ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 11, Number 3, Summer 1981

ON SUMMING SEQUENCES OF 0'S AND 1'S

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1. **Introduction.** It is known that if an infinite matrix sums all sequences of 0's and 1's, it must sum all bounded sequences. At the other extreme, it is clear that the convergence field of a matrix will contain the set of finitely nonzero sequences of 0's and 1's if and only if the columns of the matrix are convergent sequences. In this paper we investigate certain intermediate situations. Classes Φ of subsets of the positive integers called full (Definition 1) are studied, and necessary and sufficient conditions are found for a matrix to sum the set χ_Φ of characteristic functions of the sets in Φ.

Full classes include the class of all subsets, classes of sets of density zero, and the class of lacunary sets (Definition 2). In §2 we give several equivalent formulations of fullness and observe that there is no minimal full class.

Our main result (Proposition 6) includes the fact that $c_A \supseteqq c_0$ whenever $c_A \supseteqq \chi_\Phi$, where A is any matrix and Φ is full. By specifying $\Phi = 2^I$ in Proposition 6 we obtain the theorem, first proved by Hahn in 1922 ([3] Satz V), that a matrix sums all bounded sequences if it sums all sequences of 0's and 1's. By taking Φ to be the class of sets of density zero (respectively, uniform density zero) we obtain a new characterization of those matrices that sum the set of bounded strongly Cèsaro summable sequences (respectively, the strongly almost convergent sequences). These latter applications improve upon several of the results contained in [5], [6], and [7].

2. **Full classes.**

Definition 1. A class Φ of subsets of the positive integers I is *full* in case
(a) $\bigcup\{S\colon S \in \Phi\} = I$ (covering),
(b) $S \in \Phi$ whenever $S \subseteqq T$ for some $T \in \Phi$ (hereditary), and
(c) if (t_k) is a sequence of real numbers for which $\sum_{k \in S}|t_k| < +\infty$ for each $S \in \Phi$, then $\sum_{k=1}^{\infty}|t_k| < +\infty$.

Proposition 1. *Let Φ be a covering, hereditary class of subsets of I. The following are equivalent*:

Received by the editors on December 18, 1979.

(i) *Φ is full;*

(ii) *if (t_k) is a sequence of real numbers for which $\sum_{k\in S} t_k$ converges for each $S \in \Phi$, then $\sum_{k=1}^{\infty}|t_k| < +\infty$;*

(iii) *if (a_{nk}) is an infinite matrix for which $\sup_n \sum_{k\in S}|a_{nk}| < +\infty$ for each $S \in \Phi$, then $\sup_n \sum_{k=1}^{\infty}|a_{nk}| < +\infty$; and*

(iv) *if (a_{nk}) is an infinite matrix for which $\{\sum_{k\in S} a_{nk}: n = 1, 2, \ldots\}$ is bounded for each $S \in \Phi$, then $\sup_n \sum_{k=1}^{\infty}|a_{nk}| < +\infty$.*

PROOF. (i) $\Rightarrow$ (ii). If $\sum_{k=1}^{\infty}|t_k| = +\infty$, then at least one of $\sum_{k=1}^{\infty} t_k^+ = +\infty$ or $\sum_{k=1}^{\infty} t_k^- = +\infty$, say $\sum_{k=1}^{\infty} t_k^+ = +\infty$. Since Φ is full, there exists $T \in \Phi$ for which $\sum_{k\in T} t_k^+ = +\infty$. Letting $S = \{k \in T: t_k^+ > 0\}$, it follows that $\sum_{k\in S} t_k = +\infty$.

(i) $\Rightarrow$ (iii). Assume Φ is full and suppose that $\sup_n \sum_{k=1}^{\infty}|a_{nk}| = +\infty$ for some infinite matrix (a_{nk}). We may assume that $\sup_n |a_{nk}| < +\infty$ for each $k = 1, 2, \ldots$ (if $\sup_n |a_{nk_0}| = +\infty$ and $k_0 \in S \in \Phi$, then $\sup_n \sum_{k\in S} |a_{nk}| = +\infty$). We can find strictly increasing sequences (n_i) and (k_i) of positive integers for which

$$M_j = \sum_{i=k_{j-1}+1}^{k_j} |a_{n_j i}| > j^2, \quad j = 1, 2, \ldots .$$

Let $I_j = \{i: k_{j-1} + 1 \leqq i \leqq k_j\}$ and define $b = (b_i)$ by

$$b_i = \frac{a_{n_j i}}{jM_j}, \text{ if } i \in I_j.$$

Then $\sum_{i\in I_j}|(b_i)| = 1/j$ and $\sum_{i=1}^{\infty}|b_i| = +\infty$. Since Φ is full, there exists $S \in \Phi$ with $\sum_{i\in S}|b_i| = +\infty$. Consequently, we must have

$$\sum_{i\in I_j \cap S} |b_i| > 1/j^2$$

for infinitely many j and, for these j,

$$\begin{aligned} \sum_{i\in S} |a_{n_j i}| &\geqq \sum_{i\in S\cap I_j} |a_{n_j i}| \\ &= jM_j \sum_{i\in S\cap I_j} |b_i| \\ &> j\cdot j^2 \cdot \frac{1}{j^2} = j. \end{aligned}$$

It follows that (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). If $\sum_{k=1}^{\infty}|t_k| = +\infty$, the matrix (a_{nk}) defined by $a_{nk} = t_k$ if $k \leqq n$, and $a_{nk} = 0$ otherwise, satisfies $\sup_n \sum_{k=1}^{\infty}|a_{nk}| = +\infty$. By hypothesis there exists $S \in \Phi$ such that $\sup_n \sum_{k\in S}|a_{nk}| = +\infty$. Thus $\sum_{k\in S}|t_k| = +\infty$.

(iii) $\Rightarrow$ (iv). Assume that (iii) holds and let (a_{nk}) be any matrix satisfying $\sup_n \sum_{k=1}^{\infty}|a_{nk}| = +\infty$. We find a set $S \in \Phi$ for which $\{\sum_{k\in S} a_{nk}: n =$

$1, 2, \ldots\}$ is unbounded. We may assume that $\sum_{k=1}^{\infty}|a_{nk}| < +\infty$, $n = 1, 2, \ldots$, and $\sup_n|a_{nk}| < +\infty$, $k = 1, 2, \ldots$. Define (b_{nk}) by

$$b_{nk} = \begin{cases} a_{nk}^{+} & \text{if } \sum_{k=1}^{\infty} a_{nk}^{+} \geqq \sum_{k=1}^{\infty} a_{nk}^{-} \\ a_{nk}^{-} & \text{otherwise,} \end{cases}$$

and note that (b_{nk}) is a nonnegative matrix satisfying $\sup_n\sum_{k=1}^{\infty}b_{nk} = +\infty$. By hypothesis there exists $T \in \Phi$ for which $\sup_n\sum_{k\in T}b_{nk} = +\infty$. We can then find strictly increasing sequences (n_i) and (k_i) of positive integers satisfying

$$\sum_{k\in T} b_{n_jk} > j + 2 + 2\sum_{k=1}^{k_{j-1}} \alpha_k$$

and $\sum_{k=k_j+1}^{\infty}|a_{n_jk}| < 1$ (where $\alpha_k = \sup_n|a_{nk}|$, $k = 1, 2, \ldots$).

Define S by $S = \{k \in T\colon \text{if } k \in I_j, \text{ then } b_{n_jk} \neq 0\}$. Then $S \in \Phi$ and

$$\begin{aligned} \sum_{k\in S} a_{n_jk} &= \sum_{\substack{k=1\\ k\in S}}^{k_{j-1}} a_{n_jk} \pm \sum_{k\in I_j\cap T} b_{n_jk} + \sum_{\substack{k=k_j+1\\ k\in S}}^{\infty} a_{n_jk} \\ &= R_1 \pm R_2 + R_3. \end{aligned}$$

Observe that $|R_1| < \sum_{k=1}^{k_{j-1}}\alpha_k$, $|R_3| < 1$ and

$$|R_2| = \sum_{k\in I_j\cap T} b_{n_jk} \geqq \sum_{k\in T} b_{n_jk} - 1 - \sum_{k=1}^{k_{j-1}} \alpha_k.$$

From

$$\begin{aligned} \Big|\sum_{k\in S} a_{n_jk}\Big| &= |R_1 \pm R_2 + R_3| \\ &\geqq |R_2| - |R_1| - |R_3| \\ &\geqq \sum_{k\in T} b_{n_jk} - 2 - 2\sum_{k=1}^{k_{j-1}} \alpha_k > j \end{aligned}$$

it follows that $\{\sum_{k\in S}a_{nk}\colon n = 1, 2, \ldots\}$ is unbounded.

The implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are immediate, and this completes the proof.

Recall that a lacunary sequence is a strictly increasing sequence (k_r) of positive integers such that $k_r - k_{r-1} \to +\infty$ (cf. [**4**, p. 174]).

DEFINITION 2. A *lacunary set* is a set of positive integers that is finite or the range of a lacunary sequence. We denote the class of lacunary sets by Λ.

PROPOSITION 2. *The class* Λ *is full.*

PROOF. Let (t_k) be a sequence of real numbers for which $\sum_{k=1}^{\infty}|t_k| = +\infty$. We can find $i_1 < j_1 < i_2 < j_2 < i_3 < j_3 < \cdots$ satisfying $j_k - i_k = k$ and

$$\sum_{i=j_k+1}^{i_{k+1}} |t_i| \geqq 1.$$

For each k subdivide the interval $J_k = \{j_k + 1, j_k + 2, \ldots, i_{k+1}\}$ into (at most) k sets as follows:

$$\begin{aligned} B_1^k &= \{j_k + 1, (j_k + 1) + k, (j_k + 1) + 2k, \ldots\} \cap J_k, \\ B_2^k &= \{j_k + 2, (j_k + 2) + k, (j_k + 2) + 2k, \ldots\} \cap J_k, \\ B_k^k &= \{j_k + k, (j_k + k) + k, (j_k + k) + 2k, \ldots\} \cap J_k. \end{aligned}$$

Then for each $k = 1, 2, \ldots$ there exists r_k, $1 \leqq r_k \leqq k$, such that

$$\sum_{i \in B_{r_k}^k} |t_i| \geqq 1/k.$$

Finally, it is clear that $L = \bigcup_{k=1}^{\infty} B_{r_k}^k$ is a lacunary set and $\sum_{i \in L}|t_i| \geqq \sum_{k=1}^{\infty} 1/k$.

One of the motivations for this paper was interest in classes of subsets of the positive integers that properly contain the class of finite sets, but with the property that each set in the family is still, in some sense, small. The class $\wedge$ discussed above fits this general description. So also do classes of sets of zero density.

DEFINITION 3. A subset $S \subseteqq I$ has *density zero* in case

$$\delta(S) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \chi_S(i)\right)/n = 0.$$

DEFINITION 4. A subset $S \subseteqq I$ has *uniform density zero* in case

$$u(S) = \lim_{n\to\infty}\left(\sum_{i=m+1}^{m+n} \chi_S(i)\right)/n = 0$$

uniformly in $m = 0, 1, 2, \ldots$

These concepts have appeared in various places in the literature. For a study of the precise relationship between density functions and summability spaces, the reader is referred to [2]. We let η_δ^0 and η_u^0 denote the classes of sets of density zero and uniform density zero, respectively.

PROPOSITION 3. $\bigwedge \subseteqq \eta_u^0 \subseteqq \eta_\delta^0$.

PROOF. We need only prove that $\bigwedge \subseteqq \eta_u^0$ since the other containment is immediate. Clearly $u(S) = 0$ if S is finite. Let $S = \{k_r\}$, where (k_r) is a lacunary sequence, and let $0 < \varepsilon < 1$. We then choose

(i) $T > 0$ such that $2/T < \varepsilon/2$,

(ii) $W > 0$ such that $k_{r+1} - k_r > T + 1$ whenever $k_r \geqq W$,

and we let $N = \max\{2W/\varepsilon, T/2\}$. Then, if $n \geqq N$ and $m \geqq 0$, we can write

(iii) $n^{-1}\sum_{i=m+1}^{m+n}\chi_S(i) \leqq W/n + \sum_{i\in J}\chi_S(i)/n$,

where J denotes the interval $J = [\max\{m + 1, W + 1\}, m + n]$. Note that $W/n \leqq \varepsilon/2$. With regard to the second term on the right-hand side of (iii), note that the length of J is $\leqq n$ and $J \subseteqq [W + 1, +\infty)$. Letting $b = \sum_{i\in J}\chi_S(i)$ and $d = \sum_{i\in J}(1 - \chi_S(i))$, we consider two cases.

Case 1. $b = 1$. In this case $b/n = 1/n \leqq 2/T < \varepsilon/2$.

Case 2. $b > 1$. In this case we have, by (ii), that $d \geqq (b - 1)T$ and, therefore, $n \geqq$ length of $J \geqq d \geqq (b - 1)T$. Thus $b/n \leqq 2/T < \varepsilon/2$. It follows that $u(S) = 0$.

Since subsets of sets of density zero have density zero, and finite sets have density zero, Propositions 2 and 3 give the following result.

PROPOSITION 4. *The classes η_u^0 and η_δ^0 are full.*

It seems worthwhile to observe here that $\wedge$ is not a minimal full class.

PROPOSITION 5. *There is no minimal full class.*

PROOF. Let S_0 be any infinite subset of a full class Φ, and define $\Delta = \{S \in \Phi\colon S_0 \nsubseteqq S\}$. Then Δ is covering, hereditary, and properly contained in Φ. If (t_k) is any sequence with $\sum_{k=1}^{\infty}|t_k| = +\infty$, there exists $W \in \Phi$ such that $\sum_{k\in W}|t_k| = +\infty$. It follows that $\sum_{k\in W\setminus S_0}|t_k| = +\infty$ or $\sum_{k\in W\cap S_0}|t_k| = +\infty$. In the first case $T = W\setminus S_0 \in \Delta$ and satisfies $\sum_{k\in T}|t_k| = +\infty$, and in the second case we can let $T = S_0\setminus\{s\}$ for any $s \in S_0$. Again $\sum_{k\in T}|t_k| = +\infty$, and it follows that Δ is full.

3. **Applications to matrix summability.**

PROPOSITION 6. *Let Φ be any full class of subsets of the positive integers, and let $A = (a_{nk})$ be any infinite matrix. Then $c_A \supseteqq \chi_\Phi$ if and only if*

(i) $\lim_{n\to\infty}a_{nk} = a_k$ *exists*, $k = 1, 2, \ldots$;

(ii) $\sup_n\sum_{k=1}^{\infty}|a_{nk}| < +\infty$; *and*

(iii) $\lim_{n\to\infty}\sum_{k\in S}|a_{nk} - a_k| = 0$ *for each* $S \in \Phi$.

PROOF. Necessity. If $c_A \supseteqq \chi_\Phi$, then (i) holds since all singleton sets belong to Φ, and (ii) follows from Proposition 1. Assume now that (iii) does not hold so that there exists a set $S \in \Phi$ for which the limit in (iii) is not zero. Define a matrix $B = (b_{nk})$ by $b_{nk} = a_{nk} - a_k$. We know that condition (i) and (ii) hold for B. By a standard gliding hump argument we can find an $\varepsilon_0 > 0$ and strictly increasing sequences (n_i) and (K_i) of positive integers for which

$$\sum_{k\in S}|b_{n_ik}| > \varepsilon_0,$$

$$\sum_{k=1}^{K_{i-1}}|b_{n_ik}| < \varepsilon_0/8,$$

and

$$\sum_{k=K_i+1}^{\infty}|b_{n_ik}| < \varepsilon_0/8.$$

Letting $I_i = \{j\colon K_{i-1} < j \leqq K_i\}$ and defining the sequence $y = (y_k)$ by

$$y_k = \begin{cases} \operatorname{sgn} b_{n_ik} & \text{if } i \text{ is even and } k \in S \cap I_i,\\ -\operatorname{sgn} b_{n_ik} & \text{if } i \text{ is odd and } k \in S \cap I_i,\\ 0 & \text{otherwise,}\end{cases}$$

it follows that $\sum_{k=1}^{\infty} b_{n_ik}y_k > \varepsilon_0/2$ for i even and $< -\varepsilon_0/2$ for i odd so that $y \notin c_B$. However $c_A = c_B$ and y is the difference of two sequences in χ_Φ; so it follows that $c_A \not\supseteq \chi_\Phi$.

Sufficiency. Conditions (i) and (ii) imply that $\sum_{k=1}^{\infty}|a_k| < +\infty$ which, together with (iii), shows that $\lim_{n\to\infty}\sum_{k\in S}a_{nk}$ exists (and is equal to $\sum_{k\in S}a_k$) for each $S \in \Phi$.

COROLLARY 6.1. (Hahn's Theorem) *Let $A = (a_{nk})$ be any infinite matrix. The convergence field c_A of A contains all sequences of* 0's *and* 1's *if and only if it contains all bounded sequences.*

PROOF. Let $\Phi = 2^I$ in Proposition 6. Conditions (i), (ii), and (iii) (with $S = I$) are well-known (by Schur's Theorem) to be necessary and sufficient for c_A to contain all bounded sequences.

We denote by $|\sigma_1|^0$ the space of bounded sequences that are strongly Cesàro summable to zero, and by $|AC|^0$ the space of sequences that are strongly amost convergent to zero. (The space $|AC|$ was introduced in **[1]** and, independently, in **[6]**, where it is denoted by $[f]$). By letting Φ be η_δ^0 and η_u^0 in Proposition 6 we then have the following corollaries.

COROLLARY 6.2. *Let $A = (a_{nk})$ be any infinite matrix. A necessary and sufficient condition for $c_A \supseteq |\sigma_1|^0$ is that* $\lim_{n\to\infty}\sum_{k\in S}a_{nk}$ *exists for every* $S \in \eta_\delta^0$.

COROLLARY 6.3. *Let $A = (a_{nk})$ be any infinite matrix. A necessary and sufficient condition for $c_A \supseteq |AC|^0$ is that* $\lim_{n\to\infty}\sum_{k\in S}a_{nk}$ *exists for every* $S \in \eta_u^0$.

PROOF. The limit conditions in these corollaries mean that $c_A \supseteq \chi_\Phi$ where $\Phi = \eta_\delta^0$ and η_u^0 respectively. Thus, in each case, conditions (i), (ii) and (iii) hold. It follows from the proof of [**5**, Theorem 1, p. 210] and [**7**, Theorem 5, Corollary] that these conditions are equivalent to $c_A \supseteq |\sigma_1|^0$ and $c_A \supseteq |AC|^0$ respectively.

We note that in Corollaries 6.2 and 6.3 there is no prior requirement that the matrix be conservative and, in addition, there is no need to have $\sum |a_{nk}|$ in the limit conditions in place of $\sum a_{nk}$. These are to be compared with the results in [5] and [7].

Requiring that the convergence field contain the constant sequences leads to corresponding statements concerning the spaces $|\sigma_1|$ and $|AC|$. For example, we have the following corollary.

COROLLARY 6.4. *Let $A = (a_{nk})$ be a regular matrix. A necessary and sufficient condition for A to sum every strongly almost convergent sequence is that* $\lim_{n\to\infty} \sum_{k\in S} a_{nk} = 0$ *for every set S of uniform density zero.*

The conditions (i) and (ii) of Proposition 6 are clearly equivalent to the requirement that $c_A \supseteqq c_0$, and this condition is also known to be the same as the requirement that χ_{Φ_0} be bounded in the *FK*-space topology of c_A, where Φ_0 is the class of finite subsets of I (see [9, p. 597] or [8, p. 704]). Note that Φ_0 is not a full class. It is interesting to observe that, for any full class Φ, the inclusion $c_A \supseteqq \chi_\Phi$ itself guarantees the boundedness of χ_Φ. This follows from (ii) and the manner in which the *FK*-topology is generated.

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