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SEMI-STABLE KERNELS OF VALUATED GROUPS

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ABSTRACT. A characterization of semi-stable kernels of valuated abelian groups is given.

1. **Introduction.** The concept of valuated groups has recently been developed extensively by Richman and Walker [**2**]. (Throughout this paper, the term "group" will mean abelian group.) If A is a subgroup of the group B, the p-height function of B restricted to A gives rise to a valuation on A. This relation has been quite useful in determining the structure of certain classes of groups. (For a more detailed discussion, see the introduction of [**2**].) Richman and Walker [**1**] have developed a theory of Ext in pre-abelian categories, and have applied this in [**2**] to valuated groups. The notions of semi-stable kernels and semi-stable cokernels are fundamental to this theory. While semi-stable cokernels are classified in a satisfactory way in [**2**], the question of classifying semi-stable kernels is left open. In this paper, a characterization of semi-stable kernels is given.

2. **Valuated Groups.** In this section, we summarize some definitions and results on valuated groups. Most of this discussion originated in [**2**]. Let G be an abelian group and p be a prime. The *p-height function* on G is characterized by

$$h_p x = \sup\{h_p y + 1: \quad x = py\}$$

where $h_p x$ is either an ordinal or ∞. We say $\infty < \infty$ and $\alpha < \infty$ for any ordinal α.

DEFINITION. Let A be a group and p be a prime. A *p-valuation* v_p on A is a function on A satisfying the following properties:

1) $v_p x$ is an ordinal or ∞
2) $v_p(x + y) \geqq \min(v_p x, v_p y)$
3) $v_p px > v_p x$
4) $v_p nx = v_p x$ if n is not divisible by p.

If A is a subgroup of B, then the p-height function on B, restricted to A, is a p-valuation on A. We will restrict our study to p-local valuated groups,

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that is, valuated groups A such that $A[q] = 0$ and $qA = A$, whenever $q \neq p$. Thus we will drop reference to the prime p, and speak of the valuation v on A, or v_A if there is need to specify the group. If $f\colon A \to B$ is a one-to-one mapping such that $vf(x) = vx$ for each $a \in A$, we say f is an *embedding*. A subgroup A of B is a *valuated subgroup* if the inclusion map is an embedding. A valuated subgroup A of B is called *nice* if each coset of A contains an element of maximum value. If λ is a function on A whose values are ordinals or ∞, there is always a least valuation on A which dominates λ. We can describe this valuation after fixing some notation. Let

$$p^\alpha A = \{x \in A\colon hx \geqq \alpha\}$$
$$A(\alpha) = \{x \in A\colon vx \geqq \alpha\}$$
$$A_\alpha = \{\Sigma r_i x_i\colon \lambda x_i \geqq \alpha \text{ for each } i\}.$$

LEMMA 1. ([**2**], Lemma 2.) *Let A be a p-local group and suppose λ is a function on A whose value is an ordinal or ∞. Then the smallest valuation on A such that $va \geqq \lambda a$ for each $a \in A$ is given inductively by*

$$A(\alpha + 1) = p(A(\alpha)) + A_{\alpha+1}$$
$$A(\beta) = \bigcap_{\alpha<\beta} A(\alpha) \text{ if } \beta \text{ is a limit ordinal}$$
$$va = \sup\{\alpha\colon a \in A(\alpha)\}.$$

The (p-local) valuated groups form a pre-abelian category. Thus every map has a kernel and a cokernel.

THEOREM 2. ([**2**], Theorem 3.) *If $f\colon A \to B$ is a map of valuated groups, then the kernel of f is the group kernel with the induced valuation from A. The cokernel of f is the group cokernel K with the smallest valuation v such that $vx \geqq \sup\{vb\colon x = \phi b\}$, where ϕ is the natural map of B onto K.*

A map $f\colon A \to B$ in the category of valuated groups is said to be a *semi-stable kernel* if for any pushout diagram

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{f'} & D \end{array}$$

the map f' is a kernel. f is a *semi-stable cokernel* if for any pullback diagram

$$\begin{array}{ccc} C & \xrightarrow{f'} & D \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

the map f' is a cokernel. A sequence $A \to^f B \to^g C$ is *exact* if $f = \ker g$ and $g = \operatorname{coker} f$. An exact sequence is *stable* if f is a semi-stable kernel and g is a semi-stable cokernel. In this case, f is called a *stable kernel* and g is a *stable cokernel.* The stable exact sequences constitute $\operatorname{Ext}(C, A)$.

The semi-stable kernels are those subgroups $A \subseteq B$ such that every pushout(*) is an embedding. Since $C \to^{f'} D$ is one-to-one, we may consider C as a subgroup of D. Thus $A \subseteq B$ is semi-stable if, whenever we have a pushout (*), $v_C c = v_D c$ for every $c \in C$.

Semi-stable cokernels are characterized by the following concept. An onto map $\phi: A \to B$ is *semi-nice* if whenever $b \in B$ and $\alpha < vb$, then there is $a \in A$ such that $\phi a = b$ and $va > \alpha$.

LEMMA 3. ([**2**], Lemma 5.) *A cokernel is semi-stable if and only if it is semi-nice.*

THEOREM 4. ([**2**], Theorem 6.) *The inclusion $A \subseteq B$ is a stable kernel if and only if it is a nice embedding.*

The following result is a partial characterization of semi-stable kernels.

THEOREM 5. ([**2**], Theorem 7.) *If $A \subseteq B$ is a semi-stable kernel, then every coset of finite order in B/A contains an element of maximum value.*

Examples showing that the converse of Theorem 5 is false and that a semi-stable kernel need not be nice are given in [**2**]. However, the question of classifying semi-stable kernels is left open. This question will be answered by Theorem 6.

3. **Semi-Stable Kernels.** In this section we characterize semi-stable kernels.

DEFINITION. A valuated subgroup A of B is *nearly-nice* if, for each element $b \in B$ satisfying

(1) $$\alpha = \sup_{a \in A} v(b + a) > v(b + a) \text{ for all } a \in A$$

then $v(p^{k+1}b + a) \leqq \alpha + k$, for all $a \in A$, $k \geqq 0$.

A necessary and sufficient condition for A being a valuated subgroup of B that is not nearly-nice is that there is an element $b \in B$ satisfying (1) and

(2) $$v(p^{k+1}b + e) > \alpha + k$$

for some $e \in A$ and $k \geqq 0$. In lieu of replacing b by $p^i b$ and k by $k - i$, where i is the least integer such that $v(p^{i+1}b + a) \geqq \alpha$ for some $a \in A$, we may assume that there exists $e' \in A$ such that

(3) $$v(pb + e') \geqq \alpha.$$

This characterization will be useful in the following theorem.

THEOREM 6. *Let A be a valuated subgroup of B. Then $A \to B$ is a semi-stable kernel if and only if A is nearly-nice in B.*

PROOF. First assume A is not nearly-nice in B. We will show $A \to B$ is not semi-stable. There are elements $b \in B$, $e \in A$, $e' \in A$, and $k \geqq 0$ satisfying (1), (2) and (3). We now construct a group

$$C = (A/A(\alpha)) \oplus [c],$$

where $[c]$ is an infinite cyclic group. If ϕ: $A \to (A/A(\alpha)) \oplus [c]$ maps a to $(a + A(\alpha), 0)$, C will be valuated as follows:

$$v_C(\phi p^i re' + p^{i+1} rc) = \alpha + i,$$

whenever $i \geqq 0$ and $p \nmid r$;

$$v_C(\phi a + rc) = v_B(a + rb)$$

otherwise. Whenever $\phi a = 0$, we must choose $a = 0$ as the representative in B. We first show that v_C is well defined. The hypothesis on b shows that if $v_B(a + sb) \geqq \alpha$, then $p | s$. If $v_B(a + psb) \geqq \alpha$, we claim that $a = se' + a'$, where $a' \in A(\alpha)$. This is true because

$$a - se' = (a + psb) - (se' + psb),$$

the difference of two elements whose value is at least α, and so $a - se' \in A(\alpha)$. In this case, the first formula defines the valuation of $v_C(\phi a + psc)$. On the other hand, if $v_B(a_1 + sb) < \alpha$ and $\phi a_1 = \phi a_2$, then

$$v_B(a_1 + sb) = v_B(a_1 - a_2 + a_2 + sb) = v_B(a_2 + sb),$$

since $v_B(a_1 - a_2) \geqq \alpha$. So v_C is well defined.

We now verify that v_C is a valuation. The only condition that merits discussion is

$$v_C(x + y) \geqq \min\{v_C x, v_C y\}.$$

If $\min\{v_C x, v_C y\} < \alpha$, the condition clearly holds, so suppose $v_C x \geqq \alpha$, $v_C y \geqq \alpha$. We may write

$$x = \phi p^i r_1 e' + p^{i+1} r_1 c, \; p \nmid r_1$$

$$y = \phi p^j r_2 e' + p^{j+1} r_2 c, \; p \nmid r_2$$

and assume $i \leqq j$. Then

$$v_C(x + y) = v_C(\phi p^i (r_1 + p^{j-i} r_2) e' + p^{i+1}(r_1 + p^{j-i} r_2) c) \geqq \alpha + i.$$

Therefore v_C is a valuation.

Let D be the valuated group completing the pushout diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \phi\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

Then $D = (B \oplus C)/H$, where

(4) $$H = \{(a, -\phi a) \in B \oplus C : a \in A\}.$$

The valuation on D is the cokernel valuation. We wish to show $C \to D$ is not an embedding. Since $v_C(\phi p^k e' + p^{k+1}c) = \alpha + k$, it will suffice to find some $k \geqq 0$ for which $v_D(\phi p^k e' + p^{k+1}c) > \alpha + k$. We have

$$\sup\{v_{B\oplus C}(-(b + a), \phi a + c)\} = \alpha,$$

since $v_C(\phi a + c) = v_B(-(b + a))$. Therefore

$$v_D(-p^{k+1}b + p^{k+1}c + H) > \alpha + k$$

whenever $k \geqq 0$. By (2), there is $k > 0$ and $e \in A$ such that

$$v_B(p^{k+1}b + e) > \alpha + k.$$

Hence $v_D(p^{k+1}b + e + H) > \alpha + k$. Since $e - p^k e' \in A(\alpha)$, we have $\phi e = \phi p^k e'$. The element

$$-(p^{k+1}b + e) + \phi e + p^{k+1}c = -(p^{k+1}b + e) + \phi p^k e' + p^{k+1}c$$

is a representative of $-p^{k+1}b + p^{k+1}c + H$. Thus $\phi p^k e' + p^{k+1}c$ is the sum of two elements of D, each with value greater than $\alpha + k$. Therefore

$$v_D(\phi p^k e' + p^{k+1}c) > \alpha + k,$$

so $A \to B$ is not semi-stable.

Conversely, suppose $A \to B$ is not semi-stable. Then there is a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \phi\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

and an element $c \in C$ such that $\alpha = v_C c < v_D c$. $D = (B \oplus C)/H$, where H is given by (4) and D has the cokernel valuation. We may assume α is the least ordinal γ for which there is $c' \in C$ for which $v_C c' < v_D c'$. Since $v_D(c) \geqq \alpha + 1$, we have

$$c = px + y$$

where $x \in D(\alpha)$ and $y \in D_{\alpha+1}$, by Lemma 1. There is a representative $b' + c'$ of y such that $v_{B\oplus C}(b' + c') \geqq \alpha + 1$. Let $b'' + c''$ be any representative of x. Considering c as an element of $B \oplus C$ and splitting into components, we have

$$0 = pb'' + b' + e$$

$$c = pc'' + c' - \phi e,$$

where e is some element of A. Because $v_B(b') > \alpha$, we have

$$v_B(pb'' + e) > \alpha.$$

We now wish to show $v_B(b'' + a) < \alpha$ for all $a \in A$. First suppose $v_{B\oplus C}(b'' + c'') \geqq \alpha$. Then $v_C(pc'') \geqq \alpha + 1$ and $v_B(pb'') \geqq \alpha + 1$. Since

$$c = pc'' + c' + \phi(pb'' + b')$$

we have $v_C(c) \geqq \alpha + 1$, a contradiction. Therefore $v_{B\oplus C}(b'' + c'') < \alpha$. Now suppose $v_B(b'') \geqq \alpha$. Since $v_D(b'' + c'' + H) \geqq \alpha$, we have $v_D(c'') \geqq \alpha$. Therefore $v_C c'' \geqq \alpha$ by the minimality of α. This implies $v_{B\oplus C}(b'' + c'') \geqq \alpha$, which is again a contradiction. Hence $v_B(b'') < \alpha$. Since $b'' + c''$ was chosen to be an arbitrary representative of x, we have $v_B(b'' + a) < \alpha$ for all $a \in A$.

We now define a sequence $\{X_i\}$ of subsets of B inductively. Let $X_1 = \{b''\}$. If X_{n-1} has already been defined, let $X_n = \{b \in B$: $v_B(p^j b + h) > v_B(b + a) + j$, for some $h \in A, j > 0$ and every $a \in A$, $pb + g = b_1 + b_2$, where $g \in A$, $b_1 \in X_{n-1}$ and satisfies

$$v_B(p^k b_1 + a_1) > \gamma + k$$

where a_1 is an element of A and

$$\gamma = \sup_{a\in A} v_B(b + a), \tag{5}$$

and b_2 is an element of B such that $v_B b_2 > \gamma\}$. Let $X = \bigcup_{1\leqq i<\omega} X_i$. If $b \in X$, and γ is defined by (5), then there is an element $e \in A$ and $k \geqq 0$ such that $v_B(p^{k+1}b + e) > \gamma + k$. This clearly holds for b'' with $k = 0$, so suppose $b \in X_i$, $i > 1$. Then

$$p^{k+1}b + p^k g + a_1 = p^k b_1 + a_1 + p^k b_2$$

is an element of the required form. Thus every element of X satisfies (2). If there is an element of X which also satisfies (1), the proof would be complete. So suppose that no elements of X satisfy (1). Then for each $x \in X$, there is $a_x \in A$ such that $v_B(x + a_x) = \sup_{a\in A} v_B(x + a)$. Let

$$Y = \{x \in X\colon v_D(x + c + H) > v_B(x + a_x) \text{ for some } c \in C\}.$$

Since $v_B(b'' + a) < \alpha$ for all $a \in A$ but $v_D(b'' + c'' + H) \geqq \alpha$, we have $b'' \in Y$. Hence Y is a non-empty set. Let β be the least ordinal such that $v_B(x + a_x) = \beta$ and $x \in Y$. Since $v_D(x + c + H) \geqq \beta + 1$ for some $c \in C$, we have

$$x + c + H \in pD(\beta) + D_{\beta+1}.$$

Therefore we may write

$$x + c + a_2 - \phi a_2 = p(b_4 + c_4) + b_3 + c_3,$$

where $v_{B\oplus C}(b_3 + c_3) \geqq \beta + 1$, $b_4 + c_4$ is an arbitrary representative of $b_4 + c_4 + H$, and $a_2 \in A$. In particular,

$$x + a_2 = pb_4 + b_3.$$

Thus $v_B pb_4 \leqq \beta$, and $v_B b_4 < \beta$. We claim $b_4 \in X$. Since b_4 was an arbitrary representative of its coset, $v_B(b_4 + a) < \beta$ for all $a \in A$. Therefore $\sup_{a\in A} v_B(b_4 + a) \leqq \beta$. Since $x \in X$, there are $h \in A, j > 0$ such that

$$v_B(p^j x + h) > \beta + j,$$

so $pb_4 - a_2 = x - b_3$ is the required representation. Moreover,

$$v_B(p^{j+1}b_4 - p^j a_2 + h) > \beta + j \geqq v_B(b_4 + a) + j + 1$$

for all $a \in A$. Thus $b_4 \in X$. It is now easily seen that $b_4 \in Y$, contradicting the minimality of β. This completes the proof of the theorem.

References

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