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OSCILLATION THEORY FOR GENERALIZED SECOND-ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** We will consider the generalized vector-matrix differential system

$$
(1)\qquad \begin{aligned} R(t)u'(t) &= v(t) \\ v(t) &= v(a) - \int_a^t dM(t)u(t),\ t \in [a, \infty), \end{aligned}
$$

where the n-dimensional vector-valued function $u(t)$ is assumed to be absolutely continuous on compact subintervals of $[a, \infty)$, and the real $n \times n$ matrices R and M satisfy

$$
(2)\qquad \begin{aligned} &R^* = R,\ M^* = M,\ R(t) > 0 \text{ (positive definite) for all } t \in [a, \infty), \\ &R \text{ and } R^{-1} \text{ are locally } L^\infty, \text{ and } M \text{ is locally of bounded variation.} \end{aligned}
$$

By A^* we mean the transpose of the matrix A. The matrix-valued Riemann-Stieltjes integral of (1) is a direct generalization of the scalar Riemann-Stieltjes integral. The associated properties are direct consequences of the properties of the scalar integral (cf. Reid **[14]**).

If there is a function $Q(t)$ that is integrable on compact subintervals of $[a, \infty)$ such that

$$
M(t) = M(t_0) + \int_{t_0}^t Q(s)\, ds
$$

for some $t_0 \in [a, \infty)$, then (1) reduces to the vector-matrix differential equation

$$
(3)\qquad (Ru')' + Qu = 0.
$$

If we define for $n = -1, 0, 1, 2, \ldots$

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$$u(t) = \begin{cases} y_n, & t = n \\ y_n + (t - n)(y_{n+1} - y_n), & n < t < n + 1, \end{cases}$$

$$R(t) = C_n, \qquad n \leqq t < n + 1,$$

and

$$M(t) = \begin{cases} 0, & t < 0 \\ \sum_{k=0}^{[t]} (-B_k + C_k + C_{k-1}), & t \geqq 0, \end{cases}$$

where $[t]$ denotes the greatest integer of t, then (1) reduces to the second order, vector-matrix difference equation

$$-C_n y_{n+1} - C_{n-1} y_{n-1} + B_n y_n = 0, \quad n = 0, 1, 2, \ldots . \tag{4}$$

In addition, the scalar generalized second order differential equations which appear in the works of Feller [**4**], Kac and Krein [**10**], and Sz.-Nagy [**18**] are special cases of (1).

The oscillation theory associated with equation (3) is extensive, especially in the scalar case. We suggest that the reader consult [**1**, **2**, **6**, **7**, **17**, **19**] and the references contained therein. Equation (4) has also received some attention in the scalar case [**5**, **9**], but very little has been done in general.

The existence and uniqueness of solutions to equation (1) has been considered in a paper by Reid [**14**]. Additional results concerning the oscillation of (1) can be found in other papers of Reid [**15**, **16**].

Points t_1 and t_2 are said to be (*mutually*) *conjugate* with respect to (1) if there is a solution $(u; v)$ of (1) with $u \not\equiv 0$ on (t_1, t_2) and $u(t_1) = 0 = u(t_2)$. If the interval $[a, b]$ contains two distinct points which are conjugate with respect to (1), then (1) is said to be *oscillatory on* $[a, b]$. Otherwise, (1) is *nonoscillatory on* $[a, b]$. For a noncompact interval $[a, \infty)$ (or $(0, b]$), $0 \leqq a < b \leqq \infty$, (1) is said to be *oscillatory at* ∞ (*at* 0) if every neighborhood of ∞ (0) contains two distinct points which are conjugate with respect to (1). Otherwise, (1) is said to be *nonoscillatory at* ∞ (0).

The matrix equation associated with (1) is

$$\begin{aligned} R(t)\, U'(t) &= V(t) \\ V(t) &= V(a) - \int_a^t dM(t) U(t), \quad t \in [a, \infty), \end{aligned} \tag{5}$$

where U and V are $n \times n$ matrix-valued functions.

Solutions of (5) are said to be *conjoined* (or *prepared*) if

$$U^*(RU') - (RU')^*U \equiv 0.$$

By differentiation, it can be shown that the left side of the above equality must always be a constant.

Some of our results will extend those of Etgen and Lewis **[2]**, and will consequently involve the set of positive linear functions on the Banach algebra $\mathcal{M}_n$ of all $n \times n$, real-valued matrix functions. A linear functional $g: \mathcal{M}_n \to (-\infty, \infty)$ is *positive* if $g(B) \geqq 0$ whenever B is symmetric and positive semidefinite ($B \geqq 0$). If $g(B) = 0$ for all $B \in \mathcal{M}_n$, then g is said to be *trivial*. In the finite dimensional case (only), which we consider in this paper, it is known that for every nontrivial positive linear functional $g: \mathcal{M}_n \to (-\infty, \infty)$ there are nonzero, n-dimensional vectors $v_1, \ldots, v_k$, $k \leqq n$, such that for any $B \in \mathcal{M}_n$

$$g(B) = \sum_{i=1}^{k} (Bv_i, v_i),$$

where $(Bv_i, v_i) = v_i^* Bv_i$. Note that if $B > 0$, then $g(B) > 0$. The use of general positive functionals in establishing critieria for the oscillation of differential systems was first introduced by Etgen and Pawlowski **[3]**.

Let $\mathcal{A}_n(a, b) = \{u: u$ is an n-dimensional absolutely continuous, vector-valued function on $[a, b]$ satisfying $u(a) = 0 = u(b)$ and $u' \in L^2(a, b)\}$. The next theorem can be found in a paper of Reid **[14]**.

THEOREM 1.1. *The following statements are equivalent.*
(i) *Equation* (1) *is nonoscillatory on* $[a, b]$.
(ii) *There is a nonsingular, conjoined solution of* (5) *on* $[a, b]$.
(iii) *If* $u \in \mathcal{A}_n(a, b)$ *and* $u \not\equiv 0$ *on* $[a, b]$, *then*

$$\int_a^b [(Ru', u')\, dt - (dMu, u)] > 0.$$

The comparison principle for (1) follows easily from Theorem 1.1. In the case of (3) the reader may consult Lewis **[12]**.

2. **A nonoscillation theorem.** The scalar version of the next lemma can be found in **[8]** and, in the vector case, Lemma 1.1 of **[12]** is a consequence of this result. The proof is similar to that in **[8]**. The vector norm is the Euclidean norm, $\|u\| = [\sum_{i=1}^{m} u_i^2]^{1/2}$, and for any matrix A, $\|A\| = \sup|(A\xi, \xi)|$ for any unit vector ξ, $\|\xi\| = 1$.

LEMMA 2.1. *Suppose* w *is a real continuously differentiable function on* $[a, b]$ *with* $w' > 0$. *If* $u \in \mathcal{A}_n(a, b)$ *and* $u \not\equiv 0$, *then*

$$\int_a^b w' \|u\|^2 \, dx < 4 \int_a^b \frac{w^2 \|u'\|^2}{w'} \, dx.$$

In the next theorem, the letter I will denote the $n \times n$ identity matrix.

THEOREM 2.1. *Suppose there is a real continuously differentiable function* w *on* $[a, b]$ *with* $w' > 0$ *such that*

$$\|M(t)\| \leqq |w(t)|$$

and

$$R(t) \geqq \frac{4w(t)^2}{w'(t)} I,$$

then (1) *is nonoscillatory on* $[a, b]$.

PROOF. If $u \in \mathscr{A}_n(a, b)$, $u \not\equiv 0$, then

$$\int_a^b u^* dMu = u^*Mu \Big|_a^b - \int_a^b [(u')^*Mu + u^*Mu']\, dt$$

which implies that

$$\begin{aligned}\left|\int_a^b u^* dMu\right| &\leqq 2\int_a^b \|u'(t)\|\ \|u(t)\|\ \|M(t)\|\ dt \\ &\leqq 2\int_a^b \frac{|w(t)|}{(w'(t))^{1/2}} \|u'(t)\|\ (w'(t)^{1/2}\ \|u(t)\|\ dt \\ &\leqq 2\left[\int_a^b \frac{(w(t))^2}{w'(t)} \|u'(t)\|^2\, dt\right]^{1/2}\left[\int_a^b w'(t)\ \|u(t)\|^2\, dt\right]^{1/2} \\ &< 4\int_a^b \frac{(w(t))^2}{w'(t)} \|u'(t)\|^2\, dt \\ &\leqq \int_a^b (u')^*Ru'\, dt.\end{aligned}$$

Therefore,

$$\int_a^b (u')^*Ru'\, dt - \int_a^b u^* dMu > 0$$

which, by Theorem 1.1, implies that (1) is nonoscillatory.

For example, if we let $w(t) = \delta/4\ t^\delta$ for some constant $\delta \neq 0$, then (1) is nonoscillatory on $[a, b]$, $a > 0$, provided

$$\|M(t)\| \leqq \frac{|\delta|}{4} t^\delta \text{ and } R(t) \geqq t^{\delta+1} I.$$

In the case of equation (3), $R(t) = I$, and $\delta = -1$, this reduces to the Hille criterion [7]:

$$u'' + Q(t)u = 0 \text{ is nonoscillatory at } \infty \text{ if}$$

$$\left\| t \int_t^\infty Q(s)\, ds \right\| \leqq \frac{1}{4}.$$

As another example of an application of Theorem 2.1, we let $w(t) = \frac{1}{4} e^t$. The scalar equation

$$(e^t y')' + qy = 0$$

is nonoscillatory on $[a, b]$ provided

$$\left| \int_a^t q(s)\, ds \right| \leqq \frac{1}{4} e^t.$$

3. **Oscillation when** $dM \geqq 0$. We denote the maximum eigenvalue of A by νA and the minimum eigenvalue by μA.

THEOREM 3.1. *Suppose that R and M satisfy the following*:

(i) $\lim_{t\to\infty} \mu \int_a^t R^{-1}(s) ds = \infty$,

(ii) $M(t_2) \geqq M(t_1)$ *when* $t_2 > t_1$ (i.e., $M(t_2) - M(t_1)$ *is positive semidefinite*),

(iii) *for every* $t_1 \geqq a$, *there is a* $t_2 > t_1$ *such that* $M(t_2) > M(t_1)$, *and*

(iv) *there is a positive continuously differentiable function k on* $[a, \infty)$ *such that*

$$\limsup_{t\to\infty} \nu \int_a^t \left[k(s) dM(s) - \frac{k'(s)^2}{4k(s)} R(s)\, ds \right] = \infty. \tag{6}$$

Then, (1) *is oscillatory at* ∞.

PROOF. Suppose that (1) is nonoscillatory at ∞. Then there is a number $b \geqq a$ such that (1) is nonoscillatory on $[b, c]$ for all $c > b$. Difine U as the conjoined solution of (5) such that $U(b) = 0$ and $(RU')(b) = I$. By the analysis of problem 4, p. 345, of Reid [**17**] adapted to (5), U is nonsingular on (b, ∞) (see also Theorem 3.1 of Reid [**14**]) and RU' is nonsingular on $[b, \infty)$. In addition, Reid shows in this analysis that $W_0 = U(RU')^{-1}$ is positive definite on (b, ∞). Since $d(RU') = -dMU$ implies that $d(RU')\, U^{-1}k = -dMk$, we have that for $c > b$,

$$RU'U^{-1}k \Big|_c^t - \int_c^t RU' d(U^{-1}k) = - \int_c^t k dM$$

or

$$RU'U^{-1}k \Big|_c^t - \int_c^t kRU' \left[-U^{-1}U'U^{-1} + U^{-1} \frac{k'}{k} \right] ds = - \int_c^t k dM. \tag{7}$$

Since U being a conjoined solution implies that

$$(U^{-1})^*(U'^*R) = RU'U^{-1},$$

then

$$RU'U^{-1}U'U^{-1} - RU'U^{-1}\frac{k'}{k} = \left[U'U^{-1} - \frac{k'}{2k}I\right]^* R\left[U'U^{-1} - \frac{k'}{2k}I\right] - \frac{(k')^2}{4k^2}R.$$

Thus, (7) becomes

$$W_0^{-1}k\Big|_c^t + \int_c^t k\left(\left[U'U^{-1} - \frac{k'}{2k}I\right]^* R\left[U'U^{-1} - \frac{k'}{2k}I\right] - \frac{(k')^2}{4k^2}\right)ds = -\int_c^t kdM$$

from which we can conclude that for all $t \geqq c$

$$\int_c^t \left[kdM(s) - \frac{(k')^2}{4k}Rds\right] \leqq W_0^{-1}(c)k(c).$$

However, this is a contradiction to (iv).

Since

$$\mu\int_a^t R^{-1}(s)\,ds \geqq \int_a^t \|R(s)\|^{-1}\,ds,$$

then (i) holds when $\int_a^\infty \|R(s)\|^{-1}\,ds = \infty$. In addition, (6) holds when

$$\int^\infty k(t)^{-1}\,k'(t)^2\|R(t)\|\,dt < \infty$$

and $\lim_{t\to\infty} \nu \int_a^t k(s)dM(s) = \infty$.

THEOREM 3.2. *Suppose* $R \equiv I$, (1) *is nonoscillatory at* ∞, *and* M *satisfies* (ii) *and* (iii) *of Theorem* 3.1. *Then, the limit* $M(\infty)$ *exists and*

(i) $$\limsup_{t\to\infty} \nu\, t[M(\infty) - M(t)] \leqq 1$$

and

(ii) $$\liminf_{t\to\infty} \mu\, t[M(\infty) - M(t)] \leqq 1/4.$$

PROOF. The proof of Hille [7, pp. 487–489] can be adapted with the following observations. As in the last proof, if U is a solution of (5) satisfying $U(b) = 0$ and $(RU')(b) = I$ with U nonsingular on (b, ∞), then (RU') is nonsingular on $[b, \infty)$. By Theorem 3.1, νM is bounded above which implies that M is bounded above. Consequently, M being nondecreasing implies that $M(\infty)$ exists. The definition of nonoscillation given by Hille is different from the definition given in this paper; e.g., in the scalar case Hille's definition of a nonoscillatory solution y requires that y and y' be nonzero. However, the proof of Hille applies since our hypotheses require U' to be nonsingular.

4. **Comparison with scalar equations.** In this section, we establish a comparison theorem for (1) which has been previously proved for (3) by Etgen and Lewis **[2]** and recently extended to general even order differential equations by Lewis and Wright **[13]**. The proof, which is valid only in the finite dimensional case considered here, is a simplification of that given in the paper of Etgen and Lewis. This theorem allows us to apply the vast amount of already established criteria for the oscillation of scalar equations ($n = 1$) to the vector-matrix equation (3).

THEOREM 4.1. *If there is a nontrivial positive linear functional g on the set of real $n \times n$ matrices such that the scalar system*

$$\begin{aligned} g(R(t))y'(t) &= w(t) \\ w(t) &= w(a) - \int_a^t y(s)d[g(M(s))] \end{aligned} \tag{8}$$

is oscillatory on $[a, b]$, *then* (1) *is oscillatory on* $[a, b]$.

PROOF. Since (8) is oscillatory on $[a, b]$, then by Theorem 1.1 there is a function $f \in \mathscr{A}_1(a, b), f \not\equiv 0$, such that

$$\int_a^b (g(R(t))f'(t)^2\,dt - f(t)^2 d[g(M(t))]) \leqq 0.$$

Since there are vectors $v_1, \ldots, v_k$ such that

$$g(A) = \sum_{i=1}^{k} (Av_i, v_i)$$

for all $n \times n$ matrices A, the above inequality can be written as

$$\sum_{i=1}^{k} \int_a^b [(Rf'v_i, f'v_i)\,dt - (dM(t)fv_i, fv_i)] \leqq 0.$$

This implies that for some $j \in \{1, \ldots, k\}$ and $u = fv_j$,

$$\int_a^b [(Ru', u')\,dt - (dM(t)u, u)] \leqq 0.$$

By Theorem 1.1, we now have that (1) must also be oscillatory on $[a, b]$,

In order to illustrate the scope of Theorem 4.1, we state two corollaries of the general oscillation theorems of section 5, as they apply to scalar systems, and we consider some typical examples. The scalar system, which is system (1) when $n = 1$,

$$\begin{aligned} r(t)y'(t) &= w(t) \\ w(t) &= w(a) - \int_a^t y(s)dm(s) \end{aligned}$$

is oscillatory at ∞ if either (A) or (B) below holds.

(A) There is a positive $C^{(1)}[a, \infty)$ function k such that

$$\int_a^\infty r(t)\left[k'(t)^2 + \frac{k(t)^2}{t^2}\right]dt < \infty$$

and

$$\lim_{t\to\infty}\int_a^t k(s)^2 dm(s) = \infty.$$

(B) There is a positive function k on $[a, \infty)$ such that

$$\liminf_{t\to\infty}\frac{\int_a^t\left[r(s)k(s)^2\,ds - \left(\int_s^t k\right)^2 dm(s)\right]}{\left[\int_a^t k(s)\,ds\right]^2} = -\infty.$$

For $k = r^{-1}$, condition (B) reduces to

$$\liminf_{t\to\infty}\frac{-\int_a^t\left(\int_s^t r^{-1}\right)^2 dm(s)}{\left(\int_a^t r^{-1}\right)^2} = -\infty \tag{9}$$

with no monotonicity conditions required of $m(s)$.

For $k(t) = t^{\delta/2}$, condition (A) reduces to the requirement that

$$\int_a^\infty t^{\delta-2}\,r(t)\,dt < \infty \text{ and } \int_a^\infty s^\delta dm(s) = \infty \tag{10}$$

for some $a > 0$. As a consequence, we know that the scalar equation

$$(t^\alpha y')' + kt^\beta y = 0,$$

with $k > 0$, is oscillatory if $\beta > \alpha - 2$. For $\beta = \alpha - 2$, the above equation is the Eular equation which is oscillatory if and only if $k > (\alpha - 1)^2/4$. The oscillation criterion (10) for the difference equation (4) in the scalar case was proved in [**9**, Theorem 7].

If $\int^\infty r(x)^{-1}\,dx = \infty$, then by *L*'Hospital's rule condition (9) will hold if $m(t) \to \infty$ as $t \to \infty$. For the scalar equation

$$(ry')' + qy = 0,$$

$m(t) = \int^t q(s)\,ds$, so that in this case condition (9) implies the Leighton-Wintner criterion [**11**, **20**] for oscillation. The Leighton-Wintner criterion for scalar difference equations has been established in [**9**] and [**16**]. Hence, condition (B) is a generalization of the Leighton-Wintner criterion that includes both differential and difference equations in one setting.

The proof of Theorem 4.1 shows that (8) being oscillatory on $[a, b]$ implies that there is a constant vector $\xi \neq 0$ such that

$$(R(t)\xi, \xi)y'(t) = w(t)$$

$$w(t) = w(a) - \int_a^t y(s)d[(M(s)\xi, \xi)]$$

is oscillatory on $[a, b]$, and consequently, (1) is oscillatory on $[a, b]$. The paper of Lewis and Wright [**13**] shows that the analogous implications hold for the general even-order version of equation (3).

Two simple choices of a positive functional, g, are $g(A) = \sum_{i=1}^{n} c_i a_{ii}$, $c_i \geqq 0$ ($c_i = 1$ for each i yields the trace functional), and $g(A) = (A\xi, \xi)$ where ξ is a nonzero constant vector. For example, if we let $\xi = (1/\sqrt{2})$ $(1, 1)^*$ and use the latter choice for g, then the scalar equation (8) corresponding to

$$u''(x) + \begin{bmatrix} 0 & b(x) \\ b(x) & 0 \end{bmatrix} u = 0 \tag{11}$$

becomes

$$y'' + b(x)y = 0.$$

Hence, by the above theorem, equation (11) is oscillatory at ∞ if $\int^\infty s^\delta$ $b(s)\, ds = \infty$ for some $\delta < 1$.

By using the comparison principle for scalar equations, another scalar comparison equation may be derived for (3). By taking $g(A) = (A\xi, \xi)$ for any A and some constant vector ξ, $\|\xi\| = 1$, and noting that $g(R(t)) \leqq \nu(R(t))$ and $g(Q(t)) \geqq \mu(Q(t))$, then we can conclude that

$$(\nu(R(t))y')' + \mu(Q(t))y = 0 \tag{12}$$

being oscillatory on $[a, b]$ implies that system (8) (hence (3)) is oscillatory on $[a, b]$.

A similar comparison equation for the nonoscillation of (3) results by noting that $u \in \mathscr{A}_n(a, b)$ implies that

$$\int_a^b [Ru', u') - (Qu, u)]\, dt \geqq \int_a^b [\mu(R(t))\, \|u'\|^2 - \nu(Q(t))\, \|u\|^2]\, dt$$

$$\geqq \int_a^b [\mu(R(t))(f')^2 - \nu(Q(t))\, f^2]\, dt$$

where $f(t) = \|u(t)\|$ $(|f'| \leqq \|u'\|$. Therefore,

$$(\mu(R(t))y')' + \nu(Q(y))y = 0 \tag{13}$$

being nonoscillatory on $[a, b]$ implies the nonoscillation of (3) on $[a, b]$. These two comparison results are mentioned by Glazman [**6**, p. 125] for the general even-order equations.

As indicated above, using condition (9) and Theorem 4.1, it can be shown that if there is a positive functional g such that

$$\lim_{t\to\infty} g\left(\int_a^t R^{-1}(s)\,ds\right) = \lim_{t\to\infty} g\left(\int_a^t dM(s)\right) = \infty,$$

then system (1) is oscillatory at ∞, By taking, for some constant unit vector ξ, $g(A) = (A\xi, \xi)$ for all A and observing that $\mu(R^{-1}(s)) \leqq g(R^{-1}(s))$, Theorem 2.1 of Reid **[14]** follows as a corollary to Theorem 4.1 when Reid's matrix $B(t)$ is nonsingular.

The proof of Theorem 4.1 did not use the "Picone-type" identity used by Etgen and Lewis **[2]** in their proof of the special case applying to (3). Such an identity can be shown to hold for (1) and consequently, Theorem 4.1 can be proved in this way. Using this type of proof, Theorem 4.1 can be shown to hold, as well as many of the other results of this paper, for equation (1) in the more general B*-algebra setting considered in **[2]**.

5. **General oscillation theorems.** In this section we establish oscillation criteria whose corollaries include many of the previously established results for differential and difference equations.

THEOREM 5.1. *Suppose there exists a positive* $C^{(1)}[a, \infty]$ *function* k, *a sequence* $\{t_n\}$ *with* $t_n \to \infty$ *as* $n \to \infty$, *and a sequence* $\{g_n\}$ *of uniformly bounded positive functionals such that*

$$\lim_{n\to\infty} \int_a^{t_n} \left[g_n(R(t)) \left\{ \frac{d}{dt}\left(1 - \frac{t}{t_n}\right)k(t)\right\}^2 dt - d(g_n(M(t)))\left(1 - \frac{t}{t_n}\right)^2 k(t)^2 \right] = -\infty. \tag{14}$$

Then (1) *is oscillatory at* ∞.

PROOF. We will show that if $b > a$, there is an n such that

$$\begin{aligned} g_n(R(t))y' &= z \\ dz &= -d(g_n(M(t)))y \end{aligned} \tag{15}$$

is oscillatory on $[b, t_n]$. Hence, by Theorem 4.1 system (15) implies (1) is oscillatory at ∞. If $\|g_n\| \leqq B$ for all n, then for $t_n > b$

$$\left| \int_a^b \left[g_n(R(t)) \left\{ \frac{d}{dt}\left(1 - \frac{t}{t_n}\right)k(t)\right\}^2 dt - d\{g(M(t))\}\left(1 - \frac{t}{t_n}\right)^2 k(t)^2 \right] \right|$$
$$\leqq B\left[\int_a^b \|R(t)\| \left[\frac{k(t)}{t_n} + |k'(t)| \right]^2 dt + V_a^b(M) \cdot \max_{t\in[a,b]} |k(t)|^2 \right].$$

where $V_a^b(M)$ denotes the total variation of $M(t)$ on $[a, b]$. Given this upper bound, we see that the limit in (14) is independent of a; hence, it holds for a replaced by $b + 1$. For $t_n > b + 1$, let

$$y_n(t) = \begin{cases} (t - b)\left(1 - \dfrac{b+1}{t_n}\right)k(b+1), & b \leqq t \leqq b+1 \\ (1 - t/t_n)k(t), & b+1 \leqq t \leqq t_n \end{cases}$$

then

$$\begin{aligned} &\int_b^{t_n} [g_n(R(t))y_n'(t)^2 dt - d(g_n(M(t)))y_n(t)^2] \\ (16) \qquad &= \int_b^{b+1} [g_n(R(t))y_n'(t)^2 dt - d[g_n(M(t))]y_n(t)^2] \\ &\quad + \int_{b+1}^{t_n} [g_n(R(t))y_n'(t)^2 - d[g_n(M(t))]y_n(t)^2]. \end{aligned}$$

Since the above integral from b to $b+1$ is bounded independent of n, condition (14) implies that (16) is negative for sufficiently large n. Thus (15) is oscillatory on $[b, t_n]$ and the proof is complete.

COROLLARY 5.1. *Equation* (1) *is oscillatory at* ∞ *if there is a positive* $C^{(1)}[a, \infty]$ *function* k *such that*

$$\text{(i)} \qquad \int_a^\infty \|R(t)\| \left[k'(t)^2 + \frac{k(t)^2}{t^2}\right] dt < \infty$$

and

$$\text{(ii)} \qquad \limsup_{t\to\infty} \nu\left(\int_a^t k(s)^2\left(1 - \frac{t}{s}\right)^2 dM(s)\right) = \infty.$$

PROOF. By (ii) there is a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\nu\left(\int_a^{t_n} k(s)^2\left(1 - \frac{s}{t_n}\right)^2 dM(s)\right) \to \infty \quad \text{as } n \to \infty.$$

Let ξ_n be a unit eigenvector for

$$\nu\left(\int_a^{t_n} k(s)^2\left(1 - \frac{s}{t_n}\right)^2 dM(s)\right).$$

Let $g_n(A) = (A\xi_n, \xi_n)$ for all A. Then

$$\left|\int_a^{t_n} g_n(R(t))\left\{\frac{d}{dt}\left(1 - \frac{t}{t_n}\right)k(t)\right\}^2 dt\right| \leqq \int_a^{t_n} \|R(t)\| \left[|k'(t)| + \frac{k(t)}{t_n}\right]^2 dt$$

and

$$\begin{aligned} \int_a^{t_n} d(g_n(M(t)))\left(1 - \frac{t}{t_n}\right)^2 k(t)^2 &= g_n\left(\int_a^{t_n} k(t)^2\left(1 - \frac{t}{t_n}\right)^2 dM(t)\right) \\ &= \nu\left(\int_a^{t_n} k(t)\left(1 - \frac{t}{t_n}\right)^2 dM(t)\right); \end{aligned}$$

hence, conditions (i) and (ii) imply that (14) holds and the proof is complete.

In the scalar case, a weaker statement of Corollary 5.1 is to replace (ii) by

$$\lim_{t\to\infty} \int_a^t k(s)^2 dM(s) = \infty \tag{17}$$

since by two applications of L'Hospital's rule, (17) implies that condition (ii) holds.

THEOREM 5.2. *Suppose there exists a positive function k on $[a, \infty)$, a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, and a sequence $\{g_n\}$ of uniformly bounded positive functionals such that*

$$\lim_{n\to\infty} \frac{\int_a^{t_n}\left[g_n(R(x))k(x)^2dx - d(g_n(M(x)))\left(\int_x^{t_n} k(s)ds\right)^2\right]}{\left[\int_a^{t_n} k(t)dt\right]^2} = -\infty. \tag{18}$$

Then (1) *is oscillatory at* ∞.

PROOF. As in the proof of Theorem 5.1 it will suffice to show that for each $b > a$, there is an integer n such that (15) is oscillatory on $[b, t_n]$.

For each n let

$$C_n = \int_{b+1}^{t_n} k(s)ds \Big/ \int_b^{t_n} k(s)ds$$

and define

$$Z_n(x) = \begin{cases} (x - b)C_n, & x \in [b, b + 1) \\ \int_x^{t_n} k(s)ds \Big/ \int_b^{t_n} k(s)ds, & x \in [b + 1, t_n]. \end{cases}$$

Let $C_\infty = \lim_{n\to\infty} C_n$, which will be some positive number.

Now,

$$\int_b^{t_n}\left[g_n(R(s))Z_n'(s)^2\, ds - Z_n(s)^2 d[g_n(M(s))]\right] = I_1(n) + I_2(n) - I_3(n), \tag{19}$$

where

$$I_1(n) = \int_b^{b+1}\left[g_n(R(s))C_n^2\, ds - C_n^2(s - b)^2 d[g_n(M(s))]\right],$$

$$I_2(n) = \int_b^{t_n}\left[g_n(R(s))k^2(s)ds - \left(\int_s^{t_n} k(u)du\right)^2 d[g_n(M(s))]\right] \cdot \left[\int_b^{t_n} k(s)ds\right]^{-2}$$

and

$$I_3(n) = \int_b^{b+1}\left[g_n(R(s))k^2(s)ds - \left(\int_s^{t_n} k(u)du\right)^2 d[g_n(M(s))]\right] \cdot \left[\int_b^{t_n} k(s)ds\right]^{-2}.$$

As $n \to \infty$, $I_1(n)$ and $I_3(n)$ remain bounded independent of n.

Hence, there is some n such that (19) is negative and the proof is complete.

The proof of the next corollary is similar to the proof of Corollary 5.1.

COROLLARY 5.2. *Equation* (1) *is oscillatory at* ∞ *if there is a positive function* k *on* $[a, \infty)$ *such that*

$$\liminf_{t\to\infty} \frac{\mu\left\{\int_a^t \left[R(s)k(s)^2\, ds - \left(\int_s^t k(u)\,du\right)^2 dM(s)\right]\right\}}{\left(\int_a^t k(s)ds\right)^2} = -\infty.$$

COROLLARY 5.3. *Equation* (1) *is oscillatory at* ∞ *if*

$$\limsup_{t\to\infty} \frac{\nu\left\{\int_a^t \left(\int_s^t \|R(u)\|^{-1}\, du\right)^2 dM(s)\right\}}{\left(\int_a^t \|R(u)\|^{-1}\, du\right)^2} = \infty. \tag{20}$$

PROOF. Let $k(t) = \|R(t)\|^{-1}$ in Corollary 5.2 and note that as $t \to \infty$

$$\frac{\left\|\int_a^t R(s)k(s)^2\, ds\right\|}{\left(\int_a^t k(s)ds\right)^2} \leqq \left[\int_a^t \|R(s)\|^{-1}\, ds\right]^{-1} = O(1).$$

By observing that for any unit vector ξ

$$\frac{\mu\left\{\int_a^t \left[R(s)k(s)^2\, ds - \left(\int_s^t k(u)du\right)^2 dM(s)\right]\right\}}{\left(\int_a^t k(s)ds\right)^2}$$

$$\leqq O(1) - \frac{\xi^* \int_a^t \left(\int_s^t k(u)du\right)^2 dM(s)\xi}{\left(\int_a^t k(s)ds\right)^2}$$

and taking the minimum over all such unit vectors ξ, it then follows from (20) that the hypothesis of Corollary 5.2 is satisfied.

When $\int^\infty \|R(s)\|^{-1}\, ds = \infty$ and $n = 1$ (the scalar case), equation (20) may be replaced by

$$M(t) \to \infty \text{ as } t \to \infty, \tag{21}$$

since L'Hospital's rule can be used to show that (21) implies equation (20).

THEOREM 5.3. *Equation* (1) *is oscillatory at* ∞ *if*

$$(22)\qquad \limsup_{t\to\infty} \nu\left\{M(t) + \int_t^{t+1} (t+1-s)^2\, dM(s) - \int_t^{t+1} R(s)ds\right\} = \infty.$$

PROOF. Let $b > a$. Choose $\{t_n\}$ such that $\nu(T_n) \to \infty$ as $n \to \infty$ where

$$T_n = M(t_n) + \int_{t_n}^{t_n+1} (t_n + 1 - s)^2\, dM(s) - \int_{t_n}^{t_n+1} R(s)ds.$$

Let ξ_n be a unit eigenvector for νT_n. For $t_n > b+1$, define $u_n(t) = \tilde{u}_n(t)\,\xi_n$ on $[b, t_n+1]$ where

$$\tilde{u}_n(t) = \begin{cases} t-b, & b \leqq t < b+1 \\ 1 & b+1 \leqq t < t_n \\ (t_n + 1 - t) & t_n \leqq t \leqq t_n + 1. \end{cases}$$

Then

$$\begin{aligned} \int_b^{t_n+1} [(u_n')^* R u_n' - u_n^*\, dM u_n^*] &= \int_b^{b+1} [\xi_n^* R(t)\, \xi_n dt - (t-b)^2\, \xi_n^* dM(t)\xi_n] \\ &\quad - \xi_n^* [M(t_n) - M(b+1)]\xi_n \\ &\quad + \int_{t_n}^{t_n+1} \xi_n^* R(t)\xi_n dt \\ &\quad - \int_{t_n}^{t_n+1} (t_n + 1 - s)^2\, \xi_n^* dM(s)\xi_n \\ &\leqq \int_b^{b+1} \|R(t)\|dt - V_b^{b+1}(M) + \|M(b+1)\| \\ &\quad - \nu(T_n) < 0 \end{aligned}$$

for n sufficiently large (recall that $V_b^{b+1}(M)$ denotes the total variation of M on $[b, b+1]$). By Theorem 1.1, (1) is oscillatory on $[b, t_n + 1]$ for n sufficiently large. Since b is arbitrary, (1) is oscillatory at ∞.

For $u'' + Q(t)u = 0$, (22) may be replaced by

$$(23)\qquad \limsup_{t\to\infty} \nu\left[\int_a^t Q(s)ds + \int_t^{t+1} (t+1-s)^2 Q(s)ds\right] = \infty.$$

It has been conjectured that $\nu(\int_a^t Q(s)ds) \to \infty$ as $t \to \infty$ is a sufficient condition for oscillation of $u'' + Q(t)u = 0$. Clearly, it is sufficient if $\int_t^{t+1}\|Q(s)\|ds$ is bounded independent of t. Even in the scalar case, the condition

$$\limsup_{t\to\infty} \int_a^t Q(s)ds = \infty$$

is not sufficient for oscillation, cf. Willett [**19**, p. 607].

For the difference equation (4), equation (22) will hold if

$$\liminf_{n\to\infty} \mu\left\{\sum_{k=0}^{n} (B_k - 2C_{k-1})\right\} = -\infty.$$

Added in proof. For the scalar case of (1), a number of results on oscillation theory may be found in the thesis of A. Mingarelli, *Volterra-Stieltjes integral equations and generalized differential expressions*, University of Toronto, 1979. For the differential equation (3), P. Hartman in *Oscillation criteria for self-adjoint second-order differential systems and principal sectional curvatures*, J. Diff. Eqs. 34 (1979), 326–338, has employed both linear and nonlinear functionals to obtain oscillation criteria with applicaitons to differential geometry. Theorem 3.1 is a matrix analog of a theorem of Z. Opial (Ann. Pol. Mat. 6 (1959), p. 100) for scalar differential equations.

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