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# MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. Let $\Lambda^*(p)$ the class of functions $f(z)$ univalent and meromorphic in $\Delta = \{z \mid |z| < 1\}$ with simple pole at $z = p$, $0 < p < 1$, $f(0) = 1$ and which map $\Delta$ onto a domain whose complement is starlike with respect to the origin. We discuss the coefficients of the Taylor series $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $|z| < p$ and the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} b_n z^n$, $p < |z| < 1$. We also obtain best possible order estimates on $L(r)$, the length of the image of $\{z : |z| = r\}$ for a function in $\Lambda^*(p)$. Estimates on the integral means of higher order derivatives are also obtained and in the last section a question of Holland [5] is answered.

1. **Introduction.** Let $\Sigma(p)$ denote the class of functions $f(z)$ which are meromorphic and univalent in $\Delta = \{z \| z| < 1\}$ with a simple pole at $z = p$, $0 < p < 1$, and with $f(0) = 1$. If, further, there exists $\delta$, $p < \delta < 1$, such that

$$\text{Re}\,\frac{zf'(z)}{f(z)} < 0 \tag{1.1}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \text{Re}\,\frac{zf'(z)}{f(z)}\,d\theta = -1 \tag{1.2}$$

for $\delta < |z| < 1$ with $z = re^{i\theta}$, we say that $f(z)$ is in $\Lambda(p)$. Functions in $\Lambda(p)$, which have been discussed in [10, 11], map $\Delta$ onto a domain whose complement is starlike with respect to the origin. However, there exist functions with pole at $p$ having this mapping property which do not satisfy (1.1) if $p > 1/2$. The function

$$F(z) = \frac{-p(1 + z)^2}{(z - p)(1 - pz)}$$

maps $\Delta$ onto the complement of the interval $[-4p/(1 - p)^2, 0]$ but does not satisfy (1.1) if $p > 1/2$ [10].

Let $\Lambda^*(p)$ denote the class of functions $f(z)$ which have the representation

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$$f(z) = \frac{-pzg(z)}{(z-p)(1-pz)} \tag{1.3}$$

where $g(z)$ is in $\Sigma^*$, the class of normalized meromorphic starlike functions with pole at the origin. The class $\Lambda^*(p)$ contains $\Lambda(p)$ as a dense subset **[10]**.

The following theorem, although obvious, was never explicitly stated in **[10]** or **[11]**.

THEOREM 1. *A function $f$ in $\Sigma(p)$ is in $\Lambda^*(p)$ if and only if it maps $\Delta$ onto a domain whose complement is starlike with respect to the origin.*

PROOF. If $f \in \Lambda^*(p)$, it has the representation (1.3). Using the fact that $-pz/(z-p)(1-pz)$ is real for $|z| = 1$, it is easily seen that $f(z)$ has the desired mapping property.

Conversely, suppose that $f$ in $\Sigma(p)$ maps $\Delta$ onto a domain whose complement is starlike with respect to the origin. Letting $\alpha$ denote the residue of $f$ at $z = p$, it follows that

$$h(z) = \frac{1-p^2}{\alpha} f\left[\frac{z+p}{1+pz}\right]$$

belongs to $\Sigma^*$. Defining $g(z)$ by

$$\begin{aligned} g(z) &= \frac{(z-p)(1-pz)}{-pz} f(z) \\ &= \frac{(z-p)(1-pz)}{-pz} \frac{\alpha}{1-p^2} h\left[\frac{z-p}{1-pz}\right] \end{aligned}$$

and using the fact that $(z-p)(1-pz)/(-pz)$ is real for $|z| = 1$, we see that $g \in \Sigma^*$, and consequently $f(z)$ has the representation (1.3).

We note that $\Lambda(p)$ is a proper subset of $\Lambda^*(p)$ if $p > 1/2$, while $\Lambda(p) = \Lambda^*(p)$ if $p < (3 - 2\sqrt{2})^{1/2}$ **[10]**.

2. **Coefficient bounds.** In this section we examine the coefficients in the series representations of $f(z)$ in $\Lambda^*(p)$, both the Taylor series $1 + \sum_{n=1}^{\infty} a_n z^n$, $|z| < p$, and the Laurent series $\sum_{n=-\infty}^{\infty} b_n z^n$, $p < |z| < 1$. With regard to the Taylor series let $\{\lambda_n\}$ and $\{\mu_n\}$ denote the coefficient sequences of $-p(1-z)^2/(z-p)(1-pz)$ and $-p(1+z)^2/(z-p)(1-pz)$, respectively. It is easy to check that

$$\lambda_n = \left[\frac{1-p}{1+p}\right]\left[\frac{1-p^{2n}}{p^n}\right], \quad \mu_n = \left[\frac{1+p}{1-p}\right]\left[\frac{1-p^{2n}}{p^n}\right].$$

The second author proved $a_n \geqq \lambda_n$ for all $n$ if $f \in \Lambda^*(p)$ and is real on the real axis **[11]**. He also pointed out that, under the same assumptions, $a_n \leqq \mu_n$ follows from results of Goodman **[3]**. Furthermore, the inequality $|a_n| \leqq \mu_n$, $1 \leqq n \leqq 6$, follows from some work of Jenkins **[6]** for any $f \in \Lambda^*(p)$. We suspect that the inequalities

$$\ell_n \leqq \operatorname{Re} a_n \leqq |a_n| \leqq \mu_n, \; n \geqq 1$$

hold generally for $f \in \Lambda^*(p)$. In support of this conjecture we now prove that it holds for $n = 1$ and $n = 2$. We first require the following lemma concerning $\mathscr{P}$, the class of functions $P(z)$ having positive real part in $\Delta$, $P(0) = 1$.

LEMMA 1. *If* $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *belongs to* $\mathscr{P}$ *and* $0 < p < 1$, *then*

$$\operatorname{Re}(c_2 + 2(p + p^{-1})c_1) \geqq 2 - 4(p + p^{-1}). \tag{2.1}$$

PROOF. The Herglotz representation of $P$ gives a probability measure $\mu$ such that

$$P(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t), \; |z| < 1.$$

From this we obtain $c_n = 2\int_0^{2\pi} e^{-int} d\mu(t)$. Consequently,

$$c_2 + 2(p + p^{-1})c_1 = 2\int_0^{2\pi} (e^{-i2t} + 2(p + p^{-1})e^{-it}) d\mu(t).$$

Since $p + p^{-1} > 2$, the function

$$g(t) \equiv \cos 2t + 2(p + p^{-1})\cos t$$

is decreasing on $[0, \pi]$ and increasing on $[\pi, 2\pi]$. Thus,

$$\begin{aligned} \operatorname{Re}(c_2 + 2(p + p^{-1})c_1) &= 2\int_0^{2\pi} g(t) d\mu(t) \\ &\geqq 2g(\pi) = 2 - 4(p + p^{-1}). \end{aligned}$$

THEOREM 2. *If* $f(z)$ *is in* $\Lambda^*(p)$ *and* $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, $|z| < p$, *then*

$$\ell_n \leqq \operatorname{Re} a_n \leqq |a_n| \leqq \mu_n, \; n = 1, 2.$$

PROOF. From previous remarks we need only show that $\operatorname{Re} a_n \geqq \ell_n$, $n = 1, 2$. Since $f(z)$ is in $\Lambda^*(p)$, it has the representation (1.3) with $g(z)$ in $\Sigma^*$. Let $Q(z) = -zg'(z)/g(z)$; then $Q \in \mathscr{P}$ and it is easily seen that

$$\frac{zf'(z)}{f(z)} = \frac{-p(1 - z^2)}{(z - p)(1 - pz)} - Q(z), \; |z| < 1. \tag{2.2}$$

If we let $P(z) = 1/Q(z)$, then $P \in \mathscr{P}$ and (2.2) can be rewritten as

$$f(z) = -\left[\frac{p(1 - z)^2}{(z - p)(1 - pz)} f(z) + zf'(z)\right] P(z). \tag{2.3}$$

Letting $P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, $z < 1$, expanding the right hand side of (2.3) as a power series in $|z| < p$, and comparing coefficients, we obtain

$$a_1 = c_1 + (p + p^{-1}) \tag{2.4}$$

and

$$(2.5) \qquad 2a_2 = c_2 + (p + p^{-1})c_1 + (p + p^{-1})a_1 + p^2 + p^{-2}.$$

Using (2.4), we can rewrite (2.5) as

$$(2.6) \qquad 2a_2 = c_2 + 2(p + p^{-1})c_1 + (p + p^{-1})^2 + (p^2 + p^{-2}).$$

Since $\operatorname{Re} c_1 \geqq -2$, we obtain from (2.4) that

$$\operatorname{Re} a_1 \geqq -2 + (p + p^{-1}) = \ell_1.$$

Using (2.1) we obtain from (2.6) that

$$2 \operatorname{Re} a_2 \geqq 2 - 4(p + p^{-1}) + (p + p^{-1})^2 + (p^2 + p^{-2}),$$

which gives

$$\operatorname{Re} a_2 \geqq \frac{(1 + p^2)(1 - p)^2}{p^2} = \ell_2.$$

This completes the proof of Theorem 2.

We now discuss the coefficients of the Laurent series $\sum_{n=-\infty}^{\infty} b_n z^n$, $p < |z| < 1$. Libera and the second author pointed out in **[10]** that

$$|b_n| \leqq \frac{1}{p^n}\left(\frac{1 + p}{1 - p}\right) \text{ for } n = -1, -2, \ldots,$$

that these bounds are sharp, and that $|b_n| = O(n^{-1/2})$ for $n \geqq 1$. We obtain the order estimate $|b_n| = O(n^{-1})$ and prove that this is best possible.

THEOREM 3. *If $f(z)$ is in $\Lambda^*(p)$ and $f(z) = \sum_{n=-\infty}^{\infty} b_n z^n$, $p < |z| < 1$, then $|b_n| = O(n^{-1})$, $n \geqq 1$. Furthermore, there exists $f \in \Lambda^*(p)$ with* $\lim_{n\to\infty} \sup n|b_n| > 0$.

PROOF. There exists $g \in \Sigma^*$, $g(z) = z^{-1} + \sum_{n=0}^{\infty} A_n z^n$, $0 < |z| < 1$, such that

$$(2.7) \qquad f(z) = \frac{-pz}{(z - p)(1 - pz)} g(z), \quad |z| < 1.$$

Expanding the right hand side of (2.7) for $p < |z| < 1$ and comparing coefficients we obtain

$$(2.8) \quad b_n = \frac{-p}{1 - p^2}[p^{n+1} + p^n A_0 + \cdots + pA_{n-1} + A_n + pA_{n+1} + p^2 A_{n+2} + \cdots].$$

Using the estimate $|A_n| \leqq 2(n + 1)^{-1}$, $n \geqq 0$, proven by Clunie **[1]**, we obtain from (2.8)

$$
\begin{aligned}
|b_n| &\leqq \frac{p}{1-p^2}\left[p^{n+1} + 2\sum_{k=0}^{n} \frac{1}{n+1-k} p^k + \frac{2p}{n+2}\frac{1}{1-p}\right] \\
&\leqq \frac{p}{1-p^2}\left[p^{n+1} + 2\sum_{k=0}^{n} \frac{k+1}{n+1} p^k + \frac{2p}{n+2}\frac{1}{1-p}\right. \\
&< \left[\frac{p}{1-p^2}\left[p^{n+1} + \frac{2}{n+1}\frac{1}{(1-p)^2} + \frac{2p}{(n+2)(1-p)}\right]\right. \\
&= \frac{p}{(1-p^2)(1-p)}\left[p^{n+1}(1-p) + \frac{2}{(n+1)(1-p)} + \frac{2p}{n+2}\right].
\end{aligned}
$$

Since $p^{n+1}(1-p) \leqq (n+1)^{-1}$, $0 < p < 1$, we obtain

$$
\begin{aligned}
|b_n| &\leqq \frac{1}{n+1}\frac{p}{(1-p)^3}\max_{0\leqq p\leqq 1}\left[\frac{p-2p^2+3}{1+p}\right] \\
&= \frac{3p}{(1-p)^3}\cdot\frac{1}{n+1}.
\end{aligned}
$$

Thus $|b_n| = O(n^{-1})$.

To see that this order is best possible we note that Pommerenke [17] has constructed $F(z) = z^{-1} + \sum_{n=0}^{\infty} A_n z^n$ in $\Sigma^*$ such that $\lim_{n\to\infty} \sup n\,|A_n| > 0$. For this $F$ we define $f \in \Lambda^*(p)$ by

$$(z-p)(1-pz)f(z) = -pzf(z). \tag{2.9}$$

From (2.9) we obtain for $n \geqq 0$

$$b_{n+1} - (p + p^{-1})b_n + b_{n-1} = A_n$$

and so it follows that we must also have $\lim_{n\to\infty} \sup n|b_n| > 0$.

3. **Arclength.** For bounded regular univalent starlike functions Keogh [7] has shown that the arc length $L(r)$ of the image of the circle $|z| = r$ under the mapping $w = f(z)$ satisfies $L(r) = O(-\log(1-r))$. Hayman [4] then proved that $O$ may not be replaced by $o$. More recently Lewis [8] gave an example of such a function satisfying $\lim_{r\to\infty}\inf L(r)/(-\log(1-r)) > 0$. It is our purpose to establish that as $r \to 1$ the same results hold for functions in $\Lambda^*(p)$. In particular we will show that $L(r) = O(|r-p|^{-1}\log 1/(1-r))$, $r \neq p$.

Miller [14] discussed a class of starlike meromorphic functions having a different normalization than $\Lambda^*(p)$. He proved that

$$L(r) = O\left(\frac{1}{|r-p|}\log\frac{1}{|r-p|(1-r)}\right),\ r \neq p. \tag{3.1}$$

We point out that this estimate comes from an examination of his proof, as the final result is stated incorrectly. For the class $\Lambda^*(p)$ we can eliminate the $|r-p|^{-1}$ term within the logarithm in (3.1). We will make use of the following results of Pommerenke [16]. For $0 < r < 1$,

$$(3.2)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|1-re^{i\theta}|^{\mu}} \sim \begin{cases} \dfrac{2^{-\mu+1}\Gamma(\mu-1)}{[\Gamma(\mu/2)]^2}\,\dfrac{1}{(1-r)^{\mu-1}}, & \mu>1 \\ \dfrac{1}{\pi}\log\dfrac{1}{1-r}, & \mu=1. \end{cases}$$

This implies the existence of a positive constant $C_\mu$ so that

$$(3.3)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta}{|1-re^{i\theta}|^{\mu}} \leqq \begin{cases} C_\mu(1-r)^{-(\mu-1)}, & \mu>1 \\ C_1\log\dfrac{1}{1-r}, & \mu=1. \end{cases}$$

When $0<\mu<1$ the integral is a bounded function of $r$, $0<r<1$.

In what follows $C$ represents a constant independent of $f(z)$ and $r$, though it may change its value from line to line.

THEOREM 4. *If $f(z)$ is in $\Lambda^*(p)$, then*

$$L(r)=O\left(\frac{1}{|r-p|}\log\frac{1}{1-r}\right),\ r\neq p.$$

PROOF. As observed in **[10]** the function

$$P(z)=\left(\frac{-zf'(z)}{f(z)}\right)\frac{(z-p)(1-pz)}{z}$$

has positive real part in $\Delta$, with $P(0)=pf'(0)$. Hence

$$(3.4)\qquad f'(z)=\frac{P(z)}{(z-p)^2}\left[\frac{-f(z)(z-p)}{1-pz}\right].$$

From the representation (1.3) of $f(z)$ there exists $g(z)$ in $\Sigma^*$ so that

$$\frac{-f(z)(z-p)}{1-pz}=\frac{pz}{(1-pz)^2}g(z).$$

Thus, for $z=re^{i\theta}$,

$$(3.5)\qquad \left|\frac{-f(z)(z-p)}{1-pz}\right|\leqq\frac{pr}{(1-pr)^2}\cdot\frac{(1+r)^2}{r}$$
$$\leqq\frac{4p}{(1-p)^2}.$$

Also, from **[10]** we have

$$(3.6)\qquad |f'(0)|\leqq\frac{(1+p)^2}{p}.$$

Since $P(z)$ is subordinate to

$$P(0)\frac{1+ze^{-2i\arg p(0)}}{1-z},$$

it follows from Littlewood's subordination theorem, (3.3), and (3.6) that

$$\begin{aligned}(3.7)\qquad \int_{-\pi}^{\pi} |P(re^{i\theta})|d\theta &\leqq \int_{-\pi}^{\pi} \frac{|P(0)||1 + re^{i(\theta - 2\arg p(0))}|}{|1 - re\ |} d\theta \\ &\leqq 2(1 + p)^2 \int_{-\pi}^{\pi} \frac{d\theta}{|1 - re^{i\theta}|} \\ &\leqq C \log \frac{1}{1 - r}.\end{aligned}$$

Thus, if $(1 + p)/2 < |z| < 1$, we obtain from (3.4), making use of (3.5) and (3.7),

$$\begin{aligned}(3.8)\qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|d\theta &\leqq C \int_{-\pi}^{\pi} |P(re^{i\theta})|d\theta \\ &\leqq \frac{C}{r} \log \frac{1}{1 - r}.\end{aligned}$$

Also, for $0 < |z| \leqq (1 + p)/2$, $|z| \neq p$, we obtain

$$\begin{aligned}(3.9)\qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|d\theta &\leqq C \int_{-\pi}^{\pi} |re^{i\theta} - p|^{-2} d\theta \\ &\leqq C/|r - p|.\end{aligned}$$

We can combine (3.8) and (3.9) in the following way. If $(1 + p)/2 < |z| < 1$, then $(r - p)^{-1}$ is bounded away from zero. It follows from (3.8) that

$$(3.10)\qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|d\theta \leqq \frac{C}{r} \frac{1}{|r - p|} \log \frac{1}{1 - r}.$$

If $0 < |z| \leqq (1 + p)/2$, $z \neq p$, then $r^{-1} \log(1/(1 - r)) \geqq 1$ and so (3.9) yields

$$(3.11)\qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|d\theta \leqq \frac{C}{|r - p|} \frac{1}{r} \log \frac{1}{1 - r}.$$

Combining (3.10) and (3.11) we have

$$L(r) = \int_{-\pi}^{\pi} r|f'(re^{i\theta})|d\theta \leqq \frac{C}{|r - p|} \log \frac{1}{1 - r}, \quad r \neq p.$$

This completes the proof of the theorem.

We now use the example of Lewis to prove that the order result of Theorem 4 is best possible in the strongest possible sense. In particular, we find $f(z)$ in $\Lambda^*(p)$ such that

$$(3.12)\qquad \inf_{\substack{r>1 \\ r \neq p}} \frac{|r - p|L(r)}{-\log(1 - r)} > 0.$$

We first note by standard estimates that since every function in $\Lambda^*(p)$ has a simple pole at $z = p$, we have

$$\inf_{\substack{\varepsilon<r<1\\ r\neq p}} |r - p|\, L(r) > 0$$

for every $\varepsilon > 0$.
Also,

$$\begin{aligned}\lim_{r\to 0} \frac{L(r)}{-\log(1-r)} &= \lim_{r\to 0} \frac{\int_{-\pi}^{\pi} r\,|f'(re^{i\theta})|\,d\theta}{[r + r^2/2 + r^3/3 + \cdots]}\\ &= 2\pi|f'(0)| > 0,\end{aligned}$$

by Theorem 2. Thus, (3.12) will be verified after we complete the next theorem.

THEOREM 5. *There exists $f(z)$ in $\Lambda^*(p)$ such that*

$$\liminf_{r\to 1} L(r)/\log\frac{1}{1-r} > 0.$$

PROOF. Lewis [8] has constructed an analytic bounded starlike function $g(z)$, $g'(0) = 1$, such that

$$\int_{-\pi}^{\pi}\left|\frac{g'(re^{i\theta})}{g(re^{i\theta})}\right| r\,d\theta \geqq C\log\frac{1}{1-r},$$

where $C$ is a positive constant. Defining $F(z)$ in $\Sigma^*$ by $F(z) = g(z)^{-1}$, we have $zF'(z)/F(z) = -zg'(z)/g(z)$. Also, if $M$ is a bound on $|g(z)|$, $z \in \Delta$, then $|F(z)| \geqq M^{-1}$, $z \in \Delta$. Consequently,

$$\begin{aligned}\int_{-\pi}^{\pi}|F'(re^{i\theta})|\,r\,d\theta &\geqq \frac{1}{M}\int_{-\pi}^{\pi}\left|\frac{F'(re^{i\theta})}{F(re^{i\theta})}\right| r\,d\theta\\ &\geqq \frac{C}{M}\log\frac{1}{1-r}.\end{aligned}$$

Finally, we define $f(z)$ in $\Lambda^*(p)$ by $f(z) = G(z)F(z)$ where $G(z) = -pz/(z-p)(1-pz)$. The inequality

$$|f'(z)| \geqq |G(z)F'(z)| - |G'(z)F(z)|$$

gives

$$\begin{aligned}\liminf_{r\to 1}\frac{L(r)}{-\log(1-r)} &\geqq \frac{p}{(1+p)^2}\liminf_{r\to 1}\frac{\int_{-\pi}^{\pi}|F'(re^{i\theta})|\,r\,d\theta}{-\log(1-r)}\\ &\geqq \frac{p}{(1+p)^2}\,\frac{C}{M}.\end{aligned}$$

This completes the proof of the theorem.

4. **Integral means of derivatives.** We begin this section by extending Theorem 4 to obtain estimates on $\int_{-\pi}^{\pi}|f'(re^{i\theta})|^\lambda d\theta$, $\lambda > 0$, where $f \in \Lambda^*(\rho)$. We note that the integral means of $f(z)$ in $\Lambda^*(p)$ were discussed in **[11]** and for $f(z)$ in $\Sigma(p)$ in **[12]**. In the statement of the next theorem and in its proof, $C_\lambda$ signifies a constant depending on $\lambda$ but independent of $f(z)$ and $r$. Its value may change from line to line.

THEOREM 6. *Let $f(z)$ be in $\Lambda^*(p)$, then for $r \neq p$,*

$$\int_{-\pi}^{\pi} |f'(re^{i\theta})|^\lambda d\theta \leqq \begin{cases} C_\lambda \dfrac{1}{|r-p|^{2\lambda-1}(1-r)^{\lambda-1}}, & \lambda > 1 \\ C_1 \dfrac{1}{r|r-p|} \log \dfrac{1}{1-r}, & \lambda = 1 \\ C_\lambda \dfrac{1}{|r-p|^{2\lambda-1}}, & 1/2 < \lambda < 1 \\ C_{1/2} \log \dfrac{1}{|r-p|}, & \lambda = 1/2 \\ C_\lambda, & 0 < \lambda < 1/2. \end{cases}$$

PROOF. By Theorem 4, we may assume $\lambda \neq 1$. Making use of (3.4) (3.5), (3.7), (3.6) and (3.3) we obtain, for $(1 + p)/2 < |z| < 1$.

$$\text{(4.1)} \qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|^\lambda d\theta \leqq C_\lambda \int_{-\pi}^{\pi} \frac{d\theta}{|1 - re^{i\theta}|^\lambda} \leqq \begin{cases} C_\lambda \dfrac{1}{(1-r)^{\lambda-1}}, & \lambda > 1 \\ C_\lambda, & 0 < \lambda < 1, \end{cases}$$

and for $0 < |z| \leqq (1 + p)/2$,

$$\text{(4.2)} \qquad \int_{-\pi}^{\pi} |f'(re^{i\theta})|^\lambda d\theta \leqq C_\lambda \int_{-\pi}^{\pi} \frac{d\theta}{|re^{i\theta} - p|^{2\lambda}} \leqq \begin{cases} C_\lambda \dfrac{1}{|r-p|^{2\lambda-1}}, & \lambda > 1/2 \\ C_{1/2} \log \dfrac{1}{|r-p|}, & \lambda = 1/2 \\ C_\lambda, & 0 < \lambda < 1/2. \end{cases}$$

Combining (4.1) and (4.2) in the same manner as in §3, the conclusion of the theorem is obtained.

We remark that the sharpness of the case $\lambda = 1$ in Theorem 6 has already been discussed. Also, since $f(z)$ has a simple pole at $z = p$, it can be seen that the factors involving $|r - p|$ in Theorem 6 are actually necessary for each function in $\Lambda^*(p)$. We will now prove that the exponent

$\lambda - 1$ on $(1 - r)$ in the case $\lambda > 1$ cannot be replaced by a smaller exponent. For this purpose we note that $F(z) = (1 - z)^t(1 - pz)/z, 0 \leqq t \leqq 1$, is easily seen to be a member of $\Sigma^*$. Now, for $0 < \delta < \lambda - 1$, choose $t$ so that $0 < t < (\lambda - 1 - \delta)\lambda^{-1}$, and define $f \in \Lambda^*(p)$ by

$$f(z) = \frac{-pz}{(z - p)(1 - pz)} F(z) = -p \frac{(1 - z)^t}{z - p}.$$

Then, for $z = re^{i\theta}$,

$$\int_{-\pi}^{\pi} |f'(z)|^\lambda d\theta = \int_{-\pi}^{\pi} \frac{p^\lambda |1 - tp + (t - 1)z|^\lambda d\theta}{|z - p|^{2\lambda} |1 - z|^{\lambda - \lambda t}} \geqq C \int_{-\pi}^{\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\lambda - \lambda t}} \geqq C \frac{1}{(1 - r)^{\lambda(1-t)-1}},$$

by (3.2). (Here, as before, $C \neq 0$ may change its value from line to line.) Thus,

$$\lim_{r \to 1} (1 - r)^\delta \int_{-\pi}^{\pi} |f'(z)|^\lambda d\theta \geqq C \lim_{r \to 1} \frac{1}{(1 - r)^{\lambda(1-t)-1-\delta}} = \infty,$$

by our choice of $t$. This completes our argument.

We can now obtain estimates on the integral means of higher order derivatives by using a method of Feng and Mac Gregor [**2**]. For this purpose we need several lemmas, which are extensions of lemmas appearing in [**13**], to allow for a pole at $z = p$.

LEMMA 2. *Let $h(z)$ be analytic in $\Delta$, except at $z = p$, and satisfy the inequality*

$$|h(z)| \leqq \frac{A}{(1 - r)^\alpha |r - p|^\beta}, \quad |z| = r \neq p,$$

*where $A$, $\alpha$, and $\beta$ are positive constants. Then there exists a positive constant $B$ so that*

$$|h'(z)| \leqq \frac{B}{(1 - r)^{\alpha+1} |r - p|^{\beta+1}}, \quad |z| = r \neq p.$$

PROOF. Let $|z| = r$, $p < r < 1$, and let $\rho = (p + r)/2$, $\delta = (1 + r)/2$. Then

$$h'(z) = \frac{1}{2\pi i} \int_{|w|=\delta} \frac{h(w)}{(w - z)^2} dw - \frac{1}{2\pi i} \int_{|w|=\rho} \frac{h(w)}{(w - z)^2} dw.$$

Thus,

$$|h'(z)| \leqq \frac{\delta A}{(1-\delta)^\alpha(\delta-p)^\beta}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\delta e^{i\theta}-z|^2} + \frac{\rho A}{(1-\rho)^\alpha(\rho-p)^\beta}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|\rho e^{i\theta}-z|^2}.$$

Using Parseval's identity to estimate the two integrals on the right side we obtain

$$\begin{aligned}|h'(z)| &\leqq \frac{\delta A}{(1-\delta)^\alpha(\delta-p)^\beta(\delta^2-r^2)} + \frac{\rho A}{(1-\rho)^\alpha(\rho-p)^\beta(r^2-\rho^2)} \\ &= \frac{2^{\alpha+1}2^{\beta+1}A\delta}{(1-r)^{\alpha+1}(1+r-2p)^\beta(1+3r)} \\ &\quad + \frac{2^{\alpha+1}2^{\beta+1}A\rho}{(r-p)^{\beta+1}(2-r-\rho)^\alpha(3r+p)} \\ &< \frac{2^{\alpha+\beta+2}A}{(1-r)^{\alpha+1}(r-p)^\beta} + \frac{2^{\alpha+\beta+2}A}{(r-p)^{\beta+1}(1-r)^\alpha p} \\ &< \frac{B}{(1-r)^{\alpha+1}(r-p)^{\beta+1}}.\end{aligned}$$

For $|z| = r < p$, we let $\rho = (p+r)/2$, write

$$h'(z) = \frac{1}{2\pi i}\int_{|w|=\rho}\frac{h(w)}{(w-z)^2}\,dw$$

and proceed as before.

LEMMA 3. *Let $h(z)$ be analytic in $\Delta$, except $p$, and different from zero. If*

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \frac{A_1}{(1-r)^\alpha|r-p|^\beta},\quad |z| = r \neq p,$$

*where $A_1$, $\alpha$, and $\beta$ are positive constants, then there exist positive constants $A_n$ depending on $\alpha$ and $\beta$ so that*

$$\left|\frac{h^{(n)}(z)}{h(z)}\right| \leqq \frac{A_n}{(1-r)^{\alpha+n-1}|r-p|^{\beta+n-1}},\quad 0<\alpha\leqq 1,\ 0<\beta\leqq 1 \tag{4.3}$$

*and*

$$\left|\frac{h^{(n)}(z)}{h(z)}\right| \leqq \frac{A_n}{(1-r)^{n\alpha}|r-p|^{n\beta}},\quad \beta\geqq 1,\ \alpha\geqq 1 \tag{4.4}$$

*for $|z| = r \neq p$.*

PROOF. Let $g(z) = h^{(n)}(z)/h(z)$. Then

$$\frac{h^{(n+1)}(z)}{h(z)} = g'(z) + \frac{h^{(n)}(z)h'(z)}{h(z)^2}.$$

Assume (4.3) holds for some $n$. By Lemma 2 there exists $B_n$ so that

$$|g'(z)| \leqq \frac{B_n}{(1-r)^{\alpha+n}|r-p|^{\beta+n}}, \quad |z| = r \neq p.$$

Therefore,

$$\left|\frac{h^{(n+1)}(z)}{h(z)}\right| \leqq \frac{B_n}{(1-r)^{\alpha+n}|r-p|^{\beta+n}} + \frac{A_1 A_n}{(1-r)^{2\alpha+n-1}|r-p|^{2\beta+n-1}}$$

$$\leqq \frac{A_{n+1}}{(1-r)^{\alpha+n}|r-p|^{\beta+n}}, \quad |z| = r \neq p.$$

This proves (4.3) by induction.

Assuming that (4.4) holds for some $n$, we obtain from Lemma 2 the existence of a constant $B_n$ so that

$$|g'(z)| \leqq \frac{B_n}{(1-r)^{n\alpha+1}|r-p|^{n\beta+1}}, \quad |z| = r \neq p.$$

Therefore,

$$\left|\frac{h^{(n+1)}(z)}{h(z)}\right| \leqq \frac{B_n}{(1-r)^{n\alpha+1}|r-p|^{n\beta+1}} + \frac{A_1 A_n}{(1-r)^{(n+1)\alpha}|r-p|^{(n+1)\beta}}$$

$$\leqq \frac{A_{n+1}}{(1-r)^{(n+1)\alpha}|r-p|^{(n+1)\beta}}, \quad |z| = r \neq p.$$

This proves (4.4) by induction.

LEMMA 4. *Let $f(z)$ be in $\Lambda^*(p)$. Then there exists a positive constant $A$ such that*

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{A}{|r-p|(1-r)}, \quad z = re^{i\theta}. \tag{4.5}$$

PROOF. Since $f(z)$ is in $\Lambda^*(p)$, we have

$$\frac{(p-z)(1-pz)f'(z)}{f(z)} = P(z)$$

where $\operatorname{Re} P(z) > 0$, $z \in \Delta$, and $P(0) = pf'(0) = pa_1$. Logarithmic differentiation then yields

$$\frac{f''(z)}{f'(z)} = \frac{P'(z)}{P(z)} + \frac{P(z)}{(p-z)(1-pz)} + \frac{1}{p-z} + \frac{p}{1-pz}.$$

An examination of this expression shows that (4.5) holds if both $P(z)$ and $P'(z)/P(z)$ are order $(1-r)^{-1}$ as $r \to 1$. We may write $P(z) = p \operatorname{Re} a_1 Q(z) + ip \operatorname{Im} a_1$, where $Q \in \mathscr{P}$. Thus,

$$|P(z)| \leqq p|a_1|\left(\frac{1+r}{1-r} + 1\right) \leqq \frac{2(1+p)^2}{1-r}.$$

Also, Lemma 1 of [9] yields

$$\left|\frac{P'(z)}{P(z)}\right| \leqq \frac{2}{1-r^2}.$$

THEOREM 7. *Let $f(z)$ be in $\Lambda^*(p)$. Then for $n \geqq 1$,*

$$\int_{-\pi}^{\pi} |f^{(n)}(re^{i\theta})|^\lambda d\theta \leqq \begin{cases} \dfrac{C_\lambda}{(1-r)^{n\lambda-1}|r-p|^{(n+1)\lambda-1}}, & \lambda > 1 \\ \dfrac{C_1}{r|r-p|^n(1-r)^{n-1}} \log \dfrac{1}{1-r}, & \lambda = 1. \end{cases} \tag{4.6}$$

PROOF. Since the case $n = 1$ is proven in Theorem 6 we assume that $n \geqq 2$. Applying Lemma 4 to $h(z) = f'(z)$ we have

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \frac{A}{(r-p)(1-r)}, \quad z = re^{i\theta}.$$

By Lemma 3

$$\left|\frac{f^{(n)}(z)}{f'(z)}\right| \leqq \frac{A_{n-1}}{(1-r)^{n-1}|r-p|^{n-1}}.$$

Thus

$$\int_{-\pi}^{\pi} |f^{(n)}(re^{i\theta})|^\lambda d\theta \leqq \frac{A_{n-1}^\lambda}{(1-r)^{(n-1)\lambda}|r-p|^{(n-1)\lambda}} \int_{-\pi}^{\pi} |f'(re^{i\theta})|^\lambda d\theta.$$

An application of Theorem 6 now gives (4.6), and the proof is complete.

We remark that it is possible to include in (4.6) estimates for the range $0 < \lambda < 1$. However, it is unlikely that our method would give the correct exponent on $(1 - r)$ for this case. We now show that for $\lambda \geqq 1$ the exponent $n\lambda - 1$ on $(1 - r)$ cannot be reduced. The extremal function $f(z)$ is the same as in Theorem 6, namely, $f(z) = -p(1 - z)^t/(z - p)$, $0 < t < 1$. The next lemma shows that the integral means of $f^{(n)}(z)$ are of the same order as those of $g^{(n)}(z)$, where $g(z) = (1 - z)^t$.

LEMMA 5. *Let $f(z) = (1 - z)^t/(z - p)$ and $g(z) = (z - p)f(z) = (1 - z)^t$, $0 < t < 1$. Then there exists a positive constant $K$ depending on $n$ and $\lambda$ such that for $\lambda \geqq 1$* and $t$ sufficiently close to zero

$$\int_{-\pi}^{\pi} |f^{(n)}(z)|^\lambda d\theta \geqq K \int_{-\pi}^{\pi} |g^{(n)}(z)|^\lambda d\theta, \tag{4.7}$$

*where $z = re^{i\theta}$ and $r(t) < r < 1$.*

PROOF. Let $h(z) = (z - p)^{-1}$; then $f(z) = g(z)h(z)$. Using the formula

$$f^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z) h^{(n-k)}(z),$$

we obtain

$$f^{(n)}(z) = (-1)^n n!(1 - z)^t(z - p)^{-(n+1)}$$
$$+ \sum_{k=1}^{n} (-1)^n \frac{n!}{k!} t(t - 1) \ldots (t - k + 1)(1 - z)^{t-k}(z - p)^{-(n-k+1)}$$
$$= \frac{(-1)^n g^{(n)}(z)P(z)}{t(t - 1) \ldots (t - n + 1)(z - p)^{n+1}},$$

where

$$P(z) = (1 - z)^n + \sum_{k=1}^{n} \frac{n!}{k!} t(t - 1) \ldots (t - k + 1)(1 - z)^{n-k}(z - p)^k.$$

Thus, if $(1 + p)/2 < |z| < 1$, there is a positive constant $C$ so that

$$|f^{(n)}(z)| \geqq C|P(z)||g^{(n)}(z)|.$$

To prove (4.7) we need only prove the existence of a positive constant $D$ so that $|P(z)| \geqq D$ for $t$ sufficiently close to zero and $|z|$ sufficiently close to one. We note that $P(1) = t(t - 1) \ldots (t - n + 1)(1 - p)^n \neq 0$. Thus there exists $\alpha$ so that $P(e^{i\theta}) \neq 0$ if $|\theta| < \alpha$. If $|\theta| \geqq \alpha$, there exists $\gamma$ so that $|1 - e^{i\theta}|^n \geqq \gamma > 0$. Also, if $|z| = 1$,

$$|P(z) - (1 - z)^n| \leqq \sum_{k=1}^{n} \frac{n!}{k!} |t(t - 1) \ldots (t - k + 1)| 2^{n-1}(1 + p)^k.$$

Thus, there exists $\delta > 0$ so that for $0 < t < \delta$ and $|z| = 1$,

$$|P(z) - (1 - z)^n| < \gamma/2.$$

Therefore if $|\theta| \geqq \alpha$, $|z| = 1$, and $0 < t < \delta$, then

$$|P(z)| \geqq |1 - z|^n - \gamma/2 \geqq \gamma/2 > 0.$$

Thus, $P(z) \neq 0$ for $|z| = 1$ if $0 < t < \delta$. Therefore, for fixed $t$, $0 < t < \delta$ there exists $r(t) > 1$ so that $P(z) \neq 0$ for $r(t) \leqq |z| \leqq 1$, and thus there exists a positive constant $C$ so that $|P(z)| \geqq C$ for $r(t) \leqq |z| \leqq 1$. This completes the proof of (4.7).

Since sharpness of the exponent $n\lambda - 1$ when $n = 1$ was discussed earlier, we restrict our attention to $n \geqq 2$. So, for fixed $n \geqq 2$, $\lambda \geqq 1$ let $\delta < n\lambda - 1$. Then choose $t$ so that $0 < t < \min\,[1, n - (\delta + 1)/\lambda]$. Proceeding as in the remarks after Theorem 6, we have that

$$\lim_{r\to 1}(1 - r)^\delta \int_{-\pi}^{\pi} |g^{(n)}(re^{i\theta})|^\lambda d\theta = \infty.$$

If we further restrict $t$ so that (4.7) holds, we obtain

$$\lim_{r\to 1}(1 - r)^\delta \int_{-\pi}^{\pi} |f^{(n)}(re^{i\theta})|^\lambda d\theta = \infty.$$

5. **An example.** In this final section we settle a question of Holland [5] concerning meromorphic starlike functions $f(z)$ and the area of the complement of $f(\Delta)$.

For $F(z)$ in $\Sigma^*$ there exists a probability measure $\mu$ on $|z| = 1$ such that

$$-\frac{zF'(z)}{F(z)} = \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t). \tag{5.1}$$

We also associate with $F(z)$ the related starlike function $g(z) = F(z)^{-1}$. Let $K$ denote the compact complement of $F(\Delta)$. Holland proved the following theorem and asked whether the converse is true.

THEOREM 8. [5]. *If the area of $K$ is zero, then*
a) *the area of $g(\Delta)$ is infinite, and*
b) *$\mu$ is singular with respect to Lebesgue measure.*

We now prove by example that the converse of Theorem 8 is false. We first observe that if $g(\Delta)$ is not dense in the plane then the area of $K$ is positive. Integration of 5.1 leads to the formula

$$g(z) = z \exp \int_{-\pi}^{\pi} \log(1 - e^{-it}z)^{-2} d\mu(t). \tag{5.2}$$

We now choose $\mu$ as follows.

Let $\sigma(x)$ be the usual Cantor function on [0, 1]; that is, to each point $x = .\, a_1a_2 \ldots$ (ternary) of the Cantor set we define $\sigma(x) = .\, b_1b_2 \ldots$, where $b_n = a_n/2$. Then we extend $\sigma$ to all of [0, 1] by defining $\sigma$ in each of the intervals complementary to the Cantor set to be the same as at the endpoints. Then, for $-\pi \leqq \theta \leqq \pi$, define

$$v(\theta) = \sigma\left(\frac{1}{2} + \frac{\theta}{2\pi}\right) - \frac{1}{2},$$

$$w(\theta) = \begin{cases} -1/2, & -\pi \leqq \theta < 0 \\ 0, & \theta = 0 \\ 1/2, & 0 < \theta \leqq \pi, \end{cases}$$

$$\mu(\theta) = (1/2)(v(\theta) + w(\theta)).$$

We first observe that $\mu$ is singular with respect to Lebesgue measure since this is true for each of $v$ and $w$. Also, from (5.2) we obtain

$$g(z) = \frac{z}{1-z}\left[\frac{h(z)}{z}\right]^{1/2}, \tag{5.3}$$

where

$$h(z) = z \exp \int_{-\pi}^{\pi} \log(1 - e^{-it}z)^{-2} dv(t).$$

Keogh [7] discusses $h(z)$ in another context and proves it to be a bounded starlike function. We now use these facts to prove that $g(\Delta)$ has infinite area but is not dense in the plane.

First we recall [15] that $V(\theta) = \lim_{r\to 1} \arg g(re^{i\theta})$ exists for each $\theta$; furthermore we must have $V(\theta) = 2\pi\mu(\theta)$ because of the way we have normalized $\mu$: $\int_{-\pi}^{\pi}\mu(t)dt = 0$ and $\mu(t) = (1/2)[\mu(t+0) - \mu(t-0)]$. Since $\mu$ has a jump discontinuity at $\theta = 0$ of magnitude 1/2, $V$ has a jump discontinuity there of magnitude $\pi$. Thus, $g(\Delta)$ contains a half plane and so the area of $g(\Delta)$ is infinite.

We now prove that $g(\Delta)$ is not dense in the plane. Since $h(z)$ is starlike, $h(z)/z$ is subordinate to $1/(1-z)^2$. So there exists $\phi(z)$, bounded and analytic in $\Delta$ with $\phi(0) = 0$, such that $[h(z)/z]^{1/2} = (1 - \varphi(z))^{-1}$. Since $h(z)$ is bounded, there exists $\delta > 0$ such that $|1 - \phi(z)| > \delta$, $z \in \Delta$. Hence there exists $\varepsilon = \varepsilon(\delta) > 0$ such that

$$|\arg[h(z)/z]^{1/2}| \leqq |\arg(1 - \phi(z))| \leqq \pi/2 - \varepsilon \tag{5.3}$$

for $z \in \Delta$. Geometric considerations allow us to choose $\eta > 0$ such that if $|z - 1| < \eta$, $|z| < 1$, then $|\arg z/(1-z)| < \pi/2 + \varepsilon/2$. Letting $D = \{z \in \Delta \,|\, |z - 1| < \eta\}$ it follows from (5.1), (5.2), and (5.3) that $|\arg g(z)| < \pi - \varepsilon/2$, $z \in D$. Consequently $g(D)$ omits an infinite wedge having central angle $\varepsilon$. Since $g(\Delta/D)$ is bounded, $g(\Delta)$ is not dense in the plane.

## References


1. J. Clunie, *On meromorphic Schlicht functions*, J. London Math. Soc. **34** (1959), 215–216.

2. J. Feng and T. H. Mac Gregor, *Estimates on integral means of the derivatives of univalent functions*, J. D'Analyse Math. **29** (1976), 203–231.

3. A. W. Goodman, *Functions typically real and meromorphic in the unit circle*, Trans. Amer. Math. Soc. **81** (1956), 92–105.

4. W. K. Hayman, *On functions with positive real part*, J. London Math. Soc. **36** (1961), 35–48.

5. F. Holland, *On the coefficients of starlike functions*, Proc. Amer. Math. Soc. **33** (1972), 463–470.

6. J. A. Jenkins, *On a conjecture of Goodman concerning meromorphic univalent functions*, Michigan Math. J. **9** (1962), 25–27.

7. F. R. Keogh, *Some theorems on conformal mapping of a bounded star-shaped domain*, Proc. London Math. Soc **9** (1959), 481–491.

8. J. L. Lewis, *Note on an arc length problem*, J. London Math. Soc. **12** (1976), 469–474.

9. R. J. Libera, *Meromorphic close-to-convex functions*, Duke Math. J. **32** (1965), 121–128.

10. ——— and A. E. Livingston, *Weakly starlike meromorphic univalent functions*, Trans. Amer. Math. Soc. **202** (1975), 181–191.

11. A. E. Livingston, *Weakly starlike meromorphic univalent functions* II, Proc. Amer. Math. Soc. **62** (1977), 47–53.

**12.** ———, *On the integral means of univalent meromorphic functions*, Pac. J. Math. **72** (1977), 167–180.

**13.** T. H. Mac Gregor, *Rotations of the range of an analytic function*, Math. Ann. **20** (1973), 113–126.

**14.** J. Miller, *Starlike meromorphic functions*, Proc. Amer. Math. Soc. **31** (1972), 446–452.

**15.** Ch. Pommerenke, *On starkoke and convex functions*, J. London Math. Soc. **37** (1962), 209–224.

**16.** ———, *On the coefficients of close-to-conves functions*, Michigan Math. J. **9** (1962), 259–269.

**17.** ———, *On meromorphic starlike functions*, Pac. J. Math. **13** (1963), 221–235.

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