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ON THE EVALUATION OF $(\varepsilon_{q_1q_2}/p)$

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Let m be a positive squarefree integer. We denote the class number of $Q(\sqrt{-m})$ by $h(-m)$ and the fundamental unit of $Q(\sqrt{m})$ by ε_m. We consider only those m for which the norm of ε_m (written $N(\varepsilon_m)$) is -1, so that the only possible primes dividing m are the prime 2 or primes congruent to 1 modulo 4. Now, if p is an odd prime such that $(m/p) = +1$, we can interpret ε_m as an integer modulo p, and ask for the value of the Legendre symbol (ε_m/p). Because of the ambiguity in the choice of $\sqrt{m}$ taken modulo p, we must ensure that (ε_m/p) is well-defined. Since

$$\left(\frac{-1}{p}\right) = \left(\frac{N(\varepsilon_m)}{p}\right) = \left(\frac{\varepsilon_m\varepsilon_m'}{p}\right) = \left(\frac{\varepsilon_m}{p}\right)\left(\frac{\varepsilon_m'}{p}\right),$$

where the prime (′) indicates conjugation ($\sqrt{m} \to -\sqrt{m}$), this will be the case if $(-1/p) = +1$, that is, if $p \equiv 1 \pmod 4$. Thus it is assumed throughout that

$$\left(\frac{-1}{p}\right) = \left(\frac{m}{p}\right) = +1.$$

Suppose m has the prime decomposition $m = q_1 \ldots q_s$, and let a denote the number of ambiguous classes of forms of discriminant $-4m$ in the principal genus. Then, from genus theory, we know that

$$b = \begin{cases} 2^s a, & \text{if } m \text{ odd}, \\ 2^{s-1} a, & \text{if } m \text{ even}, \end{cases}$$

is an integer dividing $h(-m)$, and we define a positive integer l by

$$l = h(-m)/b.$$

We restrict our attention to primes (congruent to 1 modulo 4) represented by forms in genera containing ambiguous classes, so that p^l is represented by an ambiguous form. For such primes p, when m is a prime or twice a prime, the evaluation of (ε_m/p) is known, except in one case. In these cases, the generic characters are given by (for $k > 0$)

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$$\chi_1(k) = \left(\frac{-2}{k}\right), \qquad m = 2,$$

$$\chi_1(k) = \left(\frac{-1}{k}\right),\ \chi_2(k) = \left(\frac{k}{q}\right), \qquad m = q(\text{prime}) \equiv 1 \pmod 4,$$

$$\chi_1(k) = \left(\frac{-2}{k}\right),\ \chi_2(k) = \left(\frac{k}{q}\right), \qquad m = 2q,\ q(\text{prime}) \equiv 1 \pmod 4,$$

and the ambiguous forms of discriminant $-4m$ are given by

$$\begin{array}{ll} \mathrm{I} = (1, 0, 2), & m = 2, \\ \mathrm{I} = (1, 0, q),\ A = (2, 2, \tfrac{1}{2}(q+1)), & m = q, \\ \mathrm{I} = (1, 0, 2q),\ A = (2, 0, q), & m = 2q, \end{array}$$

where (r, s, t) denotes the form $rx^2 + sxy + ty^2$.

We remark that $N(\varepsilon_m) = -1$ when $m = 2$; when $m = q$ (prime) $\equiv 1 \pmod 4$ (Dirichlet [**6**: p. 225]); and when $m = 2q$, q (prime) $\equiv 5 \pmod 8$ (Dirichlet [**6**: p. 226]). $m = 2q$, q (prime) $\equiv 1 \pmod 8$ is the only case which requires the assumption that the norm of the fundamental unit be -1. In this case, the assumption $h = h(-2q) \equiv 4 \pmod 8$ has also to be made, as Lehmer's results [**11**: Theorems 2 and 3] require that $h/4$ be odd. What happens when $h \equiv 0 \pmod 8$ remains open. Both possibilities occur as $N(\varepsilon_{2\cdot 41}) = N(\varepsilon_{2\cdot 113}) = -1$, $h(-82) = 4$, $h(-226) = 8$. Writing h for $h(-m)$ the results in the known cases can be summarized as follows:

m	Assumptions	Evaluation of $\left(\frac{\varepsilon_m}{p}\right)$	References
2	$\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=1$	$(-1)^{y/2}$, if $p=x^2+2y^2$	[**1**]
$q \equiv 1 \pmod 8$	$\left(\frac{-1}{p}\right)=\left(\frac{q}{p}\right)=1$	$+1$, if $p^{h/4}=x^2+qy^2$ -1, if $2p^{h/4}=x^2+qy^2$	[**14**] [**5**]
$q \equiv 5 \pmod 8$	$\left(\frac{-1}{p}\right)=\left(\frac{q}{p}\right)=1$	$(-1)^y$, if $p^{h/2}=x^2+qy^2$	[**11**] [**13**]
$2q$ $q \equiv 1 \pmod 8$	$\left(\frac{-1}{p}\right)=\left(\frac{2}{p}\right)=\left(\frac{q}{p}\right)=1$ $N(\varepsilon_{2q})=-1$ $h \equiv 4 \pmod 8$	$(-1)^{y/2}$, if $p^{h/4}=x^2+2qy^2$ $(-1)^{x/2}$, if $p^{h/4}=2x^2+qy^2$	[**11**]
$2q$ $q \equiv 5 \pmod 8$	$\left(\frac{-1}{p}\right)=\left(\frac{2q}{p}\right)=1$	$(-1)^{y/2}$, if $p^{h/2}=x^2+2qy^2$ $(-1)^{x/2+1}$, if $p^{h/2}=2x^2+qy^2$	[**11**] [**13**]

It is the purpose of this paper to discuss the remaining cases when m has exactly two prime factors, that is, $m = q_1q_2$, where q_1 and q_2 are distinct primes congruent to 1 (mod 4).

In the unique factorization domain $Z[i]$ of Gaussian integers, we have $q_1 = \pi_1\bar{\pi}_1$, $q_2 = \pi_2\bar{\pi}_2$, where π_1 and π_2 are primes, which we can take to be primary, that is, to satisfy $\pi_1 \equiv \pi_2 \equiv 1 \pmod{(1+i)^3}$. Now either $\varepsilon_{q_1q_2}$ or $\varepsilon_{q_1q_2}{}^3$ is of the form $T + U\sqrt{q_1q_2}$, where T and U are positive integers with T even and U odd. Since $N(T + U\sqrt{q_1q_2}) = -1$, we have, for $j = 1, 2$, $\pi_j|(T+i)(T-i)$, that is, $\pi_j|T \pm i$, as π_j is prime. Replacing π_j by its complex conjugate $\bar{\pi}_j$, if necessary, we can assume

$$\pi_j | T + i \quad (j = 1, 2).$$

Writing $[\ \ /\pi_j]_2$ (resp. $[\ \ /\pi_j]_4$) for the quadratic (resp. biquadratic) residue symbol (mod π_j), and $(\ \ /p)_4$ for the rational biquadratic symbol (mod p) (p an odd prime), we have

THEOREM 1. *If p, q_1, q_2 are distinct primes congruent to* 1 (mod 4), *such that $(q_1q_2/p) = +1$, then*

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left[\frac{p}{\pi_1}\right]_4 \left[\frac{p}{\pi_2}\right]_4 \left(\frac{q_1q_2}{p}\right)_4,$$

where π_1, π_2 are defined as above. (Compare Furuta [7: Theorem 3])

PROOF. As T is even, $(T+i)/\pi_1\pi_2$, and $(T-i)/\bar{\pi}_1\bar{\pi}_2$ are *coprime* Gaussian integers. Since their product is U^2, by the unique factorization property, $(T+i)/\pi_1\pi_2$ must be an associate of a square, say,

$$T + i = u\pi_1\pi_2\alpha^2,$$

where u is a unit of $Z[i]$, that is, $u = \pm 1, \pm i$. Reducing this equation modulo 2, we obtain $u \equiv i \pmod 2$, so that $u = \pm i$. Replacing α by $i\alpha$, if necessary, we have

$$T + i = i\pi_1\pi_2\alpha^2. \tag{1}$$

As $U > 0$, $\alpha\bar{\alpha} > 0$, this gives $U = \alpha\bar{\alpha}$. Hence, from

$$2(T+i)(T + U\sqrt{q_1q_2}) = (T + i + U\sqrt{q_1q_2})^2,$$

we have

$$(1+i)^2\pi_1\pi_2(T + U\sqrt{q_1q_2}) = (i\pi_1\pi_2\alpha + \bar{\alpha}\sqrt{q_1q_2})^2. \tag{2}$$

Let π be a primary prime factor of p in $Z[i]$, so that $p = \pi\bar{\pi}$, $\pi \equiv \bar{\pi} \equiv 1 \pmod{(1+i)^3}$. Interpreting $\sqrt{q_1q_2}$ as an integer modulo p, we have from (2)

$$\begin{aligned}
\left(\frac{\varepsilon_{q_1q_2}}{p}\right) &= \left(\frac{T+U\sqrt{q_1q_2}}{p}\right) = \left[\frac{T+U\sqrt{q_1q_2}}{\pi}\right]_2 \\
&= \left[\frac{\pi_1\pi_2}{\pi}\right]_2 = \left[\frac{\pi_1}{\pi}\right]_2\left[\frac{\pi_2}{\pi}\right]_2 \\
&= \left[\frac{\pi}{\pi_1}\right]_2\left[\frac{\pi}{\pi_2}\right]_2 \text{ (by the law of quadratic reciprocity in } Z[i]) \\
&= \left[\frac{\pi}{\pi_1}\right]_4^2\left[\frac{\pi}{\pi_2}\right]_4^2 \cdot \left[\frac{\bar\pi}{\pi_1}\right]_4\left[\frac{\bar\pi}{\pi_1}\right]_4 \cdot \left[\frac{\bar\pi}{\pi_2}\right]_4\left[\frac{\bar\pi}{\pi_2}\right]_4 \\
&= \left[\frac{\pi}{\pi_1}\right]_4\left[\frac{\bar\pi}{\pi_1}\right]_4 \cdot \left[\frac{\pi}{\pi_2}\right]_4\left[\frac{\bar\pi}{\pi_2}\right]_4 \cdot \left[\frac{\pi}{\pi_1}\right]_4\left[\frac{\pi}{\bar\pi_1}\right]_4 \cdot \left[\frac{\pi}{\pi_2}\right]_4\left[\frac{\pi}{\bar\pi_2}\right]_4 \\
&= \left[\frac{\pi\bar\pi}{\pi_1}\right]_4\left[\frac{\pi\bar\pi}{\pi_2}\right]_4\left[\frac{\pi}{\pi_1\bar\pi_1\pi_2\bar\pi_2}\right]_4 = \left[\frac{p}{\pi_1}\right]_4\left[\frac{p}{\pi_2}\right]_4\left[\frac{\pi}{q_1q_2}\right]_4 \\
&= \left[\frac{p}{\pi_1}\right]_4\left[\frac{p}{\pi_2}\right]_4\left[\frac{q_1q_2}{\pi}\right]_4 \text{ (by the law of biquadratic reciprocity in } Z[i]) \\
&= \left[\frac{p}{\pi_1}\right]_4\left[\frac{p}{\pi_2}\right]_4\left(\frac{q_1q_2}{p}\right)_4.
\end{aligned}$$

COROLLARY 1. *If* p, q_1, q_2 *are distinct primes congruent to* 1 modulo 4, *such that* $(q_1/p) = (q_2/p) = +1$, *then*

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left(\frac{p}{q_1}\right)_4\left(\frac{q_1}{p}\right)_4\left(\frac{p}{q_2}\right)_4\left(\frac{q_2}{p}\right)_4.$$

(Furuta [7: Corollary, p. 143])

PROOF. As $(q_1/p) = (q_2/p) = 1$, we have $(q_1q_2/p)_4 = (q_1/p)_4(q_2/p)_4$, and by the law of quadratic reciprocity $(p/q_1) = (p/q_2) = 1$, so $[p/\pi_1]_4 = (p/q_1)_4$, $[p/\pi_2]_4 = (p/q_2)_4$. The result now follows immediately from Theorem 1.

COROLLARY 2. *If* p, q_1, q_2 *are distinct primes congruent to* 1 modulo 4, *such that* $(q_1/p) = (q_2/p) = -1$, $(q_1/q_2) = -1$, *then*

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = -\left(\frac{pq_1}{q_2}\right)_4\left(\frac{pq_2}{q_1}\right)_4\left(\frac{q_1q_2}{p}\right)_4.$$

PROOF. As $(q_2/q_1) = -1$, we have $[q_2/\pi_1]_2 = -1$, that is,

$$[\pi_2/\pi_1]_2\,[\bar\pi_2/\pi_1]_2 = -1.$$

Now, from $T + i = i\pi_1\pi_2\alpha^2$, we have

$$2 = \pi_1\pi_2\alpha^2 + \bar{\pi}_1\bar{\pi}_2\bar{\alpha}^2,$$

so

$$2 \equiv \bar{\pi}_1\bar{\pi}_2\bar{\alpha}^2 \pmod{\pi_1},$$

giving

$$[2/\pi_1]_2 = [\bar{\pi}_1/\pi_1]_2[\bar{\pi}_2/\pi_1]_2 = [2/\pi_1]_2[\bar{\pi}_2/\pi_1]_2,$$

that is,

$$[\bar{\pi}_2/\pi_1]_2 = +1, \ [\pi_2/\pi_1]_2 = -1.$$

Hence we have

$$\begin{aligned}[\pi_1/\pi_2]_4\,[\pi_2/\pi_1]_4 &= [\pi_1/\pi_2]_4\overline{[\bar{\pi}_2/\bar{\pi}_1]_4} = [\pi_1/\pi_2]_4[\bar{\pi}_2/\bar{\pi}_1]_4^3 \\ &= [\pi_1/\pi_2]_4[\bar{\pi}_2/\bar{\pi}_1]_2[\bar{\pi}_2/\bar{\pi}_1]_4 \\ &= -[\pi_1/\pi_2]_4[\bar{\pi}_2/\bar{\pi}_1]_4,\end{aligned}$$

that is,

$$\left[\frac{\pi_1}{\bar{\pi}_2}\right]_4\left[\frac{\bar{\pi}_2}{\pi_1}\right]_4 = -(-1)^{\frac{q_1-1}{4}\cdot\frac{q_2-1}{4}}, \tag{3}$$

by the law of biquadratic reciprocity in $Z[i]$. Also, by the law of biquadratic reciprocity in $Z[i]$, we have

$$\left[\frac{\bar{\pi}_1}{\pi_2}\right]_4\left[\frac{\pi_2}{\bar{\pi}_1}\right]_4 = (-1)^{\frac{q_1-1}{4}\cdot\frac{q_2-1}{4}}. \tag{4}$$

Multiplying (3) and (4) together, we obtain

$$[q_1/\pi_2]_4[q_2/\pi_1]_4 = -1,$$

and Theorem 1 gives

$$\begin{aligned}(\varepsilon_{q_1q_2}/p) &= [p/\pi_1]_4[p/\pi_2]_4(q_1q_2/p)_4 \\ &= -[pq_1/\pi_2]_4[pq_2/\pi_1]_4(q_1q_2/p)_4, \\ &= -(pq_1/q_2)_4(pq_2/q_1)_4(q_1q_2/p)_4,\end{aligned}$$

as required.

We are now in a position to obtain the explicit evaluation of $(\varepsilon_{q_1q_2}/p)$, when p^l is represented by an ambiguous form of discriminant $-4q_1q_2$. This is done, following ideas of Lehmer [**11**: pp. 369–371], by using the representation of p^l to compute the residue symbols appearing in the expression for $(\varepsilon_{q_1q_2}/p)$ given in Theorem 1 or its corollaries. Many of the details are suppressed, as the calculations parallel those given by Lehmer. As in Lehmer's work, we require that l be odd, and an assumption to this effect is made wherever necessary. The results, which constitute Theorem 2, are given in the Table.

TABLE

Case	$m=q_1q_2$	Assumptions	Evaluation of $\left(\frac{\varepsilon_m}{p}\right)$
I	$q_1 \equiv q_2 \equiv 1 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=+1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$ $N(\varepsilon_m)=-1$ $h \equiv 16 \pmod{32}$	$+1$, if $p^{h/16}=x^2+q_1q_2y^2$ or $q_1x^2+q_2y^2$ -1, if $2p^{h/16}=x^2+q_1q_2y^2$ or $q_1x^2+q_2y^2$
II	$q_1 \equiv q_2 \equiv 1 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=-1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$ $h \equiv 8 \pmod{16}$	$+1$, if $p^{h/8}=x^2+q_1q_2y^2$ -1, if $2p^{h/8}=x^2+q_1q_2y^2$
		$\left(\frac{-1}{p}\right)=+1$, $\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=-1$ $h \equiv 8 \pmod{16}$	$+1$, if $p^{h/8}=q_1x^2+q_2y^2$ -1, if $2p^{h/8}=q_1x_2+q_2y^2$
III	$q_1 \equiv 1$, $q_2 \equiv 5 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=+1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$ $N(\varepsilon_m)=-1$ $h \equiv 8 \pmod{16}$	$(-1)^y$, if $p^{h/8}=x^2+q_1q_2y^2$ or $q_1x^2+q_2y^2$
IV	$q_1 \equiv 1$, $q_2 \equiv 5 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=-1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$	$(-1)^y$, if $p^{h/4}=x^2+q_1q_2y^2$
		$\left(\frac{-1}{p}\right)=+1$, $\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=-1$	$(-1)^y$, if $p^{h/4}=q_1x^2+q_2y^2$
V	$q_1 \equiv q_2 \equiv 5 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=+1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$ $N(\varepsilon_m)=-1$	$+1$, if $p^{h/8}=x^2+q_1q_2y^2$ -1, if $p^{h/8}=q_1x^2+q_2y^2$
		$\left(\frac{-1}{p}\right)=+1$, $\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=-1$ $N(\varepsilon_m)=-1$	$(-1)^{T/4}$, if $2p^{h/8}$ $=x^2+q_1q_2y^2$ $(-1)^{T/4+1}$, if $2p^{h/8}$ $=q_1x^2+q_2y^2$
VI	$q_1 \equiv q_2 \equiv 5 \pmod 8$ $\left(\frac{q_1}{q_2}\right)=-1$	$\left(\frac{-1}{p}\right)=\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=+1$ $h \equiv 8 \pmod{16}$	$+1$ if $p^{h/8}=x^2+q_1q_2y^2$ -1, if $2p^{h/8}=q_1x^2+q_2y^2$
		$\left(\frac{-1}{p}\right)=+1$, $\left(\frac{q_1}{p}\right)=\left(\frac{q_2}{p}\right)=-1$ $h \equiv 8 \pmod{16}$	$+1$, if $2p^{h/8}=x^2+q_1q_1y^2$ -1, if $\cdot^{h/8}=q_1x^2+q_2y^2$

N.B. T is defined by $\varepsilon_m^\lambda = T+U\sqrt{m}$, $\lambda=1$ or 3

h is the classnumber of $Q(\sqrt{-m})$.

All representations are primitive.

Let I, A, B, C denote the classes of the forms $[1, 0, q_1q_2]$, $[2, 2, \frac{1}{2}(q_1q_2+1)]$, $[q_1, 0, q_2]$, $[2q_1, 2q_1, \frac{1}{2}(q_1+q_2)]$ respectively. These are precisely the ambiguous classes of forms of discriminant $-4q_1q_2$, so that the classes of forms of discriminant $-4q_1q_2$ fall into 4 genera. The generic characters are $\chi_1(k) = (-1/k)$, $\chi_2(k) = (k/q_1)$, $\chi_2(k) = (k/q_2)$ $(k > 0)$. The six cases appearing in the table are treated below.

CASE I. $q_1 \equiv q_2 \equiv 1 \pmod 8$, $(q_1/q_2) = +1$. In this case I, A, B, C are all in the principal genus, so that $h = h(-q_1q_2) \equiv 0 \pmod{16}$ (Brown [4: Theorem 1]). Thus, if p is a prime, such that $(-1/p) = (q_1/p) = (q_2/p) = 1$, there are positive coprime integers x and y such that $p^l = x^2 + q_1q_2y^2$, $2x^2+2xy+\frac{1}{2}(q_1q_2+1)y^2$, $q_1x^2+q_2y^2$, or $2q_1x^2+2q_1xy+\frac{1}{2}(q_1+q_2)y^2$; that is, there are positive coprime integers x and y such that

$$p^l \text{ or } 2p^l = x^2 + q_1q_2y^2 \text{ or } q_1x^2 + q_2y^2,$$

where $l = h/16$. We now assume that $N(\varepsilon_{q_1q_2}) = -1$ and $h \equiv 16 \pmod{32}$ (so that l is odd). These are two independent assumptions since: $N(\varepsilon_{41\cdot 241}) = -1$ and $h(-41\cdot 241) = 112 \equiv 16 \pmod{32}$, whereas $N(\varepsilon_{17\cdot 89}) = +1$ and $h(-17\cdot 89) = 16$; also $N(\varepsilon_{17\cdot 281}) = -1$ and $h(-17\cdot 281) = 32$, whereas $N(\varepsilon_{17\cdot 137}) = +1$ and $h(-17\cdot 137) = 32$.

Taking $p^l = x^2 + q_1q_2y^2$ modulo p, q_1 and q_2, we obtain

$$(q_1/p)_4(q_2/p)_4 = (2/p)(x/p)(y/p),$$

$$(p/q_1)_4 = (x/q_1),\ (p/q_2)_4 = (x/q_2),$$

so that, by Corollary 1, we have

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\left(\frac{x}{q_1}\right)\left(\frac{x}{q_2}\right).$$

Next we set

$$x = 2^\alpha x_1,\ x_1 \equiv 1 \pmod 2,\ \alpha \geqq 0,$$

$$y = 2^\beta y_1,\ y_1 \equiv 1 \pmod 2,\ \beta \geqq 0.$$

By the law of quadratic reciprocity, we have (as l is odd)

$$(x/p) = (2/p)^\alpha(x_1/p) = (2/p)^\alpha(p/x_1) = (2/p)^\alpha(p^l/x_1) = (2/p)^\alpha(q_1/x_1)(q_2/x_1),$$

$$(y/p) = (2/p)^\beta(y_1/p) = (2/p)^\beta(p/y_1) = (2/p)^\beta(p^l/y_1) = (2/p)^\beta,$$

$$(x/q_1) = (2/q_1)^\alpha(x_1/q_1) = (x_1/q_1),\ (x/q_2) = (2/q_2)^\alpha(x_1/q_2) = (x_1/q_2),$$

giving

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left(\frac{2}{p}\right)^{1+\alpha+\beta}.$$

If $p \equiv 1 \pmod 8$, $(2/p) = +1$, so $(\varepsilon_{q_1q_2}/p) = +1$; if $p \equiv 5 \pmod 8$, then $\alpha + \beta = 1$, and again $(\varepsilon_{q_1q_2}/p) = +1$.

Similarly, using $p^l = q_1x^2 + q_2y^2$ in Corollary 1, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left(\frac{q_1}{q_2}\right)_4\left(\frac{q_2}{q_1}\right)_4.$$

But, as $N(\varepsilon_{q_1q_2}) = -1$, we have $(q_1/q_2)_4(q_2/q_1)_4 = +1$ (Brown [**2**: Lemma 4]), so that $(\varepsilon_{q_1q_2}/p) = +1$.

Using $2p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain, using the easily proved result $(2/p)(2/x)(2/y) = (-1)^{(q_1+q_2-2/8)}$,

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)}\left(\frac{2}{q_1}\right)_4\left(\frac{2}{q_2}\right)_4 = \left(\frac{e}{q_1}\right)\left(\frac{e}{q_2}\right),$$

where d, e are positive odd integers defined by $q_1q_2 = 2e^2 - d^2$. As $(q_1/q_2)_4(q_2/q_1)_4 = +1$ (since $N(\varepsilon_{q_1q_2}) = -1$) and $h(-q_1q_2) \equiv 16 \pmod{32}$, we have $(e/q_1)(e/q_2) = -1$ (Kaplan [**9**: Prop. C_1']), so that $(\varepsilon_{q_1q_2}/p) = -1$.

Using $2p^l = q_1x^2 + q_2y^2$ in Corollary 1, we obtain in a similar manner

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)}\left(\frac{2}{q_1}\right)_4\left(\frac{2}{q_2}\right)_4\left(\frac{q_1}{q_2}\right)_4\left(\frac{q_2}{q_1}\right)_4 = \left(\frac{e}{q_1}\right)\left(\frac{e}{q_2}\right) = -1,$$

CASE II. $q_1 \equiv q_2 \equiv 1 \pmod 8$, $(q_1/q_2) = -1$. In this case I, A are in the principal genus and B, C are in the non-principal genus for which $\chi_1 = +1$, so that $h = h(-q_1q_2) \equiv 0 \pmod 8$ (Brown [**4**: Theorem 1]). Thus, if p is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = 1$, there are positive coprime integers x and y such that

$$p^l \text{ or } 2p^l = x^2 + q_1q_2y^2,$$

where $l = h/8$, and, if $(-1/p) = 1$, $(q_1/p) = (q_2/p) = -1$, such that

$$p^l \text{ or } 2p^l = q_1x^2 + q_2y^2.$$

As $(q_1/q_2) = -1$ we have $N(\varepsilon_{q_1q_2}) = -1$ (Dirichlet [**6**: p. 228]), and we assume that $h \equiv 8 \pmod{16}$ (so that l is odd). The example $q_1 = 17$, $q_2 = 73$, $h = h(-1241) = 32$, shows that this is a genuine assumption.

Using $p^l = x^2 + q_1q_2y^2$ in Corollary 1 we obtain $(\varepsilon_{q_1q_2}/p) = +1$.

Using $2p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)/8}\left(\frac{2}{q_1}\right)_4\left(\frac{2}{q_2}\right)_4,$$

the right hand side of which is -1, as $h(-q_1q_2) \equiv 8 \pmod{16}$ (Kaplan [**9**: Prop. B_2']).

Using $p^l = q_1x^2 + q_2y^2$ in Corollary 2 we obtain $(\varepsilon_{q_1q_2}/p) = +1$.
Finally, using $2p^l = q_1x^2 + q_2y^2$ in Corollary 2, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)/8}\left(\frac{2}{q_1}\right)_4\left(\frac{2}{q_2}\right)_4,$$

the right hand side of which is -1, as $h(-q_1q_2) \equiv 8 \pmod{16}$ (Kaplan [**9**: Prop. B_2']).

Case III. $q_1 \equiv 1$, $q_2 \equiv 5 \pmod 8$, $(q_1/q_2) = +1$. In this case I, B are in the principal genus and A, C are in a non-principal genus for which $\chi_1 = -1$. We have $h = h(-q_1q_2) \equiv 0 \pmod 8$ (Brown [**4**: Theorem 1]). Thus, if p is a prime for which $(-1/p) = (q_1/p) = (q_2/p) = +1$, there are positive coprime integers x and y such that

$$p^l = x^2 + q_1q_2y^2 \text{ or } q_1x^2 + q_2y^2,$$

where $l = h/8$. We now assume that $N(\varepsilon_{q_1q_2}) = -1$ and $h \equiv 8 \pmod{16}$ (so that l is odd).

These are two independent assumptions since: $N(\varepsilon_{17\cdot 53}) = -1$ and $h(-17\cdot 53) = 24 \equiv 8 \pmod{16}$, whereas $N(\varepsilon_{17\cdot 229}) = +1$ and $h(-17\cdot 229) = 40 \equiv 8 \pmod{16}$; also $N(\varepsilon_{1601\cdot 5}) = -1$ and $h(-1601\cdot 5) = 48 \equiv 0 \pmod{16}$, whereas $N(\varepsilon_{17\cdot 13}) = +1$ and $h(-17\cdot 13) = 16 \equiv 0 \pmod{16}$.

Using $p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain $(\varepsilon_{q_1q_2}/p) = (-1)^y$. Using $p^l = q_1x^2 + q_2y^2$ in Corollary 1, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^y\left(\frac{q_1}{q_2}\right)_4\left(\frac{q_2}{q_1}\right)_4.$$

As $N(\varepsilon_{q_1q_2}) = -1$, we have $(q_1/q_2)_4(q_2/q_1)_4 = +1$ (Brown [**2**: Lemma 4]), so that $(\varepsilon_{q_1q_2}/p) = (-1)^y$.

Case IV. $q_1 \equiv 1$, $q_2 \equiv 5 \pmod 8$, $(q_1/q_2) = -1$. In this case I, A, B, C are each in different genera, with I in the principal genus and B in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 4 \pmod 8$ (Brown [**4**: Theorem 1]). Thus, if p is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = 1$, there exist positive coprime integers x and y such that $p^l = x^2 + q_1q_2y^2$, where $l = h/4$ is odd, and such that $p^l = q_1x^2 + q_2y^2$, if $(-1/p) = 1$, $(q_1/p) = (q_2/p) = -1$. As $(q_1/q_2) = -1$, a theorem of Dirichlet [**6**: p. 228] guarantees that $N(\varepsilon_{q_1q_2}) = -1$. Using $p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain $(\varepsilon_{q_1q_2}/p) = (-1)^y$, and using $p^l = q_1x^2 + q_2y^2$ in Corollary 2, we also obtain $(\varepsilon_{q_1q_2}/p) = (-1)^y$.

Case V. $q_1 \equiv q_2 \equiv 5 \pmod 8$, $(q_1/q_2) = +1$. In this case I, B are in the principal genus and A, C are in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 0 \pmod 8$ (Brown [**4**: Theorem 1]). Thus, if p

is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = 1$, there exist positive coprime integers x and y such that $p^l = x^2 + q_1q_2y^2$ or $q_1x^2 + q_2y^2$; and, if $(-1/p) = 1$, $(q_1/p) = (q_2/p) = -1$, such that $2p^l = x^2 + q_1q_2y^2$ or $q_1x^2 + q_2y^2$, where $l = h/8$. We assume that $N(\varepsilon_{q_1q_2}) = -1$, so that by a theorem of Brown [**2**: Lemma 4] we have $(q_1/q_2)_4 \cdot (q_2/q_1)_4 = 1$, and hence by a theorem of Kaplan [**9**: Prop. B_4'] we have $h \equiv 8 \pmod{16}$, so that l is odd. Using $p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain $(\varepsilon_{q_1q_2}/p) = +1$, and using $p^l = q_1x^2 + q_2y^2$ in the same corollary we obtain $(\varepsilon_{q_1q_2}/p) = -(q_1/q_2)_4(q_2/q_1)_4 = -1$.

When $2p^l = x^2 + q_1q_2y^2$ or $q_1x^2 + q_2y^2$ the evaluation of $(\varepsilon_{q_1q_2}/p)$ appears to be more difficult. It was originally hoped to give a third corollary to Theorem 1 expressing $(\varepsilon_{q_1q_2}/p)$ in terms of $(2p/q_1)_4(2p/q_2)_4(q_1q_2/p)_4$ when p, q_1, q_2 are distinct primes congruent to 1 modulo 4, and such that $(q_1/p) = (q_2/p) = -1$, $(q_1/q_2) = +1$, $q_1 \equiv q_2 \equiv 5 \pmod 8$. No such representation was found, and so instead we apply Theorem 1 directly.

If $2p^l = x^2 + q_1q_2y^2$ we have

$$\begin{aligned}
\left(\frac{\varepsilon_{q_1q_2}}{p}\right) &= \left[\frac{p}{\pi_1}\right]_4\left[\frac{p}{\pi_2}\right]_4\left(\frac{q_1q_2}{p}\right)_4 \\
&= \left[\frac{2}{\pi_1}\right]_4^3\left[\frac{p^{l-1}}{\pi_1}\right]_4^3\left[\frac{2p^l}{\pi_1}\right]_4 \cdot \left[\frac{2}{\pi_2}\right]_4^3\left[\frac{p^{l-1}}{\pi_2}\right]_4^3\left[\frac{2p^l}{\pi_2}\right]_4 \cdot \left(\frac{q_1q_2}{p}\right)_4 \\
&= \left[\frac{2}{\pi_1}\right]_4\left[\frac{2}{\pi_1}\right]_2\left[\frac{p}{\pi_1}\right]_2^{3(l-1)/2}\left[\frac{x}{\pi_1}\right]_2 \cdot \left[\frac{2}{\pi_2}\right]_4\left[\frac{2}{\pi_2}\right]_2\left[\frac{p}{\pi_2}\right]_2^{3(l-1)/2}\left[\frac{x}{\pi_2}\right]_2 \cdot \left(\frac{q_1q_2}{p}\right)_4 \\
&= \left[\frac{2}{\pi_1}\right]_4\left(\frac{2}{q_1}\right)\left(\frac{p}{q_1}\right)^{3(l-1)/2}\left(\frac{x}{q_1}\right) \cdot \left[\frac{2}{\pi_2}\right]_4\left(\frac{2}{q_2}\right)\left(\frac{p}{q_2}\right)^{3(l-1)/2}\left(\frac{x}{q_2}\right) \cdot \left(\frac{q_1q_2}{p}\right)_4 \\
&= \left[\frac{2}{\pi_1}\right]_4\left[\frac{2}{\pi_2}\right]_4\left(\frac{x}{q_1}\right)\left(\frac{x}{q_2}\right)\left(\frac{2}{p}\right)\left(\frac{x}{p}\right)\left(\frac{y}{p}\right),
\end{aligned}$$

as $\left(\frac{2}{q_1}\right) = \left(\frac{2}{q_2}\right) = \left(\frac{p}{q_1}\right) = \left(\frac{p}{q_2}\right) = -1$, $\left(\frac{q_1q_2}{p}\right)_4 = \left(\frac{2}{p}\right)\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$.

Now, by Jacobi's form of the law of quadratic reciprocity, we have (as l is odd)

$$\left(\frac{x}{p}\right) = \left(\frac{p}{x}\right) = \left(\frac{2}{x}\right)\left(\frac{2p^l}{x}\right) = \left(\frac{2}{x}\right)\left(\frac{q_1q_2y^2}{x}\right) = \left(\frac{2}{x}\right)\left(\frac{x}{q_1}\right)\left(\frac{x}{q_2}\right),$$

$$\left(\frac{y}{p}\right) = \left(\frac{p}{y}\right) = \left(\frac{2}{y}\right)\left(\frac{2p^l}{y}\right) = \left(\frac{2}{y}\right)\left(\frac{x^2}{y}\right) = \left(\frac{2}{y}\right),$$

so

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = \left[\frac{2}{\pi_1}\right]_4\left[\frac{2}{\pi_2}\right]_4\left(\frac{2}{p}\right)\left(\frac{2}{x}\right)\left(\frac{2}{y}\right) = (-1)^{(q_1+q_2-2)/8}\left[\frac{2}{\pi_1\pi_2}\right]_4.$$

Setting $\alpha = g + hi$, where α is defined by (1), we have

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)/8}\left[\frac{2}{\pi_1\pi_2\alpha^2}\right]_4\left[\frac{2}{\alpha}\right]_2$$
$$= (-1)^{(q_1+q_2-2)/8}\left[\frac{2}{1-Ti}\right]_4\left[\frac{2}{g+hi}\right]_2 \text{(by (1))}$$
$$= (-1)^{(q_1+q_2-2)/8+T/4+h/2},$$

by the supplements to the laws of quadratic and biquadratic reciprocity in $Z[i]$, since $T \equiv 0 \pmod 4$ in this case. As π_j $(j = 1, 2)$ is a primary prime factor of q_j $(j = 1, 2)$, we have $\pi_j = a_j + ib_j$, $a_j \equiv 1 \pmod 2$, $b_j \equiv 0 \pmod 2$, $a_j + b_j - 1 \equiv 0 \pmod 4$, $a_j^2 + b_j^2 = q_j$. Since $q_j \equiv 5 \pmod 8$, we have, for $j = 1, 2$,

$$\begin{cases} a_j \equiv 7 \pmod 8, & b_j \equiv 2 \pmod 4, \quad \text{if } q_j \equiv 5 \pmod{16}, \\ a_j \equiv 3 \pmod 8, & b_j \equiv 2 \pmod 4, \quad \text{if } q_j \equiv 13 \pmod{16}. \end{cases}$$

Set $a + ib = \pi_1\pi_2$, so we have

$$a = a_1a_2 - b_1b_2,\ b = a_1b_2 + a_2b_1.$$

Clearly we have

$$a \equiv 5 \pmod 8, \quad b \equiv 0 \pmod 4, \quad \text{if } q_1 + q_2 \equiv 10 \pmod{16},$$
$$a \equiv 1 \pmod 8, \quad b \equiv 0 \pmod 4, \quad \text{if } q_1 + q_2 \equiv 2 \pmod{16}.$$

From $1 - Ti = \pi_1\pi_2\alpha^2 = (a + ib)(g + ih)^2$, we have

$$1 = a(g^2 - h^2) - b(2gh),$$

so that

$$g \equiv 1 \pmod 2, \quad h \equiv 2 \pmod 4, \quad \text{if } q_1 + q_2 \equiv 10 \pmod{16},$$
$$g \equiv 1 \pmod 2, \quad h \equiv 0 \pmod 4, \quad \text{if } q_1 + q_2 \equiv 2 \pmod{16},$$

giving

$$h/2 \equiv (q_1 + q_2 - 2)/8 \pmod 2,$$

so that

$$(\varepsilon_{q_1q_2}/p) = (-1)^{T/4}.$$

Similarly one can prove that $(\varepsilon_{q_1q_2}/p) = (-1)^{T/4+1}$, when $2p^l = q_1x^2 + q_2y^2$, using $(q_1/q_2)_4(q_2/q_1)_4 = +1$.

CASE VI. $q_1 \equiv q_2 \equiv 5 \pmod 8$, $(q_1/q_2) = -1$. In this case I and C are in the principal genus and A and B are in the non-principal genus with $\chi_1 = +1$. We have $h = h(-q_1q_2) \equiv 0 \pmod 8$ (Brown [**4**: Theorem 1]). Thus, if p is a prime such that $(-1/p) = (q_1/p) = (q_2/p) = +1$, there are positive coprime integers x and y such that $p^l = x^2 + q_1q_2y^2$ or $2p^l =$

$q_1x^2 + q_2y^2$, and if $(-1/p) = 1, (q_1/p) = (q_2/p) = -1$, such that $p^l = q_1x^2 + q_2y^2$ or $2p^l = x^2 + q_1q_2y^2$, where $l = h/8$. As $(q_1/q_2) = -1$, by Dirichlet's theorem [**6**: p. 228], we have $N(\varepsilon_{q_1q_2}) = -1$, and we assume that $h \equiv 8 \pmod{16}$, so that l is odd. The example $q_1 = 5$, $q_2 = 37$, $h = h(-185) = 16$, shows that this is a genuine assumption.

Using $p^l = x^2 + q_1q_2y^2$ in Corollary 1, we obtain $(\varepsilon_{q_1q_2}/p) = +1$, and using $2p^l = q_1x^2 + q_2y^2$ in Corollary 1, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2-2)/8}\left(\frac{2q_1}{q_2}\right)_4\left(\frac{2q_2}{q_1}\right)_4,$$

the right hand side of which is -1, as $h \equiv 8 \pmod{16}$ (Kaplan [**9**: Prop. B_1'). Using $2p^l = x^2 + q_1q_2y^2$ in Corollary 2, we obtain

$$\left(\frac{\varepsilon_{q_1q_2}}{p}\right) = (-1)^{(q_1+q_2+6)/8}\left(\frac{2q_1}{q_2}\right)_4\left(\frac{2q_2}{q_1}\right)_4 = +1.$$

Finally using $p^l = q_1x^2 + q_2y^2$ in Corollary 2, we obtain $(\varepsilon_{q_1q_2}/p) = -1$.

This completes the proof of Theorem 2. We remark that parts of II and VI of Theorem 2 have been proved without the restriction $h(-q_1q_2) \equiv 8 \pmod{16}$ using class field theory [**5**].

We conclude with a few examples to illustrate the theorem.

EXAMPLE 1. (Compare Kuroda [**10**: pp. 155–156]) Choose $q_1 = 5$, $q_2 = 13$, so that $(q_1/q_2) = -1$, and $h = h(-q_1q_2) = h(-65) = 8$. By part VI of Theorem 2, if p is a prime such that

$$\left(\frac{-1}{p}\right) = \left(\frac{5}{p}\right) = \left(\frac{13}{p}\right) = +1,$$

then

$$\left(\frac{\varepsilon_{65}}{p}\right) = \left(\frac{8+\sqrt{65}}{p}\right) = \begin{cases} +1, & \text{if } p = x^2 + 65y^2, \\ -1, & \text{if } 2p = 5x^2 + 13y^2; \end{cases}$$

and if p is such that

$$\left(\frac{-1}{p}\right) = +1, \left(\frac{5}{p}\right) = \left(\frac{13}{p}\right) = -1,$$

then

$$\left(\frac{\varepsilon_{65}}{p}\right) = \left(\frac{8+\sqrt{65}}{p}\right) = \begin{cases} +1, & \text{if } 2p = x^2 + 65y^2, \\ -1, & \text{if } p = 5x^2 + 13y^2. \end{cases}$$

Thus, for example, we have

$$\left(\frac{\varepsilon_{65}}{601}\right) = +1, \quad \text{as } 601 = 4^2 + 65\cdot 3^2,$$

$$\left(\frac{\varepsilon_{65}}{29}\right) = -1, \quad \text{as } 2\cdot 29 = 5\cdot 3^2 + 13\cdot 1^2,$$

$$\left(\frac{\varepsilon_{65}}{37}\right) = +1, \quad \text{as } 2\cdot 37 = 3^2 + 65\cdot 1^2,$$

$$\left(\frac{\varepsilon_{65}}{193}\right) = -1, \quad \text{as } 193 = 5\cdot 6^2 + 13\cdot 1^2.$$

These are easily verified directly:

$$\left(\frac{\varepsilon_{65}}{601}\right) = \left(\frac{8 + 234}{601}\right) = \left(\frac{242}{601}\right) = \left(\frac{2}{601}\right) = +1,$$

$$\left(\frac{\varepsilon_{65}}{29}\right) = \left(\frac{8 + 6}{29}\right) = \left(\frac{14}{29}\right) = -1,$$

$$\left(\frac{\varepsilon_{65}}{37}\right) = \left(\frac{8 + 18}{37}\right) = \left(\frac{26}{37}\right) = +1,$$

$$\left(\frac{\varepsilon_{65}}{193}\right) = \left(\frac{8 + 114}{193}\right) = \left(\frac{122}{193}\right) = -1.$$

EXAMPLE 2. Choose $q_1 = 5$, $q_2 = 29$, so that $(q_1/q_2) = +1$, $N(\varepsilon_{q_1q_2}) = N(\varepsilon_{145}) = N(12 + \sqrt{145}) = -1$, $h = h(-q_1q_2) = h(-145) = 8$. By part V of Theorem 2, we have

$$\left(\frac{-1}{p}\right) = \left(\frac{5}{p}\right) = \left(\frac{29}{p}\right) = +1,$$

then

$$\left(\frac{\varepsilon_{145}}{p}\right) = \left(\frac{12 + \sqrt{145}}{p}\right) = \begin{cases} +1, & \text{if } p = x^2 + 145y^2, \\ -1, & \text{if } p = 5x^2 + 29y^2; \end{cases}$$

and if p is such that

$$\left(\frac{-1}{p}\right) = +1, \left(\frac{5}{p}\right) = \left(\frac{29}{p}\right) = -1,$$

then

$$\left(\frac{\varepsilon_{145}}{p}\right) = \left(\frac{12 + \sqrt{145}}{p}\right) = \begin{cases} +1, & \text{if } 2p = 5x^2 + 29y^2, \\ -1, & \text{if } 2p = x^2 + 145y^2. \end{cases}$$

EXAMPLE 3. Choose $q_1 = 17$, $q_2 = 5$, so that $(q_1/q_2) = -1$, and $h = (-q_1q_2) = h(-85) = 4$. By part IV of Theorem 2, we have that if p is a prime such that

$$\left(\frac{-1}{p}\right) = \left(\frac{85}{p}\right) = +1,$$

then

$$\left(\frac{\varepsilon_{85}}{p}\right) = \left(\frac{\frac{1}{2}(9+\sqrt{85})}{p}\right) = \begin{cases} (-1)^y, \text{ if } \left(\frac{17}{p}\right) = \left(\frac{5}{p}\right) = 1, & p = x^2 + 85y^2, \\ (-1)^y, \text{ if } \left(\frac{17}{p}\right) = \left(\frac{5}{p}\right) = -1, & p = 17x^2 + 5y^2. \end{cases}$$

Thus, for example, we have

$$\left(\frac{\varepsilon_{85}}{349}\right) = \left(\frac{\frac{1}{2}(9+145)}{349}\right) = \left(\frac{77}{349}\right) = +1, \quad 349 = 3^2 + 85\cdot 2^2,$$

$$\left(\frac{\varepsilon_{85}}{89}\right) = \left(\frac{\frac{1}{2}(9+21)}{89}\right) = \left(\frac{15}{89}\right) = -1, \quad 89 = 2^2 + 85\cdot 1^2,$$

$$\left(\frac{\varepsilon_{85}}{37}\right) = \left(\frac{\frac{1}{2}(9+23)}{37}\right) = \left(\frac{16}{37}\right) = +1, \quad 37 = 17\cdot 1^2 + 5\cdot 2^2,$$

$$\left(\frac{\varepsilon_{85}}{73}\right) = \left(\frac{\frac{1}{2}(9+31)}{73}\right) = \left(\frac{20}{73}\right) = -1, \quad 73 = 17\cdot 2^2 + 5\cdot 1^2.$$

EXAMPLE 4. Choose $q_1 = 17$, $q_2 = 53$, so that $(q_1/q_2) = +1$, $h = h(-q_1q_2) = h(-901) = 24$, $N(\varepsilon_{q_1q_2}) = N(\varepsilon_{901}) = -1$. By part III of Theorem 2, we have that if p is a prime such that

$$\left(\frac{-1}{p}\right) = \left(\frac{17}{p}\right) = \left(\frac{53}{p}\right) = +1,$$

then

$$\left(\frac{\varepsilon_{901}}{p}\right) = \left(\frac{30+\sqrt{901}}{p}\right) = (-1)^y,$$

where

$$p^3 = x^2 + 901y^2 \text{ or } p^3 = 17x^2 + 53y^2.$$

Thus, for example, we have

$$\left(\frac{\varepsilon_{901}}{89}\right) = \left(\frac{30+79}{89}\right) = \left(\frac{5}{89}\right) = +1, \quad 89^3 = 587^2 + 901\cdot 20^2,$$

$$\left(\frac{\varepsilon_{901}}{13}\right) = \left(\frac{30+2}{13}\right) = \left(\frac{2}{13}\right) = -1, \quad 13^3 = 36^2 + 901\cdot 1^2,$$

$$\left(\frac{\varepsilon_{901}}{149}\right) = \left(\frac{30+93}{149}\right) = \left(\frac{123}{149}\right) = +1, \quad 149^3 = 17\cdot 269^2 + 53\cdot 198^2,$$

$$\left(\frac{\varepsilon_{901}}{1753}\right) = \left(\frac{30+253}{1753}\right) = \left(\frac{283}{1753}\right) = -1, \quad 1753^3 = 17\cdot 15410^2 + 53\cdot 5047^2.$$

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