y)$ happens to be positive definite, then E is *Hilbertable* and the equivalent norm topology induced by g gives E the structure of a complete Hilbert space. If E is isomorphic to a Hilbert space, then it can be shown (Chernoff and Marsden [2]) that for any symplectic form ω, E admits a complex-symplectic structure (J, ω) such that

$$(1)\quad g(x, y) = -\omega(Jx, y)$$

$$(2)\quad h(x, y) = g(x, y) + i\omega(x, y)$$

and g is a complete real inner product structure, while h defines a complete hermitian or complex inner product on E. (Note that any Hilbert space H admits a symplectic (and Darboux) structure, as there are isomorphisms $H \simeq H \times H \simeq H \times H^*$.)

Now let us suppose that $\mathscr{H}$ denotes a fixed complex Hilbert space with a pair (J, ω), satisfying (1) and (2). We may form the lagrangian grassmannian $\Lambda(\mathscr{H})$. Weinstein [16] has shown that in Hilbert space all maximal isotropic subspaces are lagrangian, i.e., admit isotropic complements.

PROPOSITION 3.2. *If H is a lagrangian subspace of $\mathscr{H}$, then JH is lagrangian and $\mathscr{H} = H \oplus JH$.*

PROOF. Since J is symplectic, JH is lagrangian. The latter assertion follows, since $JH = H^{\perp}$, the orthogonal complement of H via the real inner product $g(x, y)$ defined in (1).

The decomposition $H \oplus JH$ splits $\mathscr{H}$ into "real and imaginary" lagrangian subspaces. Suppose now that $\mathscr{U} = \mathscr{U}(\mathscr{H})$ denotes the unitary group on $\mathscr{H}$, which consists of isometries of $h = g + i\omega$. Fixing some lagrangian subspace H, let $\mathscr{O} = \mathscr{O}(H)$ denote the real orthogonal group on H which is comprised of isometries of the real inner product g, restricted to $H \times H$. Identifying $H \oplus JH$ with $H \times H$, there is a natural injection $\mathscr{O} \to \mathscr{U}$, defined in block matrix form as

$$A \rightsquigarrow \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

Suppose $GL_J(\mathscr{H})$ is the subgroup of $GL(\mathscr{H})$, consisting of the automorphisms which commute with J; that is, $GL_J(\mathscr{H})$ is the "complex linear group."

PROPOSITION 3.3. $\mathscr{U} = GL_J(\mathscr{H}) \cap \mathrm{Sp}(\mathscr{H})$.

PROOF. From the decomposition $h = g + i\omega$, any isometry of h is also an isometry of both g and ω. Hence, $\mathscr{U} \subset \mathrm{Sp}(\mathscr{H})$. If A is an isometry of g, then

$$\begin{aligned} g(x, y) = -\omega(Jx, y) &= -\omega(JAx, Ay) \\ &= -\omega(A^{-1}JAx, y) \qquad \text{for all } x, y \in \mathscr{H}. \end{aligned}$$

Hence, $J = A^{-1}JA$.

PROPOSITION 3.4. *If G_H is the isotropy group of symplectic maps which fix H, then identifying $\mathscr{O}$ as a subgroup of $\mathscr{U}$,*

$$\mathscr{O} = \mathrm{GL}_J(\mathscr{H}) \cap G_H.$$

PROOF. From Proposition 2.4, operators in G_H have the block form

$$\begin{bmatrix} A & BA \\ 0 & A \end{bmatrix}.$$

Note that in the identification of H^* with JH, via the Darboux decomposition of $\mathscr{H}$ as $H \times H^*$, $(A^*)^{-1}$ is identified with A. Elements of $\mathrm{GL}_J(\mathscr{H})$ have the usual form

$$\begin{bmatrix} A & C \\ -C & A \end{bmatrix}.$$

By intersecting these two groups, we obtain, canonically,

$$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

which must be unitary and, therefore, is orthogonal.

The next result gives another characterization of the lagrangian grassmannian $\Lambda(\mathscr{H})$ as a homogeneous space—this time by restricting the orbit map α_H on $\mathrm{Sp}(\mathscr{H})$ to the unitary subgroup.

THEOREM 3.5. *$\Lambda(\mathscr{H})$ is diffeomorphic to the Banach homogeneous space $\mathscr{U}/\mathscr{O}$, and the natural projection $\pi : \mathscr{U} \longrightarrow \mathscr{U}/\mathscr{O}$ (or equivalently $\alpha_H : \mathscr{U} \longrightarrow \Lambda(\mathscr{H})$) defines a smooth fibre bundle.*

PROOF. We first show $\mathscr{U}$ acts transitively on $\Lambda(\mathscr{H})$ with the isotropy subgroup $\mathscr{O}$. If H_1 and H_2 are lagrangian, suppose the linear map A carries an orthonormal basis of H_1 onto an orthonormal basis for H_2. Then A may be extended to map JH_1 onto JH_2 by multiplying basis elements by J. By Proposition 3.3, A is a unitary operator on $\mathscr{H}$.

Since $\mathscr{U} \cap G_H = \mathrm{GL}_J(\mathscr{H} \cap G_H = \mathscr{O}$, where G_H is the stabilizer subgroup of the symplectic group, $\mathscr{O}$ must be the isotropy subgroup of the unitary group. Since $\Lambda(\mathscr{H})$ is diffeomorphic to $\mathrm{Sp}(\mathscr{H})/G_H$, which is in turn diffeomorphic to $\mathscr{U}/\mathscr{O} = \mathscr{U} \cap \mathrm{Sp}(\mathscr{H})/\mathscr{U} \cap G_H$, the correspondence $A(H) \dashrightarrow \pi(A)$ is the desired diffeomorphism (see Theorem 2.3). Note that $\mathscr{U}$ is a closed smooth subgroup of $\mathrm{Sp}(\mathbf{E})$.

To conclude that π is a locally trivial fibering, we show that it is a split submersion. The tangent space $T_I\mathscr{U}$ consists of skew-hermitian operators on $\mathscr{H}$, while $T_I\mathscr{O}$ is the vector space of skew-symmetric operators on H. But the space of skew-hermitian operators splits canonically into symmetric and skew-symmetric parts, according to the well-known decomposition (in block form on $H \times H$):

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \oplus \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

In terms of the tangent map $T_I\pi$, this splitting corresponds to

$$T_I\mathscr{U} = \operatorname{Ker} T_I\pi \oplus \operatorname{Im} T_I\pi.$$

The splitting obtains at every element of $\mathscr{U}$, since left multiplication is a diffeomorphism on $\mathscr{U}$. That π is locally trivial now follows from the implicit function theorem.

We come now to the principal result of this section.

THEOREM 3.6. *Suppose* $(\mathbf{E}, \omega)$ *is a Hilbertable symplectic structure. Then all of the following topological spaces are contractible*:
(i) $\Lambda(\mathbf{E})$ (ii) G_H (iii) $\mathrm{Sp}(\mathbf{E})$ (iv) $\mathscr{A}_\mathscr{O}(\mathbf{E})$.

PROOF. (i) This is a corollary of the last theorem, for, by a well-known result of Kuiper [9], the unitary and orthogonal groups are contractible spaces and we may apply Lemma 3.1 (Corollary 1).

(ii) From §2, G_H is diffeomorphic to $\mathrm{GL}(H) \times Q(H)$. $Q(H)$ is linear and hence contractible. The linear group $\mathrm{GL}(H)$ is contractible, since H is a Hilbert space, and $\mathrm{GL}(H)$ is diffeomorphic to $\mathscr{U} \times \mathscr{P}$, where $\mathscr{P}$ is the contractible cone of positive operators. (This is simply the polar decomposition theorem.)

(iii) Since $\Lambda(\mathscr{H})$ is diffeomorphic to $\mathrm{Sp}(\mathbf{E})/G_H$, by (i) and (ii) and Lemma 3.1 (Corollary 1), the assertion follows.

(iv) Recall that $\mathscr{A}_{\mathscr{O}}(\mathbf{E})$ denotes the totality of (strongly) non-degenerate alternating forms on $\mathbf{E}$. From §1, orbits under pull-back by the general linear group are diffeomorphic to the homogeneous space $\mathrm{GL}(\mathbf{E})/\mathrm{Sp}(\mathbf{E})$. Hence, each orbit is contractible. However, it is easy to show that any two symplectic forms are isometric. Let Ω_1 and Ω_2 be symplectic forms associated with Darboux decompositions $H_1 \times H_1^*$ and $H_2 \times H_2^*$. Let $A : H_1 \to H_2$ be an isomorphism. Then the isomorphism $B = A \times (A^*)^{-1} : H_1 \times H_1^* \to H_2 \times H_2^*$ is easily shown to be symplectic; i.e., $B^*\Omega_2 = \Omega_1$.

Remarks. The contractibility of the symplectic group is a sort of companion result to Kuiper's proof [**9**] of the contractibility of the unitary group. The result has repercussions for the *non-linear* Banach symplectic theory. For example, one can conclude that the tangent bundle of a symplectic Banach manifold M modeled on a Hilbertable space $\mathbf{E}$ is trivial, and in fact isomorphic to $\mathbf{E} \times \mathrm{Sp}(\mathbf{E})$ (see Husemoller [**8**]). Consequently, the cotangent bundle T^*M is trivial, and, hence the principal part of the natural two-form Ω on T^*M is described by a smooth map $\Omega : T^*M \to \mathscr{A}_{\mathscr{O}}(\mathbf{E})$ (Lang [**10**], p. 108).

It might be queried whether some form of Theorem 3.6 holds in the general Banach case. That this is far from the case is clear from a number of examples. The stabilizer group G_L is not even connected, if $L = \ell_p \times \ell_q$ and $\mathbf{E} = L \times L^*$, since there is an elegant proof (Douady [**3**]), using the continuity of the Fredholm index, that the linear group of $\ell_p \times \ell_r$ is disconnected whenever $p \neq r$. A variation of Douady's proof has been constructed by the author to show that the symplectic group $\mathrm{Sp}(\ell_p \times \ell_q)$ is disconnected. If one allows weak symplectic forms, the symplectic group is not connected, even in Hilbert space. A proof can be constructed using Douady's method applied to the space $H^1 \times L_2$, where L_2 consists of square integrable functions on [0, 1], and H^1 denotes the Sobolev space of absolutely continuous functions in L_2. The space $H^1 \times L_2$ arises as the phase space in some infinite dimensional Hamiltonian systems (Chernoff and Marsden [**2**]).

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