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COUNTABILITY PROPERTIES OF FUNCTION SPACES

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ABSTRACT. A study is made of those function spaces which have such properties as first and second countability, separability, the Lindelöf property, and the properties of $\aleph_0$-spaces and cosmic spaces.

Key words and phrases: Function spaces, first countable, metrizable, separable, Lindelöf, $\aleph_0$-space, cosmic space.

The purpose of this paper is to organize and extend results concerning countability properties of function spaces. The term "countability properties" refers to those topological properties which involve some countable set in their definitions, such as first and second countable spaces, separable spaces, or Lindelöf spaces.

We shall be concerned with function spaces having topologies which are of "closed-open" form. That is, if X and Y are topological spaces, and if $C(X, Y)$ denotes the space of continuous functions from X into Y, then a topology of "closed-open" form is one generated by the sets of the form $[C, V] \equiv \{f \in C(X, Y) \mid f(C) \subseteq V\}$, where C is from some predetermined collection of nonempty closed subsets of X, and V is open in Y. We will call Γ a *closed collection from X* (*compact collection from X*, respectively) if it is a family of nonempty closed (compact, respectively) subsets of X; and we will use the notation $C_\Gamma(X, Y)$ to denote the space $C(X, Y)$ with the topology generated by the subbase $\{[C, V] \mid C \in \Gamma$ and V is open in $Y\}$. When Γ consists of the singleton subsets of X, then the topology on $C_\Gamma(X, Y)$ is the *topology of pointwise convergence* and will be specifically denoted by $C_\pi(X, Y)$. The symbol π will then be used to mean the family of all singleton subsets of X. Also when Γ consists of the nonempty compact subsets of X, then the topology on $C_\Gamma(X, Y)$ is the *compact-open topology* and denoted by $C_\kappa(X, Y)$, and κ will be used to mean the family of all nonempty compact subsets of X.

The range space Y can naturally be embedded in $C_\Gamma(X, Y)$ by associating points of Y with the constant functions; and if Y is a Hausdorff space, then this is a closed embedding. Therefore, for a hereditary property (or a

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closed hereditary property, if Y is a Hausdorff space), it is necessary that Y have this property in order that $C_\Gamma(X, Y)$ have the property. This fact will then pertain to many of the properties which we will investigate.

A collection $\mathscr{P}$ of subsets of X is called a Γ-network for X if whenever $C \in \Gamma$ and U is open in X with $C \subseteq U$, then there exists a $P \in \mathscr{P}$ such that $C \subseteq P \subseteq U$. A π-network is generally called a network, We will call Γ a *proper* closed collection from X if Γ is a closed collection from X which is a κ-network for X; and we will call Γ *point-proper* if Γ is a network. All spaces will be T_1-spaces.

1. **First and second countable spaces and metric spaces.** The following two facts can be established in a straightforward manner (see for example [**5**, page 121]).

PROPOSITION 1.1. *If Γ is a point-proper closed collection from X, then $C_\Gamma(X, Y)$ is a Hausdorff space if and only if Y is a Hausdorff space.*

PROPOSITION 1.2. *If Γ is a point-proper compact collection from X, then*
(a) *$C_\Gamma(X, Y)$ is regular if and only if Y is regular, and*
(b) *$C_\Gamma(X, Y)$ is completely regular if and only if Y is completely regular.*

There is no corresponding result for normality. In fact, Borges has given an example in [**2**] of a paracompact Hausdorff (and hence normal) space Y such that neither $C_\kappa(I, Y)$ nor $C_\kappa(I, \beta Y)$ are normal, where I is the closed unit interval in the real numbers R. If Y is a regular Hausdorff space, then whenever $C_\Gamma(X, Y)$ is second countable, Y is necessarily a second countable metric space. Conversely, if X is a locally compact second countable space and Y is a second countable metric space, then $C_\Gamma(X, Y)$ is a second countable metric space whenever Γ is a proper compact collection. This follows from 1.1, 1.2, and the following easily proved fact (see [**7**, page 152]).

PROPOSITION 1.3. *Let Γ be a proper compact collection from a locally compact Hausdorff space X. If both X and Y are second countable, then $C_\Gamma(X, Y)$ is second countable.*

If $C_\Gamma(X, Y)$ is to be first or second countable, then it will be necessary for X and Γ to have certain properties. This will be illustrated by the next theorem. We will say that Γ has the *countable covering property* if there exists a countable subset $\Gamma' \subseteq \Gamma$ such that for every $C \in \Gamma$, there exist $C_1, \ldots, C_n \in \Gamma'$ with $C \subseteq C_1 \cup \cdots \cup C_n$.

THEOREM 1.4. *Let X be a completely regular space, let Y contain a nontrivial path, and let Γ be a closed collection from X. If $C_\Gamma(X, Y)$ is first countable, then Γ has the countable covering property.*

PROOF. Let $\alpha: I \to Y$ be a continuous function such that $\alpha(0) \neq \alpha(1)$.

Let f be the constant function from X onto $\{\alpha(0)\}$. So f has a countable base $\{W_i\}$, where each

$$W_i = [C_{il}, V_{il}] \cap \cdots \cap [C_{ik_i}, V_{ik_i}].$$

Define $\Gamma' = \{C_{ij} | i \geqq 1 \text{ and } 1 \leqq j \leqq k_i\}$, which is a countable subset of Γ. Now suppose that there exists a $C \in \Gamma$ such that for every i, $C \nsubseteq C_{il} \cup \cdots \cup C_{ik_i}$. Let $V = Y \backslash \{\alpha(1)\}$, and note that $[C, V]$ is a neighborhood of f. To get a contradiction, we will show that no W_i is contained in $[C, V]$. Let i be a fixed natural number. Since X is completely regular, there exists a $g \in C(X, Y)$ such that $g(x) = \alpha(1)$ and $g(C_{il} \cup \cdots \cup C_{ik_i}) = \{\alpha(0)\}$, where x is some element of $C \backslash (C_{il} \cup \cdots \cup C_{ik_i})$. Then $g \in W_i$, while $g \notin [C, V]$. With this contradiction, we see that Γ must have the countable covering property.

If Γ consists of the singleton subsets of X, then Γ has the countable covering property if and only if X is countable. Also, if Γ consists of the nonempty compact subsets of X, then Γ has the countable covering property if and only if X is hemicompact. This concept of hemicompactness was introduced in **[1]** and means that there exists a countable family of compact subsets such that every compact subset of the space is contained in some member of this family. These facts, along with the fact that first countability and metrizability are countably productive properties, can be used to establish the following corollary of 1.4.

COROLLARY 1.5. *Let X be a completely regular space, and let Y contain a nontrivial path. Then*

(a) *$C_\pi(X, Y)$ is first countable if and only if Y is first countable and X is countable,*

(b) *$C_\pi(X, Y)$ is metrizable if and only if Y is metrizable and X is countable, and*

(c) *if $C_\kappa(X, Y)$ is first countable, then Y is first countable and X is hemicompact.*

Part (c) of 1.5 appears in **[1]** for the case where $Y = \mathbf{R}$. An alternate proof of part (b) of 1.5 is given in [**4**, page 273] making strong use of a metric on Y; however, part (a) of 1.5 does not appear there.

There is no converse to 1.4 using first countability. However, there is a converse using metrizability, but only for certain kinds of Γ—which we introduce with the following definition. A closed collection Γ from X will be called *hereditary* if every nonempty closed subset of a member of Γ is a member of Γ. The following theorem and its proof are then generalizations of Theorem 7 in **[1]** and its proof.

THEOREM 1.6. *Let Γ be a hereditary compact collection from X which has* *the countable covering property. If Y is metrizable, then $C_\Gamma(X, Y)$ is metrtzable.*

PROOF. Let $\Gamma' = \{C_i\}$ be a countable subset of Γ such that for every $C \in \Gamma$, there exits an n such that $C \subseteqq C_1 \cup \cdots \cup C_n$. Let d be a compatible metric on Y which is bounded by 1. For each i, define ρ_i by $\rho_i(f, g) = \sup \{d(f(x), g(x)) \mid x \in C_i\}$. Define ρ by

$$\rho(f, g) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(f, g).$$

Now ρ is a metric on $C(X, Y)$.

To see that ρ is compatible with the topology on $C_\Gamma(X, Y)$, let $f \in C_\Gamma(X, Y)$ and let $B = [D_1, V_1] \cap \cdots \cap [D_n, V_n]$ be a basic open subset of $C_\Gamma(X, Y)$ containing f. Then there exists an integer m such that

$$D_1 \cup \cdots \cup D_n \subseteqq C_1 \cup \cdots \cup C_m.$$

Also, since each D_i is compact, there exists an $\varepsilon > 0$ such that $N_\varepsilon(f(D_i)) \subseteqq V_i$ for each $1 \leqq i \leqq n$. Now suppose that $\rho(f, g) < \varepsilon/2^n$. Then for each $1 \leqq i \leqq n$,

$$\rho_i(f, g) \leqq 2^i \rho(f, g) \leqq 2^n \rho(f, g) < \varepsilon.$$

Let i be between 1 and n, and let $x \in D_i$. Then for some j between 1 and m, $x \in C_j$. Thus

$$d(f(x), g(x)) \leqq \rho_j(f, g) < \varepsilon,$$

so that $g(x) \in V_i$. Hence, $g(D_i) \subseteqq V_i$ for each i, and thus $g \in B$. This means that $N_{\varepsilon/2^n}(f) \leqq B$.

Going in the other direction, let $\varepsilon > 0$ be arbitrary. Let i be temporarily fixed. For each $x \in C_i$, there exists a neighborhood $U(x)$ of x such that $\overline{f(U(x))} \subseteqq N_{\varepsilon/4}(f(x))$. Since C_i is compact, there exist $x_1, \ldots, x_{k_i} \in C_i$ such that $C_i \subseteqq U(x_1) \cup \cdots \cup U(x_{k_i})$. For each j between 1 and k_i, let $C_{ij} = \overline{U(x_j)} \cap C_i$, and let $V_{ij} = N_{\varepsilon/4}(f(x_j))$.

Now let m be such that $\sum_{i=m+1}^{\infty} 2^{-i} < \varepsilon/2$, and let

$$W = \bigcap_{i=1}^{m} \bigcap_{j=1}^{k_i} [C_{ij}, V_{ij}].$$

First, note that $f \in W$ by construction. Next, if $g \in W$, then for each $1 \leqq i \leqq m$ and for each $z \in C_i$, $z \in C_{ij}$ and hence $g(z) \in V_{ij}$ for some j. But

$$V_{ij} = N_{\varepsilon/4}(f(x_j)) \subseteqq N_{\varepsilon/2}(f(z))$$

since $f(z) \in N_{\varepsilon/4}(f(x_j))$. Therefore $d(g(z), f(z)) < \varepsilon/2$, so that $\rho_i(f, g) \leqq \varepsilon/2$. Then $\sum_{i=1}^{m} 2^{-i} \rho_i(f, g) < \varepsilon/2$, and thus $\rho(f, g) < \varepsilon$. Therefore $W \subseteqq N_\varepsilon(f)$.

We now have the following corollary of 1.6.

COROLLARY 1.7. *Let X be a completely regular space, and let Y contain a nontrivial path. Then $C_\kappa(X, Y)$ is metrizable if and only if Y is metrizable and X is hemicompact.*

Corollary 1.7 has no analog for first countability as suggested by 1.5. This can be seen by letting X and Y both be the unit square in the plane with the order topology gotten from lexicographic ordering. This space, call it Z, is a first countable compact Hausdorff space which contains a nontrivial path. However, $C_\kappa(Z, Z)$ is not first countable. To see this, let $e \in C_\kappa(Z, Z)$ be the identity, and let $\{W_i\}$ be a sequence of neighborhoods of e, where each

$$W_i = [C_{i1}, V_{i1}] \cap \cdots \cap [C_{ik_i}, V_{ik_i}].$$

Since each C_{ij} is compact, we may assume that each V_{ij} is the finite union of basic open sets. So for each i and j, the set $S_{ij} \equiv \{s \in I \mid V_{ij} \cap (\{s\} \times I)$ is a nonempty proper subset of $\{s\} \times I\}$ is finite. Let $S = \bigcup \{S_{ij} \mid i = 1, 2, \ldots$ and $j = 1, \ldots, k_i\}$, which is a countable subset of I. Therefore, there exists an $s_0 \in I \setminus S$. Let I_0 be the interval (0, 1/3) in I, and let $V = \{s_0\} \times I_0$, which is an open subset of Z. For each i and j, either $V \subseteq V_{ij}$ or $V \cap V_{ij} = \emptyset$. Now let $\varphi\colon I \to I$ be a continuous function such that $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(1/3) = 2/3$. Define $f \in C(Z, Z)$ by taking $f(\langle s, t\rangle) = \langle s, \varphi(t)\rangle$ if $s = s_0$ and $f(\langle s, t\rangle) = \langle s, t\rangle$ if $s \neq s_0$. If we define $W = [\{\langle s_0, 1/3\rangle\}, V]$, then $e \in W$. But $f \in W_i \setminus W$ for each i, so that $W_i \nsubseteq W$ for each i. Therefore, since $\{W_i\}$ was arbitrary, e has no countable base, and hence $C_\kappa(Z, Z)$ is not first countable.

This example raises an interesting problem. Find necessary and sufficient conditions for $C_\Gamma(X, Y)$ (and in particular for $C_\kappa(X, Y)$) to be first countable for a large class of spaces X and Y, such as X being completely regular and Y containing a nontrivial path. We do not have a complete answer to this problem, but there is an interesting analog of Theorem 1.6 for first countability of $C_\Gamma(X, Y)$, where we weaken the metrizability of Y to Y having a point-countable base. This is a generalization of a theorem in **[13]**, and it has a similar proof. Therefore, we omit its proof, but merely note that it hinges on the fact that $f(C)$ is second countable for each $f \in C_\Gamma(X, Y)$ and $C \in \Gamma$.

THEOREM 1.8. *Let Γ be a hereditary compact collection from X which has the countable covering property. If Y has a point-countable base, then $C_\Gamma(X, Y)$ is first countable.*

Therefore, if X is completely regular and if Y has a point-countable base and contains a nontrivial path, then $C_\kappa(X, Y)$ is first countable if and only if X is hemicompact. However, it is not necessary for Y to have a point-countable base in order that $C_\kappa(X, Y)$ be first countable. This can be seen from the following example. As above, let Z be the unit square with the lexicographic ordering topology. Since we have already shown that $C_\kappa(Z, Z)$ is not first countable, then we know that Z cannot have a point-countable base because of Theorem 1.8. But $C_\kappa(I, Z)$ is first countable since the image of each element of $C_\kappa(I, Z)$ lies in a vertical slice of Z. That is, if $f \in C_\kappa(I, Z)$, then $f(I)$ is actually second countable; so an argument like that in the proof of Theorem 1.8 will work to construct a countable base for f.

2. **Separable and Lindelöf spaces.** Characterizations of a function space being separable or Lindelöf have been given for certain classes of topological spaces. For example, Vidossich has shown in **[10]** that if X is a completely regular space and Y is a nontrivial separable convex subset of a locally convex Hausdorff space, then the following are equivalent.

(1) $C_\pi(X, Y)$ is separable.

(2) $C_\kappa(X, Y)$ is separable,

(3) X is submetrizable and has a dense subset of cardinality less than or equal to $2^{\aleph_0}$.

Also, Warner showed in **[11]** that $C_\kappa(X, R)$ is separable if and only if the topology on X is stronger than a separable metrizable topology on X; for example, if X is the union of a countable family of compact metrizable subsets.

As for the Lindelöf property, Corson and Lindenstrauss showed in [3] that if X is metrizable, then the following are equivalent.

(1) $C_\pi(X, R^\omega)$ is Lindelöf.

(2) $C_\kappa(X, R^\omega)$ is Lindelöf,

(3) X is separable.

Along the same lines, Zenor showed in **[12]** that if X is completely regular, then the following are equivalent.

(1) $C_\pi(X, R^\omega)$ is hereditary Lindelöf.

(2) $C_\pi(X, Y)$ is hereditary Lindelöf for every second countable space Y.

(3) X^ω is hereditary separable.

This was also shown to be true with the concepts of hereditary Lindelöf and hereditary separable interchanged.

In this section we prove general results which include these latter two results as corollaries. We will use the same basic tool which was used in **[12]**, and which we state as the following lemma.

LEMMA 2.1. *Let X, Y, and Z be topological spaces with Y second countable, and let $f: X \times Z \to Y$ be a function such that*

(a) *f is continuous on X, and*

(b) *Z has the weakest topology for which f is continuous on Z. Then,*

(1) *if X^ω is hereditary Lindelöf, Z is hereditary separable, and*
(2) *if X^ω is hereditary separable, Z is hereditary Lindelöf.*

For the next theorem, we will need to consider a closed collection Γ from X as a topological space. The topology on Γ will be the Vietoris topology (see for example [8]). Basic open sets in Γ will be those of the form $\langle U_1, \ldots, U_n \rangle \equiv \{C \in \Gamma \mid C \subseteqq U_1 \cup \cdots \cup U_n \text{ and } C \cap U_i \neq \emptyset$ for each $i\}$, where $U_1, \ldots, U_n$ are open in X. The notation $\Gamma(Y)$ will refer to the space $\{f(C) \mid f \in C(X, Y) \text{ and } C \in \Gamma\}$ with the Vietoris topology. Finally, the notation Γ^ω, $\Gamma^\omega(Y)$, and $C_\Gamma^\omega(X, Y)$ in the following theorems will be used to denote the countable infinite product spaces.

THEROEM 2.2. *Let X and Y be topological spaces, and let Γ be a point-proper closed collection from X such that $\Gamma(Y)$ is second countable. Then,*
(1) *if Γ^ω is hereditary Lindelöf, $C_\Gamma^\omega(X, Y)$ is hereditary separable, and*
(2) *if Γ^ω is hereditary separable, $C_\Gamma^\omega(X, Y)$ is hereditary Lindelöf.*

PROOF. Define $\Phi: \Gamma \times C_\Gamma^\omega(X, Y) \to \Gamma^\omega(Y)$ as follows. If $C \in \Gamma$ and $F = (f_i)_{i \in \omega} \in C_\Gamma^\omega(X, Y)$, then take $\Phi(C, F) = (f_i(C))_{i \in \omega}$. For each $C \in \Gamma$, let $\Phi_C: C_\Gamma^\omega(X, Y) \to \Gamma^\omega(Y)$ be defined by $\Phi_C(F) = \Phi(C, F)$. Also for each $F \in C_\Gamma^\omega(X, Y)$, let $\Phi_F: \Gamma \to \Gamma^\omega(Y)$ be defined by $\Phi_F(C) = \Phi(C, F)$. Then 2.2 will be established if we show that Φ satisfies the hypotheses of 2.1.

To see that each Φ_F is continuous, let $C \in \Gamma$ and let $B = \pi_i^{-1}(\langle V_1, \ldots, V_k \rangle)$ be a subbasic open subset of $\Gamma^\omega(Y)$ containing $\Phi_F(C)$. If $F = (f_j)_{j \in \omega}$, then define $W = \langle f_i^{-1}(V_1), \ldots, f_i^{-1}(V_k) \rangle$. It is straightforward to verify that $C \in W$ and that $\Phi_F(W) \subseteqq B$.

For each $C \in \Gamma$, each $i \in \omega$, and each open V in Y, the subbasic open set $\pi_i^{-1}([C, V])$ in $C_\Gamma^\omega(X, Y)$ can be written

$$\pi_i^{-1}([C, V]) = \Phi_C^{-1}(\pi_i^{-1}(\langle V \rangle)).$$

Therefore, part (b) of Lemma 2.1 will follow if we can show that each Φ_C is continuous. So with C fixed, let $F = (f_i)_{i \in \omega} \in C_\Gamma^\omega(X, Y)$ and let $B = \pi_i(\langle V_1, \ldots, V_k \rangle)$ be a basic open subset of $\Gamma^\omega(Y)$ containing $\Phi_C(F)$. Then $C \subseteqq f_i^{-1}(V_1 \cup \cdots \cup V_k)$, and there exist $x_1, \ldots, x_k \in C$ such that $x_j \in f_i^{-1}(V_j)$ for each $j = 1, \ldots, k$. Since Γ is point-proper, there exist $C_1, \ldots, C_k \in \Gamma$ with $x_j \in C_j \subseteqq f_i^{-1}(V_j)$ for each j. Now define

$$V = [C_1, V_1] \cap \cdots \cap [C_k, V_k] \cap [C, V_1 \cup \cdots \cup V_k].$$

Then $F \in \pi_i^{-1}(V)$, and $\Phi_C(\pi_i^{-1}(V)) \subseteqq B$.

We now give a partial converse of 2.2.

THEOREM 2.3. *Let X be a completely regular space, let Y be a second countable space which contains a nontrivial path, and let Γ be a point-proper closed collection from X. Then,*

(1) *if $C^{\omega}_{\Gamma}(X, Y)$ is hereditary Lindelöf, X^{ω} is hereditary separable, and*

(2) *if $C^{\omega}_{\Gamma}(X, Y)$ is hereditary separable, X^{ω} is hereditary Lindelöf.*

PROOF. Define $\Psi\colon C_{\Gamma}(X, Y) \times X^{\omega} \to Y^{\omega}$ as follows. If $f \in C_{\Gamma}(X, Y)$ and $x = (x_i)_{i \in \omega} \in X^{\omega}$, then take $\Psi(f, x) = (f(x_i))_{i \in \omega}$. For each $f \in C_{\Gamma}(X, Y)$, let $\Psi_f\colon X^{\omega} \to Y^{\omega}$ be defined by $\Psi_f(x) = \Psi(f, x)$. Also for each $x \in X^{\omega}$, let $\Psi_x\colon C_{\Gamma}(X, Y) \to Y^{\omega}$ be defined by $\Psi_x(f) = \Psi(f, x)$. Then 2.3 will be established if we show that Ψ satisfies the hypotheses of 2.1.

To see that each Ψ_x is continuous, let $f \in C_{\Gamma}(X, Y)$ and let $B = \pi_i^{-1}(V)$ be a subbasic open subset of Y^{ω} containing $\Psi_x(f)$. If $x = (x_j)_{j \in \omega}$, then $f(x_i) \in V$. Since Γ is point-proper, there exists a $C \in \Gamma$ such that $x_i \in C \subseteq f^{-1}(V)$. Then define $W = [C, V]$, which contains f. It is now easy to see that $\Psi_x(W) \subseteq B$.

Let $S = \{\Psi_f^{-1}(\pi_i^{-1}(V)) \mid f \in C_{\Gamma}(X, Y),\ i \in \omega$, and V is open in $Y\}$. Each member of S is open since $\Psi_f^{-1}(\pi_i^{-1}(V)) = \pi_i^{-1}(f^{-1}(V))$. Finally, it remains to show that S generates all the open subsets of X^{ω}. So let $x = (x_i)_{i \in \omega} \in X^{\omega}$, and let $B = \pi_i^{-1}(U)$ be a subbasic open subset of X^{ω} containing x. Let $\alpha\colon I \to Y$ be a continuous function such that $\alpha(0) \neq \alpha(1)$. Since X is completely regular, there exists an $f \in C_{\Gamma}(X, Y)$ such that $f(X) \subseteq \alpha(I)$, $f(x_i) = \alpha(0)$, and $f(X \backslash U) = \{\alpha(1)\}$. Let $V = Y \backslash \{\alpha(1)\}$, and let $W = \Psi_f^{-1}(\pi_i^{-1}(V))$. Since $x_i \in f^{-1}(V) \subseteq U$, then $x \in W \subseteq B$.

The next two statements are then corollaries of 2.2 and 2.3.

COROLLARY 2.4. *Let X be a completely regular space, and let Y be a second countable space with contains a nontrivial path. Then*

(1) *$C^{\omega}_{\pi}(X, Y)$ is hereditary Lindelöf if and only if X^{ω} is hereditary separable, and*

(2) *$C^{\omega}_{\pi}(X, Y)$ is hereditary separable if and only if X^{ω} is hereditary Lindelöf.*

COROLLARY 2.5. *Let X be a metric space, and let Y be a second countable space which contains a nontrivial path. Then the following are equivalent.*

(1) *X is separable.*

(2) *$C^{\omega}_{\pi}(X, Y)$ is hereditary separable.*

(3) *$C^{\omega}_{\pi}(X, Y)$ is hereditary Lindelöf.*

(4) *$C^{\omega}_{\kappa}(X, Y)$ is hereditary separable.*

(5) *$C^{\omega}_{\kappa}(X, Y)$ is hereditary Lindelöf.*

PROOF. If X is a separable metric space, then the space of all nonempty compact subsets of X under the Vietoris topology is a separable metric space (see **[8]**).

Two questions are suggested by these results. First, if X^{ω} is hereditary separable (hereditary Lindelöf, respectively), is Γ^{ω} hereditary separable (hereditary Lindelöf, respectively)? Second, if $C_{\Gamma}(X, Y)$ is hereditary separable (hereditary Lindelöf, respectively), is $C_\Gamma^\omega(X, Y)$ hereditary separable (hereditary Lindelöf, respectively)? The converses of these questions are, of course, true.

3. **Cosmic spaces and $\aleph_0$-spaces.** The concepts of cosmic spaces and $\aleph_0$-spaces are strengthenings of the concepts of hereditary Lindelöf and hereditary separable spaces. A cosmic space is a regular space having a countable network, and an $\aleph_0$- space is a regular space having a countable κ-network (see [9]). Basic properties for these concepts include:

(1) every separable metric space is an $\aleph_0$-space,

(2) every $\aleph_0$-space is a cosmic space,

(3) every cosmic space is both hereditary Lindelöf and hereditary separable, and

(4) every first countable $\aleph_0$-space is a separable metric space.

Michael showed in [9] that if X and Y are cosmic spaces, then $C_\pi(X, Y)$ is a cosmic space; and that if X and Y are $\aleph_0$-spaces, then $C_\kappa(X, Y)$ is an $\aleph_0$-space. In fact, when X is completely regular, the following are equivalent.

(1) $C_\kappa(X, R)$ is a cosmic space.

(2) $C_\kappa(X, R)$ is an $\aleph_0$-space.

(3) X is an $\aleph_0$-space.

Also, when X is completely regular, $C_\pi(X, R)$ is a cosmic space if and only if X is a cosmic space. On the other hand, $C_\pi(X, R)$ is an $\aleph_0$-space if and only if X is countable.

In this section we consider closed-open topologies for more general compact collections Γ. The key concept here is that of what we shall call a *Γ-cosmic space*, where Γ is a compact collection from the space. By this we mean a space X having a countable Γ-network. If Γ is point-proper, the concept of a Γ-cosmic space is intermediate between the concepts of a cosmic space and an $\aleph_0$-space; but it may not be strictly intermediate. For example, if Σ denotes the nonempty compact countable subsets of the space, then Guthrie has shown in [6] that a regular space is a Σ-cosmic space if any only if it is an $\aleph$-space.

If Γ is a compact collection from X, then Γ will be called *countably full* if whenever $C \in \Gamma$ and D is a compact subset of X containing C such that $D \setminus C$ is countable, then $D \in \Gamma$. Also, if X and Y are spaces and Δ is a compact collection from Y, then the notation $\Delta^{-1}(X)$ will stand for the compact collection $\{C|C$ is a compact subset of X such that $f(C) \in \Delta$ for every $f \in C(X, Y)\}$. The notation Δ^{-1} will be used for $\Delta^{-1}(X)$ if X is understood.

THEOREM 3.1. *Let Γ and Δ be compact collections from spaces X and Y, respectively, such that $\Gamma \subseteqq \Delta^{-1}(X)$ and Δ is countably full. If X is a Γ-cosmic space and Y is a Δ-cosmic space, then $C_\Gamma(X, Y)$ is a cosmic space.*

PROOF. Let $\mathscr{P}$ be a countable Γ-network for X, and let $\mathscr{S}$ be a countable Δ-network for Y. We may assume without loss of generality that $\mathscr{P}$ is closed under finite intersections. Let $\mathscr{F} = \{[P, S] \mid P \in \mathscr{P} \text{ and } S \in \mathscr{S}\}$. Now let $C \in \Gamma$, let V be open in Y, and let $f \in [C, V]$. We shall find $P \in \mathscr{P}$ and $S \in \mathscr{S}$ such that $C \subseteqq P$, $S \subseteqq V$, and $f(P) \subseteqq S$. Then $f \in [P, S] \subseteqq [C, V]$, so that $\mathscr{F}$ would generate a countable network for $C_\Gamma(X, Y)$.

Let $\{P_n\}$ be the members P of $\mathscr{P}$ such that $C \subseteqq P \subseteqq f^{-1}(V)$, and let $\{S_n\}$ be the members of $\mathscr{S}$ which are contained in V. Suppose that for each n, there exists an $x_n \in P_1 \cap \cdots \cap P_n$ such that $f(x_n) \notin S_n$. Then let A be the union of C and the sequence $\{x_n\}$. It can be seen that $\{x_n\}$ is eventually in each neighborhood of C, so that A is compact. Since Δ is countably full, $f(A) \in \Delta$; and since $A \subseteqq f^{-1}(V)$, $f(A) \subseteqq V$. Thus for some k, $f(A) \subseteqq S_k \subseteqq V$. But $f(x_k) \notin S_k$, which is a contradiction. Therefore, there exists an n such $f(P_1 \cap \cdots \cap P_n) \subseteqq S_n$, so that $P = P_1 \cap \cdots \cap P_n$ and $S = S_n$ are the desired elements of $\mathscr{P}$ and $\mathscr{S}$.

The converse of 3.1 is not true without some restrictions. For example, let X be a connected space which is not a cosmic space (say not separable), and let Y be a cosmic space which is totally disconnected (say the rationals). Then $C_\pi(X, Y)$ is homeomorphic to Y, and hence, is a cosmic space.

THEOREM 3.2. *Let X be a completely regular space, let Y contain a nontrivial path, and let Γ be a compact collection from X. If $C_\Gamma(X, Y)$ is a cosmic space, then X is a Γ-cosmic space.*

PROOF. Let $\mathscr{P}$ be a countable network for $C_\Gamma(X, Y)$, and let $\alpha\colon I \to Y$ be a continuous function such that $\alpha(0) \neq \alpha(1)$. Also let $A = \alpha(I)$, and let $F = \{f \in C_\Gamma(X, Y) \mid f(X) \subseteqq A\}$. For every $P \in \mathscr{P}$, let $P^* = \{x \in X \mid \alpha^{-1}g(x) > 0 \text{ for every } g \in P \cap F\}$. To see that $\{P^* \mid P \in \mathscr{P}\}$ is a Γ-network for X, let $C \in \Gamma$ and let U be open in X with $C \subseteqq U$. Since X is completely regular, there exists a continuous function $\beta\colon X \to I$ such that $\beta(C) = \{1\}$ and $\beta(X \backslash U) = \{0\}$. Let $f = \alpha \circ \beta$, and let $V = Y \backslash \{\alpha(0)\}$. Now $f \in [C, V]$, so that there exists a $P \in \mathscr{P}$ such that $f \in P \subseteqq [C, V]$. To see that $C \subseteqq P^*$, let $x \in C$. If $g \in P \cap F$, then $g(C) \subseteqq V \cap A$, so that $g(x) \in V \cap A$. Thus $g(x) \in A \backslash \{\alpha(0)\}$, so that $\alpha^{-1}g(x) > 0$, and hence $x \in P^*$. Finally, to see that $P^* \subseteqq U$, let $x \in P^*$. Then, since $f \in P \cap F$, $\alpha^{-1}f(x) > 0$. Thus $f(x) \in A \backslash \{\alpha(0)\} \subseteqq V$, so that since $f(X \backslash U) = \{\alpha(0)\}$, then $x \in U$.

As a corollary of 3.1 and 3.2 we have the following.

COROLLARY 3.3. *Let X be a completely regualr space, and let Y be an $\aleph_0$-space which contains a nontrivial path. Then*

(1) *$C_\kappa(X, Y)$ is a cosmic space if and only if X is an $\aleph_0$-space, and*

(2) *$C_\pi(X, Y)$ is a cosmic space if and only if X is a cosmic space.*

As in section 2, we can consider a compact collection Γ from X as a topological space by putting the Vietoris topology on it. Using this topology, we have the following relationship.

THEOREM 3.4. *Let Γ be a compact collection from X. If Γ is a cosmic space, then X is a Γ-cosmic space. Conversely, if Γ is hereditary and X is Γ-cosmic space, then Γ is a cosmic space.*

PROOF. Let $\mathscr{P}$ be a countable network for Γ. To see that $\{\bigcup P \mid P \in \mathscr{P}\}$ is *a* Γ-network for X, let $C \in \Gamma$ and let U be an open subset of X with $C \subseteqq U$. Then $\langle U \rangle$ is an open neighborhood of the element C in Γ, so that there exists a $P \in \mathscr{P}$ such that $C \in P \subseteqq \langle U \rangle$. Clearly $C \subseteqq \bigcup P$ and $\bigcup P \subseteqq U$.

Conversely, let $\mathscr{R}$ be a countable Γ-network for X. For each $R_1, \ldots, R_n \in \mathscr{R}$, let $(R_1, \ldots, R_n)^* = \{C_1 \cup \cdots \cup C_n \mid \text{for each } 1 \leqq i \leqq n,\ C_i \in \Gamma$ and $C_i \subseteqq R_i\}$. To see that $\{(R_1, \ldots, R_n)^* \mid n$ is an integer and $R_1, \ldots, R_n \in \mathscr{R}\}$ is a network for Γ, let $C \in \Gamma$ and let $\langle U_1, \ldots, U_n \rangle$ be a basic open set in Γ containing C. Then there exist closed subsets $C_1, \ldots, C_n$ of C such that each $C_1 \subseteqq U_i$ and $C_1 \cup \cdots \cup C_n = C$. Since Γ is hereditary, each $C_i \in \Gamma$. So for each i, there exists an $R_i \in \mathscr{R}$ such that $C_i \subseteqq R_i \subseteqq U_i$. Then $C \in (R_1, \ldots, R_n)^*$, which in turn is contained in $\langle U_1, \ldots, U_n \rangle$.

A corollary of 3.4 is that X is an $\aleph_0$-space if and only if the space of compact subsets of X is a cosmic space.

If we use the Cantor set K, we can relate Γ being a cosmic space to a certain function space being a cosmic space.

THEOREM 3.5. *Let Γ be a hereditary compact collection from X. If $C_{\Gamma^{-1}}(K, X)$ is cosmic, then Γ is cosmic.*

PROOF. Define $F = \{f \in C_{\Gamma^{-1}}(K, X) \mid f(K) \in \Gamma\}$. Since $C_{\Gamma^{-1}}(K, X)$ is cosmic, then F is cosmic. Also define $\varphi: F \to \Gamma$ by $\varphi(f) = f(K)$. Now since X is cosmic (being a subspace of $C_{\Gamma^{-1}}(K, X)$) and since a (locally) compact cosmic space is a (separable) metrizable space, then each element of Γ is metrizable. Therefore φ must be onto. To see that φ is continuous, let $f \in F$ and let $\langle U_1, \ldots, U_n \rangle$ be a basic neighborhood of $\varphi(f)$ in Γ. Then there exist closed subsets $C_1, \ldots, C_n$ of $f(K)$ such that each $C_i \subseteqq U_i$ and $C_1 \cup \cdots \cup C_n = f(K)$. Since Γ is hereditary, each $C_i \in \Gamma$, so that $f^{-1}(C_i) \in \Gamma^{-1}$. Then $F \cap [f^{-1}(C_1), U_1] \cap \cdots \cap [f^{-1}(C_n), U_n]$ is an open neighborhood of f in F which maps into $\langle U_1, \ldots, U_n \rangle$ under φ. Therefore, φ is continuous; and since the continuous image of a cosmic space is a cosmic space, then Γ is a cosmic space.

Now as a corollary to 3.1, 3.4, and 3.5 we have the following.

COROLLARY 3.6. *Let Γ be a countably full hereditary compact collection* *from X. Then X is a Γ-cosmic space if and only if $C_{\Gamma^{-1}}(K, X)$ is a cosmic space.*

In order to determine when a function space is an $\aleph_0$-space, we need to introduce the idea of a k_Γ-space, which is a strengthening of the concept of a k-space. If Γ is a compact collection from X, then X will be called k_Γ-space if the open subsets of X are precisely those sets U such that $U \cap C$ is open in C for every $C \in \Gamma$. For example, if Γ consists of all compact subsets, then a k_Γ-space is precisely a κ-space; if Γ consists of the singleton subsets, then a κ_Γ-space is precisely a discrete space; and if Γ consists of the compact countable subsets, then a k_Γ-space is precisely a sequential space (for Hausdorff spaces).

If Γ is a compact collection from X, and if X is not a k_Γ-space, we can enlarge the topology on X to turn it into a k_Γ-space as follows. Let $k_\Gamma(X)$ be the set X with the topology consisting of those subsets W of X such that $W \cap C$ is open in C (with respect to the topology inherited from X) for every $C \in \Gamma$. Then it is easily seen that the elements of Γ are compact as subsets of $k_\Gamma(X)$, and that $k_\Gamma(X)$ is a k_Γ-space. Now if $k_\Gamma(X)$ is a Γ-cosmic space, then X is a Γ-cosmic space. For certain Γ, the converse of this is true.

THEOREM 3.7. *Let Γ be a countably full compact collection from X. If X is a Γ-cosmic space, then $k_\Gamma(X)$ is* a *Γ-cosmic space.*

The proof of 3.7 is the same as the proof of Proposition 8.2 in **[9]**, and involves an argument much like that used in proving 3.1.

THEOREM 3.8. *Let Γ be a countably full hereditary compact collection from X. If X is a Γ-cosmic space and Y is an $\aleph_0$-space, then $C_\Gamma(X, Y)$, is an $\aleph_0$-space.*

PROOF. We outline where this proof differs from that of 3.1. First, let $\mathscr{P}$ be a countable Γ-network for $k_\Gamma(X)$ (whose existence is guaranteed by 3.7) and let $\mathscr{S}$ be a countable κ-network for Y. Instead of taking $f \in [C, V]$, take a compact subset D of $C_\Gamma(X, Y)$ contained in $[C, V]$. In the second paragraph, instead of using $f^{-1}(V)$, use the set $\{x \in X \mid f(x) \in V$ for every $f \in D\}$, which can be shown to be an open subset of $k_\Gamma(X)$. Also instead of using the compact set $f(A)$, use the set $\{f(x) \mid f \in D$ and $x \in A\}$, which can be shown to be a compact subset of Y by using the hypotheses that Γ is countably full and hereditary.

We can put all the above results together and make the following statement.

THEOREM 3.9. *Let X be a completely regular space, let Y be an $\aleph_0$-space which contains a nontrivial path, and let Γ be a countably full hereditary* *compact collection from X. Then the following are equivalent.*

(1) *X is a Γ-cosmic space.*

(2) *$k_\Gamma(X)$ is a Γ-cosmic space.*

(3) *Γ is a cosmic space.*

(4) *$C_{\Gamma^{-1}}(K, X)$ is a cosmic space.*

(5) *$C_\Gamma(X, Y)$ is an $\aleph_0$-space.*

(6) *$C_\Gamma(X, Y)$ is a cosmic space.*

(7) *$C_\Gamma(k_\Gamma(X), Y)$ is an $\aleph_0$-space.*

(8) *$C_\Gamma(k_\Gamma(X), Y)$ is a cosmic space.*

Finally, we state another application of the k_Γ-space concept. This is an Ascoli theorem for the more general closed-open function space topologies. The proof is similar to that of the standard Ascoli theorem involving the compact-open topology.

THEOREM 3.10. *Let F be a subset of $C(X, Y)$, and let Γ be a compact collection from X. Then the following are equivalent.*

(1) *F is compact in $C_\Gamma(X, Y)$.*

(2) *F is compact in $C_\Gamma(k_\Gamma(X), Y)$.*

(3) *F is compact in $C_\kappa(k_\Gamma(X), Y)$.*

(4) (i) *F is closed in $C_\Gamma(k_\Gamma(X), Y)$,*

(ii) *$F[x]$ is compact for every $x \in X$, and*

(iii) *F is evenly continuous on every element of Γ.*

As a result, we see that if X is a k_Γ-space, then F is compact in $C_\Gamma(X, Y)$ if and only if it is compact in $C_\kappa(X, Y)$.

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