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RINGS WITH INVOLUTION AS PARTIALLY ORDERED ABELIAN GROUPS

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Let $(S, *)$ be a ring with involution $*$. The involution is *positive definite* if, for all finite subsets $\{r_i\}$ of S, $\sum r_i r_i* = 0$ implies all the r_i are zero. Then the set of self-adjoint elements of S, denoted S^s, possesses a natural partial ordering, with positive cone consisting of elements of the form $\sum r_i r_i*$; with this ordering, S^s is a directed partially ordered abelian group. Let S_b denote the set of bounded elements, that is, the set of elements s such that $ss*$ is less than an integral multiple of 1 in this ordering. Then S_b is a $*$-subalgebra of S whenever S is an algebra over the rationals. We will be studying the objects S_b, S^s, and $(S_b)^s$, from the point of view of their ordered structures.

For instance, suppose S is a field, and $*$ is the identity. Then S is formally real, and S_b must be Prüfer domain, all of whose residue fields are themselves formally real (and in fact, are embeddable in the reals). Viewing S_b as a partially ordered abelian group with order unit 1 (indeed, S_b is the convex subgroup of S generated by 1), S_b has the Riesz decomposition property, and its normalized extremal states are precisely the ring homomorphisms into the reals. There is a natural mapping from the collection of total orderings of S to the set of extremal states of S_b, and this in turn maps to Spec S_b (the prime ideal space of S_b); when S is even a real algebra much more can be said.

If either S is a field and $*$ is not the identity, or S is a quaternionic division algebra with the natural involution, essentially the same properties hold, with the appropriate modifications. A useful tool here is an involutory version of the Artin Schreier Theorem, about the existence of sufficiently many total orderings finer than the natural ordering.

Studies are made of several specific bounded subrings. For instance, if S is the rational function field in one variable over the reals, then S_b is a Dedekind domain with class group of order 2, with spectrum the unit circle (in the point-open topology), and all of its maximal ideals are not principal.

Expanding the scope of S somewhat, we next allow S to be a division

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ring (or an artinian ring), and consider the possibility that $S = S_b$. It is shown that any real $*$-algebra with this property must be finite dimensional; in particular, this applies to real division algebras (with positive definite involution) generated by their unitaries. Developing a recent result of Holland, we show that if S is a division ring, and S^s is monotone sigma-complete in its natural ordering, then again S must be one, two, or four-dimensional over the reals. These results rely heavily on ideas and results from the theory of $C*$ algebras.

Proceeding in other directions, we employ the notion of the bounded subring to determine necessary and sufficient conditions on prime PI rings so that their matrix rings are Baer $*$ with respect to $*$-transpose.

I would like to thank the referee of this paper for a very thorough and detailed examination of the contents.

TABLE OF CONTENTS

1. **Definitions and elementary properties.** Unless specified otherwise, all involutions will be *positive definite*, that is, for all finite subsets $\{s_i\}$ of the ring S, $\sum s_i s_i* = 0$ implies all the s_i are zero. (When there is doubt as to whether an involution is positive definite, it will be referred to as an involutory antiautomorphism).

Let $(S, *)$ be a rational algebra with involution $*$. An element s of S is *bounded* if there exists a finite subset $\{t_i\}$ of S, and a positive integer n so that

$$ss* + \sum t_i t_i* = n \cdot 1. \tag{1}$$

The collection of all bounded elements is a rational $*$-subalgebra of S [**18**], or [**2**, p. 243], and it is easy to see that if S is a real or complex $*$-algebra, so is S_b.

It is clear that S_b contains all the *unitary* elements of S (elements u such that $uu* = u*u = 1$), more generally, all the *partial isometries* (elements w such that $(ww*)^2 = ww*$), and thus includes all the *projections* (elements p such that $p^2 = p = p*$) of S. There is a very intimate relationship between S and S_b, if for example, for all s in S, $1 + s*s$ is invertible in S. This is a natural and weak condition—satisfied if nonzero divisors of S are invertible, and more generally still (viz. 1.12).

LEMMA 1.1. *Let* $(S, *)$ *be a ring with involution. Suppose* $\{x_i\}_{i=1}^{m}$ *is a subset of* S, *and* $1 + \sum x_i x_i*$ *has an inverse in* S. *Then there exist elements* $\{t_j\}_{j=1}^{2m+1}$ *so that*

$$[(1 + \sum x_i x_i*)^{-1}]^2 + \sum t_j t_j* = 1;$$

further, all of $(1 + \sum x_i x_i*)^{-1}$, $x_k*(1 + \sum x_i x_i*)^{-1} (k = 1, 2, \ldots, m)$ *belong to* S_b.

PROOF. [**18**; Lemma 6]. Set $a = \sum x_i x_i*$, and $z = (1 + \sum x_i x_i*)^{-1}$. Pre- and post-multiply the identity $1 = (1 + a)^2 - 2a - a^2$ by z. Then with t_j defined as follows,

$$t_j = \begin{cases} zx_i & \text{for } j = 2i - 1 \text{ or } 2i \\ za & \text{for } j = 2m + 1, \end{cases}$$

the equation is satisfied.

COROLLARY 1.2. (Pointed out to me by Sterling Berberian). *Let* $(S, *)$ *be a ring with involution such that* $1 + xx*$ *is invertible* (*in* S) *for all* x *in* S. *Then* S_b *is a right and left order in* S (*that is, every element of* S *can be written as* ab^{-1} *and* $b_1^{-1}a_1$, *for some* a, a_1, b, b_1 *in* S_b.

PROOF. Write $x = x(1+x*x)^{-1} \cdot (1+x*x) = (1+xx*) \cdot (1+xx*)^{-1}x$, and apply 1.1; the involution may be applied, to show S_b is a left order.

(Some condition on S is required in order that S_b be an order in it, else the result is not true. For instance, if $S = \mathbf{R}[X]$, and $*$ is the identity, then $S_b = \mathbf{R}$.)

The corollary is useful when S is von Neumann regular; the positive definiteness condition on the involution assures that all elements of the form $1 + \sum x_i x_i*$ are not zero divisors.

Given the (always positive definite) involution $*$ on S, we can impose an involution $\# = *$-transpose on $M_n S$, the ring of n by n matrices over S. The positive definiteness of $\#$ follows immediately from that of $*$.

LEMMA 1.3. ([**10**] and [**11**]). *For the ring with involution $(S, *)$, for any positive integer n, $(M_nS)_b = M_nS_b$.*

PROOF. Let e_{ij} denote the element of M_nS with a one in the i, j entry, and zeroes elsewhere. Then the e_{ij}'s are all partial isometries (with respect to $\sharp$), so $\{S_b, e_{ij}\} \subset (M_nS)_b$; thus $M_nS_b \subset (M_nS)_b$.

On the other hand, if A belongs to $(M_nS)_b$, there exist B_i in M_nS such that $AA\sharp + \sum B_iB_i\sharp = mI$, for some positive integer m. Reading off the n equations from the diagonal entries of the left hand side, we see all the entries of A are bounded.

The ring with involution, $(S, *)$ is a *Rickart $*$-ring*, if for all s in S, the right annihilator of s is of the form pS for some projection p of S. If $1 + ss*$ is always invertible in S, then by [**14**; Theorem 26], for every idempotent e in S, there exists a projection p so that $eS + pS$; hence if S is a $p\,.\,p.$ ring (a ring in which principal right ideals are projective as S-modules), and $1 + ss*$ is always invertible in S, then S is a Rickart $*$-ring.

The ring, $(S, *)$ is a *Baer $*$ ring*, if the annihilator of any subset of S is of the form pS.

LEMMA 1.4. *Let $(S, *)$ be a ring with positive definite involution.*

(a) *If S is a Rickart $*$-ring (Baer $*$ ring), so is S_b.*

(b) *If $(M_nS, \sharp)$ is a Rickart $*$-ring (Baer $*$ ring), so is M_nS_b.*

(c) *If S is semihereditary, and for all X in M_nS, $I + XX\sharp$ is invertible in M_nS, then M_mS_b, M_mS_b are Rickart $*$-rings for all $m \leqq n$, with respect to $\sharp$.*

PROOF: (a). All the projections of S lie in S_b, and if the annihilator of s in S is pS, and s belongs to S_b, it is easy to see that the annihilator in S_b of s is pS_b.

(b). Follows immediately from (a) and 1.3.

(c). As S is semihereditary, M_nS is $p.\ p.$, hence by [**14**; Theorem 26], M_nS is a Rickart $*$-ring, with respect to $\sharp$. If p is a projection then pM_nSp also a Rickart $*$-ring, and the result follows from (b).

A reasonable conjecture, is that if $1 + \sum x_ix_i*$ is invertible for all finite subsets $\{x_i\}$ of S, and S is semihereditary, then $(S, *)$ satisfies the conditions of 1.4(c). This is certainly true in the commutative case.

Now we consider some order properties of $(S, *)$ and $(S_b, *)$. Recall that a *partially ordered* (abelian) *group* G is a group with a translation invariant partial order. We denote the positive cone (the set of elements greater than or equal to zero), G^+. The partially ordered group G is directed if $G = G^+ - G^+$. An element u of G^+ is called an *order unit* (also called a *strong unit*) if for all g in G, there exists a positive integer n so that $-nu \leqq g \leqq nu$. If A is a subset of G^+, the *convex subgroup generated* by A is $\{g \in G \mid$ there exist n_i in $\mathbf{N}$, a_i in A such that $-\sum n_i a_i \leqq g \leqq \sum n_i a_i\}$. (It is easy to check that the so-defined set is a subgroup). If $A = \{a\}$, then a is automatically an order unit for the convex subgroup it generates.

If $(S, *)$ is a ring with involution, denote by S^s the set of symmetric elements, that is, elements such that $x* = x$. We call S^s the *symmetric part* of S.

PROPOSITION 1.5. *For a rational algebra with involution* $(S, *)$, *the symmetric part of S is a directed partially ordered vector space* (*over the rationals*) *with positive cone*

$$(S^s)^+ = \{\textstyle\sum s_i s_i * \mid \{s_i\} \text{ a finite subset of } S\}.$$

Further, $(S_b)^s$ *(denoted S_b^s) is the convex subgroup of S^s generated by* 1; *it thus has* 1 *as an order unit. The relative ordering on S_b^s induced by the inclusion $S_b^s \subset S^s$ agrees with the ordering on S_b^s obtained by considering it as the symmetric part of a ring with involution.*

(We refer to the ordering described above as the *natural ordering* on S^s, or on S_b^s, or by abuse of language, on S or S_b).

PROOF. It is clear that since $*$ is positive definite,

$$(S^s)^+ \cap -(S^s)^+ = \{0\};$$

$(S^s)^+$ is obviously closed under addition. If k is a positive integer, $xx^*/k = k(x/k)(x/k)* \in (S^s)^+$; thus division by positive integers leaves $(S^s)^+$ invariant. So $(S^s)^+$ is a positive cone for a partially ordered rational vector space.

To see that S^s is directed, suppose $a = a*$; then $(1 + a/2)(1 + a*/2) = (1 + a/2)^2$ belongs to $(S^s)^+$, and $a = (1 + a/2)^2 - (1 + a^2/4)$ belongs to $(S^s)^+ - (S^s)^+$.

If s belongs to S_b^s, according to the defining equation for S_b, there exist t_i in S so that for some positive integer n,

$$s^2 + \textstyle\sum t_i t_i * = n \cdot 1;$$

hence $s^2 \leqq n$. Also $s = (1 + s/2)^2 - (1 + s^2/4)$, whence

$$-1 - n/4 \leqq -1 - s^2/4 \leqq s \leqq (1 + s/2)^2 \leqq m$$

(for some positive integer m); the last inequality holds as $1 + s/2$ belongs to S_b. Thus s belongs to the convex subgroup generated by 1.

Conversely, if $s = s*$ and $-n \leqq s \leqq n$, then there exist x_i, y_j in S so that $s + \sum x_i x_i * = n$ and $-s + \sum y_j y_j * = n$. Adding these two equations, we see that all of the x_i, y_j lie in S_b. As S_b is a ring, s belongs to it.

Finally, suppose t lies in S_b^s and t belongs to $(S^s)^+$. Then $t = \sum s_i s_i *$ for some subset $\{s_i\}$ of S, and $t \leqq n$. Thus there exist x_j in S so that $\sum s_i s_i * + \sum x_j x_j * = n$, whence $\{s_i\}$ is a subset of S_b, and thus t lies in $(S_b^s)^+$.

LEMMA 1.6. *Let $(S, *)$ be a ring with positive definite involution, and suppose a is an element of $(S^s)^+$ having an inverse in S. Then a^{-1} also lies in $(S^s)^+$.*

PROOF. If $a = \sum s_i s_i *$, then $(a^{-1})* = a^{-1}$ and

$$a^{-1} = a^{-1}aa^{-1} = \sum (a^{-1}s_i)(a^{-1}s_i)* \in (S^s)^+.$$

For s in S_b, define the nonnegative real number $\|s\|$, as in [18] as follows:

$$\|s\|^2 = \inf\{q \in Q \,|\, ss* \leqq q \text{ in the natural ordering}\}.$$

Although only $*$-regular rings are considered in [18], the proofs there only require (for the purposes of (i) through (v) below) that $*$ be positive definite, to obtain that the function $\| \quad \|: S_b \to \mathbf{R}^+$ has many of the properties of a $C*$ algebra norm:

(i) $\|0\| = 0$; $\|1\| = 1$.
(ii) $\|ab\| \leqq \|a\| \cdot \|b\|$, for a, b in S_b.
(iii) $\|a + b\| \leqq \|a\| + \|b\|$.
(iv) $\|aa*\| = \|a\|^2 = \|a*\|^2$.
(v) $\|qa\| = |q| \cdot \|a\|$, if q is a rational.

Further, if S is a real or complex $*$-algebra, (v) holds if q is allowed to vary over the reals or complexes respectively. It is not generally true that $\|a\| = 0$ implies $a = 0$, so we refer to $\| \quad \|$ as a (the) seminorm.

LEMMA 1.7. *Let $(S, *)$ be a ring with involution that is an algebra over the rational numbers. Suppose $a = a*$ is an element of S, and $a^2 \leqq q^2$ with q a positive rational. Then $-q \leqq a \leqq q$.*

PROOF. [18; Equation (7)]. We have

$$a = q - \frac{1}{2q}(a - q)^2 - \frac{1}{2q}(q^2 - a^2),$$

so $a \leqq q$; applied to $-a$, we obtain $-a \leqq q$, or $a \geqq -q$.

One easily checks that for a in S_b (and S a rational algebra), $\|a\| = r \in \mathbf{R}$ implies $aa* \leqq q$ for all rational $q > r^2$.

If A, B are square (but not necessarily of equal dimensions) matrices, then $A \oplus B$ denotes the matrix direct sum.

LEMMA 1.8. *Let $(S, *)$ be a rational algebra with positive definite involution. For a in S_b, for all n, we have*

$$\|a\| = \|a \oplus 0_{n-1}\| = \|a \oplus a \oplus \cdots \oplus a\|,$$

where the norm on the latter two elements is the natural seminorm on $M_n S_b$ (viz. 1.3).

PROOF. Set $E = 1 \oplus 0_{n-1}$, $A = a \oplus 0_{n-1}$, $B = a \oplus a \oplus \cdots \oplus a$ in

M_nS_b. Obviously any equation of the form $aa* + \sum c_ic_i* = q$ (q in Q, c_i in S) yields an equation $BB\# + \sum C_iC_i\# = qI$ in M_nS, so $\|a\|^2 \geqq \|B\|^2$. Since $A = EBE$ and $\|E\| = 1$, $\|B\|^2 \geqq \|A\|^2$. Finally, if $AA\# + \sum C_iC_i\# = qI$, then the (1, 1) entry of each of the matrices $C_iC_i\#$ is a positive element in S, and so $aa* \leqq q$; hence $\|A\|^2 \geqq \|a\|^2$, completing the proof.

PROPOSITION 1.9. *Let* $(S, *)$ *be a rational algebra with positive definite involution such that* $\|a\| = 0$ *implies* $a = 0$, *where* $\| \quad \|$ *is the natural seminorm. Then the completion of* S_b *at* $\| \quad \|$, S_B, *is a Banach* $*$*-algebra, and* $*$ *extends to a positive definite involution on* S_B. *Further, for all* x *in* S_B, $\|xx*\| = \|x\|^2 = \|x*\|^2$, *and* $(S_B)_b = S_B$; *additionally, the natural seminorm on* S_B *agrees with the extension of* $\| \quad \|$. *Finally, if* $1 + \sum x_ix_i*$ *is invertible in* S_b *for all finite subsets* $\{x_i\}$ *of* S_b, *then* S_B *satisfies the same condition.*

PROOF. The completion of $Q.1$ (inside S_b) is isometrically isomorphic to $\mathbf{R}.1$, and its action on the completed ring S_B obviously makes the latter into a real vector space. Since $\| \quad \|$ is subadditive on S_b and extends continuously to S_B, it is a norm on the latter, and so S_B becomes a Banach space in it. Since $\|xy\| \leqq \|x\| \cdot \|y\|$ on S_b, the same holds on S_B; thus S_B is a Banach algebra.

If x is an element of S_B and is written as the limit of a sequence $\{x_i\}$ in S_b, from $\|a\| = \|a*\|$ for all a in S_b, we deduce that $\{x_i*\}$ converges, and declare the limit to be $x*$. It is routine to check that this extends $*$ to an involutory anti-automorphism of S_B and $\|x\|^2 = \|x*\|^2 = \|xx \|$ follows directly. Now we verify that $*$ is positive definite on S_B.

Suppose $\{x_j\}_{j=1}^n$ is a subset of S_B, and $\sum x_jx_j* = 0$. We may assume $\|x_j\| < 1$ for all j. For any positive ε, we may find $\{y_j\} \subset S_b$ with $\|x_j - y_j\| < \varepsilon/2n$. Then $\|\sum y_jy_j*\| < \varepsilon$, whence from the definition of the natural norm, for each j, $\|y_j\|^2 < \|\sum y_jy_j*\| < \varepsilon$. Hence for each j, $\|x_j\| < 2\sqrt{\varepsilon}$ for all ε, so $x_j = 0$.

Now we show $(S_B)_b = S_B$ and that the natural norm on S_B, $\| \quad \|_1$, agrees with $\| \quad \|$. We first observe that all the considerations of the previous paragraphs apply to M_2S_B (notice that if $\|A\| = 0$ for A in M_2S_b, then reading off the (1, 1) and (2, 2) entries of $AA\#$ would yield that they have norm zero, so all the entries of A would be zero). Define $J = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ in M_2S_B, and a subring $C = \{bI + b'J | b, b' \in S_B\}$. As $J^2 = -I$, $C = S_BI + S_BJ$, and J is central in C, we see that C is (algebraically) isomorphic to $S_B \otimes_{\mathbf{R}} \mathbf{C}$. If we norm C as a subalgebra of M_2S_B, C is a norm closed $\#$-subalgebra ($J\# = -J$), and the norm satisfies $\|xx\#\| = \|x\|^2 = \|x\#\|^2$. Since C is a complex $*$-algebra, C is thus a $C*$ algebra; hence $C_b = C$ and there is only one such norm on C. In particular, the natural norm on C agrees with $\| \quad \|$.

Let a be an element of C. If $\|a\|^2 = r \in \mathbf{R}$, there exists d in C such that

$aa\sharp + dd\sharp = rI$. If $a = bI$ (b in S_B), observing that the (1, 1) entry of $dd\sharp$ is positive, we deduce that $aa\sharp \leqq r$; so $(S_B)_b = S_B$ and $\|b\| \geqq \|b\|_1$ (by 1.8).

On the other hand, if $bb* + \sum^n x_i x_i* = q \in Q$ with b in S_b and x_i in S_B, set $s = \sum x_i x_i*$. Then $s = s*$ and s belongs to S_b. We may find y_i in S_b with $\|y_i - x_i\| < \varepsilon/4qn$. Then $\|q - bb* - \sum y_i y_i*\| < \varepsilon$; since the restriction of $\|\quad\|$ to S_b is the natural norm, by 1.7 we have in S_b.

$$-\varepsilon \leqq q - bb* - \sum y_i y_i* \leqq \varepsilon.$$

Therefore, $\|b\|^2 \leqq q$, and so for b in S_b, $\|b\|_1 \geqq \|b\|$; thus on S_b, the two norms agree. Since $\|\quad\|_1$ is dominated by $\|\quad\|$, the former is continuous with respect to the latter, and thus from the density of S_b in S_B, the norms agree on S_B.

Finally, suppose $1 + \sum y_i y_i*$ is invertible in S_b for all finite subsets $\{y_i\}$ of S_b. Consider $\{x_i\}$ in S_B. Approximate each x_i by a sequence $\{x_{ij}\}_j$ in S_b; for each j, there exists z_j in S_b so that

$$(1 + \sum x_{ij} x_{ij}*) z_j = z_j(1 + \sum x_{ij} x_{ij}*) = 1.$$

By 1.1, $\|z_j\| \leqq 1$, and so $1 + \sum x_i x_i*$ cannot be a topological divisor of zero; thus it must be invertible.

PROPOSITION 1.10. *Let $(S, *)$ be a rational algebra with positive definite involution. The collection of elements of S_b such that $\|s\| = 0$ is a two-sided $*$-ideal of S_b, denoted $J*(S)$. Suppose that all elements of S of the form $1 + \sum x_i x_i*$ are invertible. Then*

(a) *$J*(S)$ is contained in the Jacobson radical of S_b, $J(S_b)$*

(b) *$*$ induces a positive definite involution on the quotient algebra $S_b/J*(S)$; and*

(c) *The natural seminorm on S_b induces a norm ($\|x\| = 0$ implies $x = 0$) on $S_b/J*(S)$ which agrees with the natural norm on the quotient algebra.*

(The inclusion $J*(S) \subset J(S_b)$ may be strict; the example in 2.25 is a nontrivial local domain (so $J(S_b)$ is a proper maximal ideal), but its $J*$ is zero).

PROOF. By (i) through (iv) above, $J*(S)$ is a two-sided $*$-ideal of S_b.

(a) It suffices to show, for all s in $J*(S)$, that $1 - s$ is invertible in S_b. Let r be a small positive rational number. We deduce from the positivity of $(s - r)(s - r)*$ that $r(s + s*) \leqq ss* + r^2$, so $s + s* \leqq r^{-1} ss* + r$. Hence,

$$\begin{aligned}(1 - s)(1 - s)* &= 1 - (s + s*) + ss* \\ &\geqq 1 - r + (1 - r^{-1})ss* \\ &\geqq 1 - 2r\end{aligned}$$

(the latter inequality holds since $mss* \geqq r$ for all positive integers m).

Thus if r is less than 1/2, applying the hypothesis to a positive rational multiple of a sum of the form $1 + \sum x_i x_i*$, we see that $(1 - s)(1 - s)*$ is invertible in S, and therefore by 1.1, is invertible in S_b. Since $s*$ also lies in $J*(S)$, $(1 - s)*(1 - s)$ is invertible as well, whence $1 - s$ has an inverse in S.

(b) Suppose $a = \sum s_j s_j*$ belongs to $J*(S)$, with s_i in S_b. Then $a^2 \leqq q^2$ for all positive rationals q; by 1.7 and the positivity of a, $0 \leqq a \leqq q$. As $s_i s_i* \leqq a$ for each i, all the s_i belong to $J*(S)$, so the anti-automorphism of $S_b/J*(S)$ induced by $*$ is positive definite.

(c) The induced seminorm on the quotient algebra is defined by $\|a + J*(S)\|_1 = \|a\|$; this is well-defined, since if $a - b$ belongs to $J*(S)$, then $\|a\| \leqq \|b\| + \|a - b\| = \|b\|$, and similarly $\|a\| \geqq \|b\|$. If $\|a + J*(S)\|_1$ is zero, then obviously a belongs to $J*(S)$, so this seminorm is a norm, and the submultiplicative and subadditive properties are also inherited by $\| \quad \|_1$.

If $\| \quad \|_2$ is the intrinsically defined seminorm on $\bar{S} = S_b/J*(S)$, from $aa* \leqq q$ implying $aa* + J*(S) \leqq q + J*(S)$, we obtain $\| \quad \|_1 \geqq \| \quad \|_2$. On the other hand, if $aa* + J*(S) \leqq q + J*(S)$, say $aa* + \sum x_i x_i* + s = q$ (with s in $J*(S)$), then $s = s*$, so $s^2 \leqq 2^{-2n}$ for all n, whence by 1.7, $-s \leqq 2^{-n}$; thus $aa* \leqq q - s \leqq q + 2^{-n}$, so $\|a\|^2 \leqq q + 2^{-n}$. Thus $\|a + J*(S)\|_1^2 = \|a\|^2 \leqq q$.

COROLLARY 1.11. *If $(S, *)$ is real $*$-algebra with involution such that all elements of the form $1 + \sum x_i x_i*$ are invertible, then $J*(S) = J(S_b)$; that is, $J*(S)$ equals the Jacobson radical of S_b.*

PROOF. If s belongs to $J(S_b)$, then so does $ss*$, whence $ss* - r$ is invertible for all nonzero real r. Completing $S_b/J*(S)$ to a real $C*$-algebra, the image of $ss* - r$ is invertible for all nonzero r, so the spectrum of $ss* + J*(S)$ consists of 0 only. However, a positive element in a $C*$ algebra with one point spectrum is that scalar, so $ss*$ belongs to $J*(S)$. Since the induced involution is positive definite, s belongs to $J*(S)$.

Although we will not be considering K_0 (except in a brief discussion of a class of examples), it is worthwhile recording another consequence of the invertibility of all terms of the form $1 + \sum x_i x_i*$.

LEMMA 1.12. *Let $(S, *)$ be a ring with involution, such that all terms of the form $1 + \sum x_i x_i*$ are invertible in S. Then*

(a) *If, for x in S_b, $\|x - 1\| < \sqrt{2} - 1$, then x is invertible in S_b.*

(b) *If the image of x in $S_b/J*(S)$ is invertible in the norm completion, x is invertible in S_b (so $\sqrt{2} - 1$ in* (a), *may be replaced by* 1).

(c) *The induced map on K_0, $K_0(S_b) \to K_0(S_B)$ (S_B denotes the norm completion of $S_b/J*(S)$) is an embedding.*

PROOF. (a). First suppose $x = x*$ and $\|x - \| = q < 1$. Then there exist d_i in S_b so that $(x - 1)^2 + \sum d_i d_i * = r$, r a rational less than 1. Hence $2x = 1 - r + \sum d_i d_i * + xx*$. Multiplying by the appropriate rational, $(1 - r)^{-1}$, we see x is invertible. For general x with $\|x - 1\| < \sqrt{2} - 1$, we see that both $\|xx* - 1\|$ and $\|x*x - 1\|$ are less than 1, so both $xx*$ and $x*x$ are invertible; thus x is invertible.

(b). Pick y in S_b whose image in $S_b/J*(S)$ approximates sufficiently closely the inverse of the image of x. Then both $\|xy - 1\|$ and $\|yx - 1\|$ are small, so both xy and yx are invertible, whence x is invertible.

(c). The map $S_b \to S_B$ satisfies criteria (1), (2), (3′) of [**17**; p. 204, 208], so [**17**; Theorem 3.1] applies.

The map of 1.12 (c) need not be an isomorphism, even when S is a field and a real algebra.

We previously commented that all elements of the form $1 + \sum x_i x_i *$ are not zero divisors, so if all nonzero divisors are invertible (in S), all such sums are invertible in S_b. Another, somewhat different source of rings for which such sums are invertible, is the following proposition.

PROPOSITION 1.13. *Let* $(S, *)$ *be a ring with involution.*

(a) *If* M_2S *is a Rickart* $*$*-ring with respect to* $\# = *$*-transpose, then* $1 + aa*$ *is invertible in* S *for all* a *in* S.

(b) *If* $M_{2^n}S$ *is a Rickart* $*$*-ring with respect to* $\# = *$*-transpose, then* $1 + \sum x_i x_i *$ *in invertible in* S *for all subsets* $\{x_i\}$ *of* 2^{n-1} *elements of* S.

PROOF. (a). Set $R = M_2S$, and define A in R as $A = [\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}]$. Let P be the projection of R such that PR is the right annihilator of A. Setting $B = [\begin{smallmatrix} -a & 0 \\ 1 & 0 \end{smallmatrix}]$, we have $P = P\#$; $AP = 0$, $AB = 0$, and therefore $PB = B$. Writing $P = [\begin{smallmatrix} c & e^* \\ e & d \end{smallmatrix}]$, we have $c = c*$ and $d = d*$. From $AP = (0)$, we deduce (i) $c = -ae$, and from $PB = B$, we have (ii) $e* = -(1 - c)a$. Applying the involution to (ii) and substituting in it (i), we have $c = aa*(1 - c)$, so adding $1 - c$ to both sides yields $1 = (1 + aa*)(1 - c)$; applying the involution, we see that $1 - c$ is the inverse of $1 + aa*$. (The same method also shows that d is the inverse of $1 + a*a$.)

(b). Define A in $M_{2^{n-1}}S$ to be matrix whose top row is $(x_1, x_2, \ldots)$, and whose remaining rows consist of zeroes. Then $I + AA\#$ is invertible in $M_{2^{n-1}}S$ because $M_{2^n}S = M_2 M_{2^{n-1}}S$, and the (1, 1) entry of the inverse of $I + AA\#$ is the inverse of $1 + \sum x_i x_i *$.

Let (G, u) be a partially ordered group with order unit u (our model is $(s_b^s, 1)$). A *state* is an order-preserving group homomorphism $f\colon G \to \mathbf{R}$ to the additive group of the reals, so that $f(u) = 1$. This is often referred to as a normalized state, but we will not deal with unnormalized states. The collection of states is a compact subset of the space of functions from G to $\mathbf{R}$, topologized via the point-open topology ([7] or [**1**], for further details). Given two states f, g, and a real number r between 0 and 1, we may form a third state, $h = rf + (1 - r)g$; h is a *convex linear combination* of f and g. A state is *extremal* if it cannot be written as a convex linear combination of two distinct states. It is well-known, for example, that for t in G, $\sup\{|f(t)| \mid f \text{ a state of } (G, u)\} = \sup\{|f(t)| \mid f \text{ an extremal state of } (G, u)\}$, and it follows immediately from [7; 3.1 and 3.2], that if t is a positive element of G, then $\sup\{f(t) | f \text{ a state of } (G, u)\} = \inf\{n/m |\ mt \leqq nu, m \in \mathbf{N} \cup \{0\}, n \in \mathbf{N}\}$. Since we are always assuming S, and hence S_b, is a rational algebra, S_b^s is a rational vector space, so the infimum on the right (above) becomes (for t positive) $\inf\{q \in \mathbf{Q}^+ |\ t \leqq q\}$. To see that this is the same as what we have previously defined as $\|t\|$, we require a lemma.

LEMMA 1.14. *Let $(S, *)$ be a ring with positive definite involution that is an algebra over the rationals. Let t be an elements of S_b such that $t \geqq 0$. Then*

$$\inf\{q \in Q | t \leqq q\}^2 = \inf\{\lambda^2 \in Q | t^2 \leqq \lambda^2\}.$$

PROOF. By 1.7, the left infimum is dominated by the right infimum. Write $t = \sum^n a_i a_i{*}$. Define A in $M_n S_b$ to be the matrix whose first row is $(a_1, a_2, \ldots, a_n)$, and whose remaining rows are zero. Then $AA\# = t \oplus 0_{n-1}$, which we shall call T. If $t \leqq q$, then $T \leqq qI$. In $M_n S_b$, $\|AA\#\|^2 = \|T\|^2 = \inf\{\lambda \in Q | T^2 \leqq \lambda I\}$; by (iv), $\|A\|^4 = \|T\|^2$. By definition, $\|A\|^2 = \inf\{q \in Q | T \leqq qI\}$; therefore

$$\inf\{q \in Q | T \leqq qI\}^2 = \inf\{\lambda \in Q | T^2 \leqq \lambda I\}. \tag{1}$$

But cutting down by the projection $E = e_{11}$ in $M_n S_b$ preserves the order relations in (1), and $ETE = t$, so the desired equality holds.

For (G, u) a partially ordered group with order unit u, let $S(g, u)$ denote the collection of states of (G, u) topologized by the point-open topology, and with the obvious convex structure. Let $\operatorname{Ker} G = \bigcap\{\ker f \mid f \in S(G, u)\}$ — in otherwords, Ker G is the collection of elements of G that vanish at every state. On the quotient group $\bar{G} = G/\operatorname{Ker} G$, we can impose a partial ordering, namely $x + \operatorname{Ker} G$ belongs to $\bar{G}^+$ if and only if $G^+ \bigcap (x + \operatorname{Ker} G)$ is nonempty. One readily checks that $\bar{G}$ becomes a partially ordered group with this ordering, with order unit $u + \operatorname{Ker} G$ (simply observe that if $x + \operatorname{Ker} G \in (\bar{G}^+) \bigcap (-\bar{G}^+)$, then $f(x) = 0$ for all states f of (G, u)).

LEMMA 1.15. *Let (G, u) be a partially ordered group with order unit. Set* $\operatorname{Ker} G = \bigcap\{\ker f \mid f \in S(G, u)\}$. *Then $\bar{G} = G/\operatorname{Ker} G$ has the structure of a partially ordered group with order unit $u + \ker G$, and the mapping*

$$\alpha\colon S(G, u) \to S(\bar{G}, u + \operatorname{Ker} G)$$
$$f \mapsto \tilde{f}\ ; f(a + \operatorname{Ker} G) = f(a)$$

is a well-defined affine homeomorphism.

PROOF. The argument above shows that $(\bar{G}^+) \bigcap (-\bar{G}^+) = 0 + \operatorname{Ker} G$, and clearly $\bar{G}^+ + \bar{G}^+ \subseteqq \bar{G}^+$; even more obvious is the fact that u + Ker G is an order unit. Certainly f: $(G, u + \text{Ker } G) \to (\mathbf{R}, 1)$ is a well-defined group homomorphism, as $f(\text{Ker } G) = \{0\}$. If $x + \text{Ker } G \geqq 0$, we may find h in Ker G with $x + h$ in G^+, so $f(x + \text{Ker } G) = f(x) = f(x + h) \geqq 0$. Thus f is a state, and that α is affine, continuous and one-to-one follows by inspection. Given a state t of $(\bar{G}, u + \text{Ker } G)$, define $\hat{t}$: $G \to \mathbf{R}$, $\hat{t}(g) = t(g + \text{Ker } G)$. If $g \geqq 0$, then $g + \text{Ker } G \geqq 0$, so $\hat{t}$ is a state of (G, u), and $\alpha(\hat{t}) = t$. Hence α is onto, and as the state spaces are compact, the mapping is a homeomorphism.

COROLLARY 1.16. *Let $(S, *)$ be a ring with positive definite involution such that all sums of the form $1 + \sum x_i x_i*$ are invertible. Then there are natural affine homeomorphisms between the state spaces*:

$$S(S_s^b, 1) \to S(S_b^s/J*(S)) \leftarrow S(S_B^s, 1).$$

PROOF. The invertibility hypothesis ensures that S is an algebra over the rational numbers. It follows from 1.14 and 1.7 that $J*(S)$ plays the role in S_b^s of "Ker G" in 1.15; so the first two state spaces are affinely homeomorphic. Given a state f in $S(S_B, 1)$, define $\check{f}$ on $(S_b^s/J*(S), 1 + J*(S))$ simply by restriction. Then $f \to \check{f}$ is continuous and affine. Since any state sends the unit ball (in the natural norm) to the interval $[-1, 1]$, by 1.9 (for S_B) states have norm 1, so are continuous. Since $S_b/J*(S)$ is dense in S_B, the assignment $f \to \check{f}$ is one-to-one. Any state g of $(S_b^s/J*(S), 1 + J*(S))$ has norm 1, is thus $\| \;\|$-continuous, and therefore extends to a continuous linear functional $\bar{g}$ of S_B^s. If x lies in S_B^+, there are a_i in S_B such that $x = \sum a_i a_i*$. We may approximate each a_i by a_{ij} in $S_b/J*(S)$, so x is a limit of positive elements $S_b/J*(S)$, whence $\bar{g}(x) \geqq 0$. Thus $\bar{g}$ is a state, and $\bar{g}$ is mapped to g, so the map is onto, and from compactness, is a homeomorphism.

2. **Fields.** Throughout this section, E will denote a formally real field, and F will denote either a formally real field, or a quadratic extension of a formally real field such that the Galois automorphism is positive definite, and in the latter case, E will be the fixed field.

With the identity map, the formally real field E becomes a partially ordered abelian group (with positive cone, all sums of squares). We study the connections between states, total orderings, and Spec E_b, as well as the corresponding results for the quadratic extensions F and F_b. We also consider, in some detail, a number of interesting examples of bounded subrings of fields and their possible pathologies.

Our immediate aim is to establish an involutory version of the Artin-Schreier Theorem. Recall that if E is a formally real field, an ordering on E is a partial order on the set E with positive cone P, satisfying:

(i) $P + P \subseteqq P$

(ii) $P \cap (-P) = (0)$

(iii) $P \cdot P \subseteqq P$

(iv) P contains all squares.

The ordering is *total* if $E = P \cup (-P)$. In particular, since E is formally real, the natural ordering corresponding to the identity involution (positive cone: sums of squares) is an ordering in this sense, and is the unique minimal such. The Artin-Schreier Theorem states that every ordering is contained in a total ordering, or what amounts to the same thing, if e in E is positive in every total ordering, then e is a sum of squares.

If $(F, *)$ is a field with positive definite involution, we define a *$*$-ordering of F* to be an ordering on the fixed (under $*$) subfield E (necessarily formally real, from the positive definiteness of $*$) such that in addition, all elements of E of the form $xx*$ (x in F) are positive. So all $*$-orderings contain the natural ordering, and again the natural ordering is the minimal $*$-ordering. The *$*$-ordering is total* if the corresponding ordering is total. As a caveat, it is well for the reader to bear in mind that the natural ordering on E, as a formally real field, is in general strictly weaker than the ordering on E induced by the natural ordering on F; it is even possible to construct an example (2.16) where $F = F_b$, but $E_b \neq E$, E_b being the bounded subring with respect to the ordering induced by the identity involution.

We wish to show, if a self-adjoint element of F (that is, an element of E) is not of the form $\sum x_i x_i*$ (x_i in F), then there exists a total $*$-ordering of F at which this element is negative. This is what is meant by an involutory version of the Artin-Schreier Theorem.

Associated in an essentially unique manner with every total ordering on E is an embedding into a real closed algebraic extension field, such that every element of E that is positive has a square root in this extension. One sees very easily that to each total $*$-ordering of F, we may associate, uniquely, a $*$-preserving embedding of $(F, *)$ into $(C, *)$ where C is an algebraic closure of F, and $*$ on C is positive definite; further, $C = R[i]$, where R is the real closure corresponding to the total ordering on E, and $i^2 = -1$, and $i* = -i$. It is straightforward to fill in the proof of the rest of the following proposition, so we omit its proof.

PROPOSITION 2.1. *Let $(F, *)$ be a field with (positive definite) involution, and pick d in E such that $F = E[\sqrt{d}]$. There are natural bijections between the following three sets*:

(a) {*total $*$-orderings on $(F, *)$*}

(b) {*$*$-isomorphism classes of $*$-preserving embeddings $(F, *) \to (C, *)$, C an algebraic closure of F, with $*$ positive definite on C, and $d* = -d$*}

(c) {*real closures of E in which d is a negative square*}.

A *norm* is an element of the form $xx*$; there will be no confusion with the Banach algebra type norm previously introduced.

THEOREM 2.2. (Involutory version of Artin-Schreier). *Let* $(F, *)$ *be a field with (positive definite) involution, and* E *the fixed subfield. Then an element* x *of* E *is positive in every total* $*$*-ordering if and only if* x *is a sum of norms.*

Our proof is based on the "real" proof in [**6**, Theorem 8.4].

LEMMA 2.3. *Let* Q *be the positive cone of a* $*$*-ordering on* F. *For* x *in* E *define* $P_x = \{px + q \mid p, q \in q\}$. *Then there is a* $*$*-ordering on* F *extending the* Q*-ordering in which* x *is positive if and only if*

$$\text{(A)} \qquad P_x \cap -P_x = (0).$$

PROOF. The positive cone of the $*$-ordering must contain P_x, so (A) is certainly necessary. On the other hand $P_x + P_x \subseteqq P_x$, $P_x \cdot P_x \subseteqq P_x$ (the latter, since x belongs to E, and therefore x^2 is a norm). Hence assuming (A) holds, P_x is a positive cone for *a* $*$-ordering, and Q is contained in P_x.

LEMMA 2.4. *Given a* $*$*-ordering of* F *with positive cone* Q, *then for* x *in* E, *there exists a* $*$*-ordering of* F *extending that from* Q *in which* x *is positive, if and only if for all* p, q *in* Q, $px + q = 0$ *implies* $p = q = 0$.

PROOF. Certainly, if x is positive in a finer ordering, necessity of the condition above is clear. On the other hand, if $px + q \neq 0$ for all nonzero, p, q, then $P_x \cap -P_x = \{0\}$ follows immediately. By 2.3, such an extension-ordering exists.

PROOF of 2.2. We first show that a maximal $*$-ordering of F is total. Let Q be the corresponding positive cone, and suppose x lies in E but in neither Q nor $-Q$. If $px + q = 0$ and $r(-x) + s = 0$ for some p, q, r, s in Q, we obtain $qr + ps = 0$, whence $qr = ps = 0$, and this implies either $p = q = 0$ or $r = s = 0$. By 2.4, Q may be extended so that at least one of x, $-x$ becomes positive, but this contradicts the maximality of Q. Hence $E = Q \cup -Q$.

Now let P be the subset of E consisting of sums of the form $\sum x_i x_i*$, with x_i in F. Then P is the positive cone for the natural $*$-ordering of F. If x lies in E but not in P, for p, q in $P - \{0\}$, $p(-x) + q = 0$ implies $x = q/p$, so by 1.5, x would lie in P; thus $p(-x) + q$ is never zero, so there exists a $*$-ordering with positive cone Q extending that of P in which $-x$ is positive. Now a simple Zorn's Lemma argument applies to yield a total $*$-ordering containing Q in its positive cone.

COROLLARY 2.5. *Let* $(H, *)$ *be a field with positive definite involution.*

Then there exists an algebraic closure K of H, and an extension of the involution to K that is positive definite.

PROOF. Because $*$ is positive definite, the natural $*$-ordering on H can be extended to a total $*$-ordering. Let R be the real closure of the fixed subfield on H corresponding to this $*$-ordering, and set $K = R[i]$, with the usual automorphism.

Let $(F, *)$ be a field with (positive definite) involution. Call $(F, *)$ *totally $*$-Archimedean* if every total $*$-ordering on it is Archimedean, that is, if every $*$-algebraic closure of F (see 2.1(b)) is $*$-embeddable in the complex numbers, equipped with complex conjugation. In case $*$ happens to be the identity, the property can also be referred to as *totally Archimedean* (see §3 for an explanation of "Archimedean").

THEOREM 2.6. *Let $(F, *)$ be a field with a positive definite involution. The following are equivalent.*

(i) *$(F, *)$ is totally $*$-archimedean.*

(ii) $F = F_b$.

(iii) *For all r in F, there exists a positive integer n such that $n - rr*$ is a sum of norms.*

(iv) *For all r in F, for all total $*$-orderings, there exists a positive integer n (depending on r, and apparently on the ordering) so that $n - rr*$ is positive at this total $*$-ordering.*

PROOF. (i) $\Rightarrow$ (iv). Any $*$-algebraic extension of $(F, *)$ is obviously $*$-embedded in the complexes, so (iv) follows from 2.1.

(iv) $\Rightarrow$ (iii). Suppose for a fixed r in F, there is no bound on the n's required; pick $(G_j, *)$ to be $*$-algebraic closures so that $n(j) \geqq j$, with j varying over the positive integers. Form a nonprincipal ultraproduct $(G, *)$ of the $(G_j, *)$, and observe that the inclusion $(F, *) \to (G, *)$ induces a total $*$-ordering of F in which n-$rr*$ is not positive for any n (an easy consequence of Los' Theorem). Hence there must exist an integer n so that $n - rr*$ is positive in all total $*$-orderings. By 2.2, $n - rr^*$ is a sum of norms.

(iii) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (i). Let $(G, *)$ be a $*$-algebraic closure of $(F, *)$, and let H be the fixed field of $*$ in G. Consider H_b, its ring of bounded elements (in its own natural ordering). Since H is real closed, every element is either a square or the negative of a square; thus the natural ordering on H is total. If $H = H_b$, then H is Archimedean, so is embeddable in the reals. To show $H \neq H_b$ is impossible, we first show H_b is a valuation domain.

Pick a, b in H_b. Then ab^{-1} is either positive or the negative of a sum of squares; by multiplying if necessary by -1, we may assume $ab^{-1} > 0$. If $ab^{-1} \leqq 1$, then $ab^{-1} \in H_b$, so $a = b(ab^{-1})$, and thus $aH_b \subseteqq bH_b$. If

$ab^{-1} \nleqq 1$, then $ab^{-1} > 1$ as the ordering is total. This yields $a^{-1}b < 1$, whence $bH_b \subseteqq aH_b$.

Now select a in H. Since a is algebraic over F and $F_b = F$, a is integral over H_b (as $F_b \subseteqq H_b$). Any valuation domain is integrally closed in its field of fractions, so $a \in H_b$, and thus $H = H_b$.

The foregoing applies whether or not $*$ is the identity. Examples of totally ($*$-) Archimedean fields include any field with involution that is algebraic over the rationals. The quaternionic versions of 2.2 and 2.6 also work out readily, because the symmetric elements are all central and every element commutes with its adjoint (assuming the involution is the natural one, and that it is positive definite).

Now let us decide what the states are on F_b^s, when F is a field.

PROPOSITION 2.7. *Let* $(F, *)$ *be a commutative ring with positive definite involution, and suppose* $1 + \sum x_i x_i*$ *is invertible in* F *for all finite subsets* $\{x_i\}$ *of* F. *Then*

(a) *a state of* $(F_b^s, 1)$ *is extremal if and only it if is multiplicative,*

(b) *the extremal states of* $(F_b^s, 1)$ *form a compact space* X *in the topology of pointwise convergence,*

(c) *the natural mapping* $F_b^s \to C(X, \mathbf{R})$ *is a ring homomorphism with dense image, and the kernel is* $J*(F) \cap F^s$, *and*

(d) *the norm closure of* $F_b^s/J*(F)$ *is isometrically isomorphic to* $C(X, \mathbf{R})$.

PROOF. Let S be the closure of $F_b/J*(F)$ in the natural norm, and set $T = S^s$. One easily checks that $F_b^s/J*(F) \cap F^s$ is dense in T. By 1.16, the states of (T, 1) are the "same" as those of $(F_b^s, 1)$. Now 1.9 and 1.10 can be applied to S, and thence to T. In particular, S has zero radical, so it follows that $J(T) = \{0\}$. Then the Gelfand mapping

$$\phi: T \to C(X, \mathbf{C})$$
$$t \mapsto \hat{t}, \quad \hat{t}(x) = x(t),$$

where $X = \{x: T \to \mathbf{C} \mid \text{nonzero ring homomorphisms}\}$ equipped with the point-open topology, is an embedding. Since $1 + t^2$ is invertible in T for all t in T, $x(T) \subseteqq \mathbf{R}$, so $x(T) = \mathbf{R}$. Thus $\phi(T) \subseteqq C(X, \mathbf{R})$. As $x(t^2) = (x(t))^2$, all the elements of X are states. In particular, ϕ is order-preserving, whence ϕ is continuous in the natural norms, and since $\phi(T)$ is norm complete and separates the points of X, we must have $\phi(T) = C(X, \mathbf{R})$ by the Stone-Weierstrass theorem.

Suppose $t \geqq 0$; then $s = \sqrt{\hat{t}}$ lies in $C(X, \mathbf{R})$, so there exists c in T such that $\hat{c} = s$. Thus $c^2 = t$ as ϕ is one-to-one, so t lies in T^+; hence ϕ is an order-isomorphism. As the natural norm on $C(X)$ agrees with the supremum norm, ϕ is an isometry. This proves (d).

The extremal states of $C(X, \mathbf{R})$ are exactly the points of X, which are exactly the multiplicative states. Now 1.16 and 1.10 imply (a), (b), and (c).

By 2.7(b), the state space is a Bauer simplex. In contrast, when there is some degree of noncommutativity (meaning, in the case of division rings, worse than quaternionic), the state space is almost never even a Choquet simplex (cf 3.6).

PROPOSITION 2.8. *Let $(F, *)$ is a field with involution, and suppose F is totally $(*-)$ Archimedean. Then the set of extremal states is a totally disconnected compact space in its point-open topology.*

PROOF. Let S denote the extremal state space of $(F_b^s, 1)$, and for a point x in S, let V be the intersection of all the closed and open neighbourhoods of x. Pick r in F_b^s. The topology on S is the weakest with respect to which all functions $\hat{r}$: $S \to \mathbf{R}$ defined by $\hat{r}(f) = f(r)$ are continuous. Hence $\hat{r}(V) = \{v(r) | v \in V\}$ is a connected compact set, that is, a closed interval. If $r(V)$ is not a single point, it must contain a rational, say $q = v_0(r)$ for some v_0 in V. Then $r - q$ lies in the kernel of v_0; as v_0 is multiplicative, the kernel is an ideal of F_b; but $F = F_b$ by hypothesis, so we must have $r = q$. Hence, in all cases, $\hat{r}(V)$ consists of a single point. Since the set of such $\hat{r}$'s separate the points of S, V itself must consist of a single point; thus S has a basis of closed and open sets; being compact, S is totally disconnected.

The space of extremal states appears to have a tendency to be *connected* when $F_b \neq F$ and F is a real algebra. We will now give an example where the extremal state space is the closed unit interval, and later on we will give an example where the extremal state space is homeomorphic to the unit circle.

EXAMPLE 2.9. For this example consider a formally real field E such that
(i) The extreme state space of $(E_b, 1)$ is homeomorphic to $[0, 1]$, and
(ii) E_b is a principal ideal domain.
(Several other properties of this example will also be mentioned later).
Let E be the field of (equivalence classes of) real-valued functions each defined and real meromorphic on a neighbourhood of $[0, 1]$, with addition and multiplication defined on the intersection of the domains; it is readily checked that E actually is a field, and is formally real.

Let D be the subring of E consisting of functions analytic on a neighbourhood of $[0, 1]$. If f is a meromorphic function bounded in our sense, it must be bounded in absolute value on $[0, 1]$, hence is analytic on some neighbourhood of $[0, 1]$. Thus $E_b \subseteqq D$. On the other hand, if f is analytic on a neighbourhood of $[0, 1]$, it must be bounded in some (possibly smaller) nieghbourhood, simply from the compactness of $[0, 1]$. Hence there is a neighbourhood of $[0, 1]$ with $q^2 - f^2 > r$ on this neighbourhood, where q, r are positive rationals. It follows that $q^2 - f^2$ is an exponential, hence is a square, so f belongs to E_b; thus $E_b = D$. (As some warning why we must go to neighbourhoods and consider strict positivity, observe that the function $f(x) = x$ is positive over that will turn out to be the extremal state space, but f is not positive in the natural ordering!)

Any function in D can have only finitely many zeroes on [0, 1], and so, since D and all its relatives are well-known to be Bézout domains, in this case, D is actually a principal ideal domain, and it is very easy to see simultaneously that the maximal ideals correspond precisely to the points of [0, 1]. Now D is also a real algebra, so the extremal states correspond exactly to the maximal ideals, and the point-open topology on [0, 1] is precisely the usual topology.

One can also show that if E is the field of fractions of either the ring of entire functions, or the ring of functions analytic in a neighbourhood of the reals, then the maximal ideal space of E_b is $\beta\mathbf{R}$.

It is convenient to have available a criterion that decides whether a commutative domain (with involution) is the bounded subring of its field of quotients, or contains it.

THEOREM 2.10. *Let* $(D, *)$ *be a commutative domain with positive definite involution, and let* $(K, *)$ *be its field of quotients. Then* $K_b \subseteqq D$ *if and only if* (A) *or, equivalently,* (B) *hold.*

(A) (i) $M = M*$ *for all maximal ideals* M *of* D,

(ii) *the induced involutory anti-automorphism on* D/M *is positive definite, for all maximal ideals* M, *and*

(iii) D *is a Prüfer domain.*

(B) (i) $1 + \sum x_i x_i *$ *is invertible in* D, *for all subsets* $\{x_i\}$ *of* D, *and*

(ii) D *is a Prüfer domain.*

In particular, $K_b = D$ *if and only if* (A) *holds and every element of* D *is bounded.*

PROOF. The conditions (A) and (B) are equivalent to $M_n D$ being a Baer $*$ ring (with respect to $\# = *$-transpose) for all n [**9**; Theorem 2.3], so they are equivalent to each other. If $K_b \subset D$, they by 1.3, $M_n D$ contains all the projections of $M_n K$, so if must itself be Baer $*$, whence (again by [**9**; Theorem 2.3]), (A) holds.

On the other hand, if (A) holds, $M_n D$ is a Baer $*$ ring for all n, so by [**9**; Proposition 1.1], $M_n D$ contains all the projections of $M_n K$. If x belongs to K_b, there exist $\{t_i\}_{i=1}^m$ in K, and q in the rationals such that

$$\frac{xx*}{q^2} + \sum t_i t_i * = 1.$$

Define the matrix w in $M_{m+1}K$ whose first row is $(x/q,\ t_1,\ t_2,\ \ldots)$, and whose remaining rows consist of zeroes. Define P in $M_{2m+2}K$ by

$$P = \frac{1}{2}\begin{bmatrix} ww\# & w \\ w\# & w\#w \end{bmatrix}.$$

Inasmuch as $ww\#w = w$ and $w\#ww\# = w\#$, P is a projection, so all of its entries lie in D. Thus x belongs to D, whence $K_b \subseteqq D$.

The proof above of the converse also yields that if M_nS is Baer $*$ (Rickart $*$), and S is a $*$-subring of a ring T such that T is both right and left essential over the image S, then M_nT is Baer $*$ (Rickart $*$) if and only if $S_b \subseteqq T$.

Another property of fields, as distinguished from division rings, is that of Riesz decomposition, most easily stated in terms of its equivalent property, (Riesz) interpolation. A partially ordered group G satisfies the *interpolation* property, if whenever a, b, c, d, are elements of G so that $a \geqq c$, $a \geqq d$, $b \geqq c$, $b \geqq d$ (compactly expressed $a, b \geqq c, d$), then there exists an element e of G lying between, that is $a \geqq e \geqq c$, $b \geqq e \geqq d$ $(a, b \geqq e \geqq c, d)$. Obviously, a convex subgroup of a group with the interpolation property satisfies the interpolation property itself, so if S^s has interpolation, so does S^s_b. The following result is due to my colleague, Professor H. Helfenstein.

PROPOSITION 2.11. *Let $(F, *)$ be a field with involution. Then both F^s and $(F^s_b, 1)$ satisfy the interpolation property.*

PROOF. If a, b, c, d belong to F^s, and $a, b \geqq c, d$, we may assume $d = 0$ (subtract d from everything). If either $a = c$ or $b = c$, or either a or b is zero, there is an obvious choice for e. Hence we may assume that not both a and $b - c$ are zero. Since both are positive, $a + b - c \neq 0$. Set $e = ab(a + b - c)^{-1}$. Observe that since all of a, b and $a + b - c$ are greater than or equal zero, $e \geqq 0$; since $b \leqq b + a - c$, $e \leqq a$, and similarly $e \leqq b$. Since $(a - c)(b - c) \geqq 0$, we deduce $c^2 - c(a + b) + ab \geqq 0$, so $ab \geqq c(a + b - c)$, whenec $e \geqq c$. As F^s_b is a convex subgroup of F^s, it also satisfies the interpolation property.

(Of course, the proof works for any $*$-ordering on F^s that is invariant under the operation of taking inverses.)

Let $(F, *)$ possess a total $*$-ordering, and let $(C, *)$ be the corresponding algebraic $*$-closure (cf 2.1(b)). Then C_b is a valuation domain, with unique maximal ideal $J*(C)$ (viz. 2.6(ii) $\Rightarrow$ (i)) consisting of the elements c such that $cc*$ is infinitesimal. Now $A = C_b/J*(C)$ is a field, and it its own bounded subring, and additionally the natural ordering is a total ordering. Hence A^s is a real closed subfield of the reals, and A is of course $*$-embeddable in $(\mathbf{C}, -)$ ($-$ denotes complex conjugation). From the embedding $F_b \subset C_b$, we obtain a prime ideal N of F_b, $N = J*(C) \cap F_b$, and a $*$-embedding $F_b/N \to \mathbf{C}$. This latter embedding induces an extremal state, by restriction to the symmetric parts. So to each total $*$-ordering, we can associate a prime ideal (whose quotient ring is $*$-embeddable in the complexes), and an extremal state. Every such prime ideal yields a state, and perhaps more surprisingly, the map from total $*$-orderings to extremal states is onto.

THEOREM 2.12. *Let* $(F, *)$ *be a field with a positive definite involution.*

(a) *Let* $f: F_b \to \mathbf{C}$ *be a* $*$*-homomorphism. Then* $M = \ker f$ *is a prime ideal,* $M = M*$, *and there exists a total* $*$*-ordering so that if* $(C, *)$ *is the corresponding algebraic* $*$*-closure,* $M = J*(C) \cap F_b$, *and the quotient map, composed with the imbedding into* $\mathbf{C}$

$$F_b/M \to C_b/J*(C) \subseteqq \mathbf{C}$$

agrees with f.

(b) *Given a total* $*$*-ordering, and corresponding algebraic* $*$*-closure* $(C, *)$, *set* $M = J*(C) \cap F_b$. *Then* $M = M*$, M *is a prime ideal of* F_b, *and* F_b/M *is* $*$*-embeddable in the complexes.*

(c) *If* F *is a* $\mathbf{R}$-$*$*-algebra, the set map obtained from* (b)

$$\{\textit{total } *\textit{-orderings of } F\} \to \{\textit{prime ideals of } F_b\}$$

maps onto the set of maximal ideals.

PROOF. (a). Set $E = F^s$, and define the following subset of E.

$$P = \{\textstyle\sum b_i b_i * a_i d_i^{-1} | b_i \in F;\ a_i,\ d_i \in E \cap F_b;\ f(a_i), f(d_i) > 0 \text{ in } \mathbf{R}\}.$$

Obviously $P + P \subseteqq P$, and all elements of the from $xx*$ lie in P. Thus to show P is the positive cone for a $*$-ordering on F, it remains to establish $P \cap (-P) = (0)$.

Suppose not; then we may find nonzero elements of F, b_i, c_j, and nonzero elements of $E \cap F_b$ whose values at f are greater than zero, a_i, d_i, x_j, y_j so that

$$\textstyle\sum b_i b_i * a_i d_i^{-1} + \sum c_j c_j * x_j y_j^{-1} = 0. \tag{1}$$

By multiplying through, we may assume in (1) that $d_i = y_j = 1$ for all i, j, and that $\{b_i, c_j\} \not\subset V$.

Let V be the localization of F_b at M. As F_b is a Prüfer domain, V is a valuation domain, with maximal ideal $(M) = MV$. Since f is a $*$-homomorphism, $M = M*$, so $(M) = (M)*$. Since $V/(M)$ is a field of fractions of F_b/M, and the latter is positive definite, so is the field $V/(M)$, and in addition f induces an embedding of $V/(M)$ in $\mathbf{C}$.

Since V is a valuation domain, there exists z in (M) so that $\{zb_i, zc_j\} \subset V$, $\{zb_i, zc_j\} \not\subset (M)$. Multiplying (1) through by $zz*$ and remembering that $d_i = 1 = y_j$, we obtain

$$\sum(zb_i)(zb_j)*a_i + \sum(zc_j)(zc_j)*x_j = 0.$$

Applying f and the positive definiteness of the induced $*$ on $V/(M)$, we obtain $\{zb_i, zc_j\} \subset (M)$, a contradiction.

Thus P is a bona fide positive cone for a $*$-ordering of F. Since $f(1/n) = 1/n$ for all integers n we have for all m in $M \cap E$, for all positive integers n $1/n \pm m$ belongs to P. Hence, we obtain

$$-1/n \leqq m \leqq 1/n \tag{2}$$

at any $*$-ordering finer than that of P. Let Q be any total $*$-ordering containing P, and let $(C, *)$ be the corresponding algebraic $*$-closure. By (2), $M \subset J*(C) \cap F_b$; on the other hand, if x belongs to F_b but not to M, then $xx*$ also does not (as $x*$ does not, and M is prime), and so $f(xx*) > 0$. Hence there exists a positive integer n such that $xx* - 1/n$ is positive in the P-ordering, and thus in the Q-ordering. Therefore $xx*$ cannot lie in $J*(C)$, so neither can x, whence $M = J*(C) \cap F_b$.

Since $C_b/J*(C)$ is uniquely $*$-embeddable in the complexes, we obtain a composite mapping $F_b \subseteqq C_b \to C_b/J*(C) \subseteqq \mathbf{C}$, with kernel M. Now the ordering induced on $D = F_b/M$ by either Q or P is total since $\bar{f}(D^s) \subseteqq \mathbf{R}$ (where $\bar{f}$: $D \to \mathbf{C}$ is the map induced by f). Let G denote the field of fractions of D. Then $\bar{f}$, being an embedding, extends to an embedding, also called $\bar{f}$: $G \to \mathbf{C}$, as does the composite mapping; call its induced mapping f_0. The $*$-orderings induced on D (and hence on G) by $\bar{f}$ and by Q (equivalently by the composite mapping, hence by f_0) agree and are total. Since both $\bar{f}$ and f_0 are order-preserving, they are states sending 1 to 1; but a totally ordered group can have just one normalized state. Thus $f_0 = \bar{f}$, and so the composite map agrees with f on F_b.

(b). This follows from the discussion immediately preceding.

(c). All the prime images of F_b that are $*$-embeddable in $\mathbf{C}$ must contain a copy of the reals, so must be either the reals or the complexes, hence must be fields and the corresponding ideals are maximal. Ontoness follows from (a).

COROLLARY 2.13. *Let $(F, *)$ be a field with a positive definite involution, and suppose $(F, *)$ possesses no infinitesimals in the natural ordering (that is, $J*(F) = (0)$). Then $J(F_b) = (0)$ if any of the following conditions hold.*

(a) *All nonzero prime ideals are maximal (as occurs if F_b is a union of Dedekind domains), and F has no Archimedean total $*$-orderings.*

(b) *F is a real algebra.*

(c) *If M is a prime ideal of F_b such that F_b/M $*$-embeds in the complexes, then M is maximal.*

PROOF. Conditions (a) and (b) each imply (c), so assume (c) holds. Given a nonzero element x of F_b, there exists a total $*$-ordering at which

$xx*$ is not infinitesimal. Let M be the prime ideal corresponding to this total $*$-ordering, by (b) of the previous result. Then $xx*$ does not belong to M; as F_b/M embeds in the complexes, by hypothesis M is maximal, and x does not lie in M, whence x does not belong to $J(F_b)$.

(Example 2.15 demonstrates why the hypothesis on no Archimedean orderings is necessary.)

To summarize Theorem 2.12 and its consequences, there are set maps,

$$\begin{Bmatrix}\text{total } *\text{-orderings} \\ \text{of } F\end{Bmatrix} \xrightarrow{K} \begin{Bmatrix}\text{extremal state} \\ \text{space of } (F_b^s, 1)\end{Bmatrix} \xrightarrow{L} \text{Spec } F_b.$$

The middle set is a compact Hausdorff space in the point-open topology (equivalently, the relative product topology, the topology of pointwise convergence). The latter set is compact in the Zariski topology. The map L is readily seen to be continuous. The element (0) of Spec F_b is precisely the image under LK of the Archimedean total $*$-orderings (as readily follows from 2.12(a)), if such exist. With this exception, the range of L tries to be Max Spec F_b; H. Schültung observed, if $F = \mathbf{Q}(x, y)$, the image of L is not Max Spec $F_b \cup \{(0)\}$. Of course, if M is a maximal ideal of F_b, F_b/M is a totally $*$-Archimedean field, so its symmetric part admits an extremal state, inducing an extremal state on F_b; thus Max Spec F_b is always in the image of L.

The mapping K is always onto, by 2.12(a). Neither K nor L need be one-to-one. For L, the easiest example of this behaviour occurs with any formally real totally Archimedean field with more than one ordering, $\mathbf{Q}[\sqrt{2}]$, being the simplest possibility.

For an example with K not one-to-one, begin with E, a real closed subfield of the reals ($E = \mathbf{R}$ is admissible), and form the ring of Laurent power series in one variable

$$\{\sum_{i=n}^{\infty} a_i x^i | a_i \in E,\ n \in \mathbf{Z}\},$$

denoted $G = E((x))$, and consider the ring of formal power series, $D = E[[x]]$. Since we obviously have $D[x^{-1}] = G$, G is the field of fractions of D, and we will now show $G_b = D$ (computed with respect to the identity involution).

Clearly $D/(x) = E$ is formally real, and since D is a local principal ideal domain, by 2.10, $G_b \subseteqq D$. Now to show something, say $r = x^m(a_0 + \sum a_i x^i)$, ($m \geqq 0$, $a_i \in E$, $a_0 \neq 0$), is a square in D, it suffices to show m is even and a_0 is positive, i.e., a square in E. Thus $a_0^2 + 1 - r^2$ is always asquare, so, since E is Archimedean, r is bounded by any positive integer bounding $a_0^2 + 1$. Hence $D \subseteqq G_b$, whence equality holds.

(If E is merely real closed, one would expect that $G_b = E_b[[x]]$; unfortunately, if J is an order in a field K, it does not follow that $J[[x]]$ is an order in $K((x))$.)

Now the element x itself is neither a sum of squares nor a negative sum of squares, so G possesses more than one total ordering. On the other hand, $1/n \pm x$ is a square for all positive integers n, so x belongs to $J*(G)$, and thus $J*(G) = (x) = J(D)$. Hence all states annihilate (x), so induce states on the quotient field E; being real closed, E admits only one state. Hence K is rather far from being one-to-one. We also have an example with many distinct total orderings, all having the same set of infinitesimals (the ideal (x)), and the same induced ordering on the residue field.

EXAMPLE 2.14. For this example we will study $E(x)_b$. Here E is a real closed subfield of the reals, and x is an indeterminate. It turns out that $E(x)_b$ is fairly easy to describe, has extremal state space the one point compactification of the reals, is a Dedekind domain with class group of order 2, and none of its maximal ideals are principal but they correspond to the points of E and a point at infinity in a natural way.

Define a subring D or $E(x)$, $D = \{f/g \mid f, g \in E[x], \deg f \leqq \deg g, g \text{ has no linear factors}\}$. It is clear that D is a ring, and an E-algebra, in fact D is precisely the set of elements of $E(x)$ that are positive at all the valuations corresponding to linear polynomials and the infinite valuation. The elements of the form

$$(**) \qquad \frac{(x+1)^2 + b^2}{x^2+1}\, c$$

b, $c \neq 0$, a, b, $c \in E$, are units in D, and because every polynomial in $E[x]$ is a product of linear and irreducible quadratic polynomials, it follows that every element of D is a product of a unit with elements of the forms

$$(*) \qquad \mathrm{I}\colon \frac{1}{x^2+1};\ \mathrm{II}_a\colon \frac{x-a}{x^2+1};\ \mathrm{III}_{b,c}\colon \frac{(x-b)(x-c)}{x^2+1}.$$

Now every unit is a product of elements of the form $(**)$, and $(**)^{-1}$ and it is easy to check that all of the elements of the form $(*)$ or $(**)$ are bounded (view them as continuous functions on the real line, observe they are bounded as functions by rational numbers, and show the difference is a sum of squares in D; alternately, one can use 1.1 and its proof to obtain that I, II_0 belong to $E(x)_b$, and $\mathrm{II}_a = \mathrm{II}_0 - a\mathrm{I}$, and $\mathrm{III}_{b,c}$ and the units in $(**)$ are E-linear combinations of I and II_a). Thus $D \subseteqq E(x)_b$.

On the other hand, suppose $a = f/g$ is a quotient of relatively prime polynomials, and a is bounded. If $\deg f > \deg g$, then a has a polynomial part, and so cannot be bounded as a function on the reals, let along in our sense. If $g(e) = 0$ for some e in E, then a is obviously not bounded on any neighbourhood of e (in the reals), so cannot be bounded in the natural ordering. Thus $E(x)_b \subseteqq D$, so equality holds; in particular, D is a Prüfer domain (2.10).

It follows easily from the factorization of elements of $(*)$ that D is Noetherian, so is a Dedekind domain. Let M be a maximal ideal of D. As M is generated by products of the elements of $(*)$, being maximal, M can be generated by elements of $(*)$ themselves. We first note that any ideal containing I and $\mathrm{III}_{b,c}$ (any choice for b, c) is improper.

Define a certain set of ideals, as follows:

$$M_0 = (\mathrm{I}, \mathrm{II}_0),\ M_a = (\mathrm{II}_a, \mathrm{III}_{a,-a}),\ a \in E.$$

We will show these are proper ideals, then that these are maximal and distinct, and finally that these are all the maximal ideals of D. If M_0 were not proper we could find polynomials f_i, g_i with $\deg f_i \leqq \deg g_i$ $(i = 1, 2)$ such that

$$\frac{1}{1 + x^2} \cdot \frac{f_1}{g_1} + \frac{x}{1 + x^2} \cdot \frac{f_2}{g_2} = 1;$$

that is, $f_1 g_2 + x f_2 g_1 = (1 + x^2) g_1 g_2$. But the degree on the right hand side is greater by 2 than $\deg g_1 + \deg g_2$, which is in turn at most one less than the degree of the left side, a contradiction. If M_a were not proper, there would exist polynomials f_i, g_i so that $g_i(a) \neq 0$ and

$$\frac{x - a}{1 + x^2} \cdot \frac{f_1}{g_1} + \frac{x^2 - a^2}{1 + x^2} \cdot \frac{f_2}{g_2} = 1,$$

so $(x - a) f_1 g_2 + (x^2 - a^2) f_2 g_1 = (1 + x^2) g_1 g_2$. But the left side vanishes at a, while the right side does not.

To see that M_∞ is maximal, observe that $(\mathrm{I}, \mathrm{II}_0) = (\mathrm{I}, \mathrm{II}_a)$ for any a in E, so adding any element of $(*)$ to M_∞ will either leave it unchanged, or blow it up to being improper, since $(\mathrm{I}, \mathrm{III}_{b,c})$ is always improper. Since $(\mathrm{II}_a, \mathrm{II}_b)$ contains I, adjoining I or $\mathrm{II}_b (b \neq a)$ to M_a makes it improper, and from the identity

$$\mathrm{III}_{b,c} - \mathrm{III}_{a,-a} + (b + c)\mathrm{II}_a = (a - b)(a - c)\mathrm{I}$$

the only elements $\mathrm{III}_{b,c}$ we can adjoin to M_a that do not make it improper are those with either $a = b$, or $a = c$; we may assume $a = b$. But from

$$\mathrm{III}_{a,c} - \mathrm{III}_{a,-a} = -(a + c)\mathrm{II}_a,$$

$\mathrm{III}_{a,c}$ already belongs to M_a. Thus M_a is maximal.

Next we observe that the following relations hold:

$$(***) \qquad (\mathrm{I}) = M_\infty^2;\ (\mathrm{II}_a) = M_\infty M_a;\ (\mathrm{III}_{b,c}) = M_b M_c.$$

Because unique factorization of ideals holds in D, no element in $(*)$ is contained in any maximal ideal other than the M's, so these must be the only maximal ideals. That these ideals are all distinct is an immediate consequence of some of the relations established above. So the assignment $M_s \to s \in E \cup \infty$ is a bijection from the set of maximal ideals.

We deduce from (***) that the class group consists of at most two elements, and to show it has exactly two elements, it suffices to prove M_∞ is not principal. We do this by showing, if r belongs to D, and $rs = I$, $rt = \mathrm{II}_0$ are solvable in D, then r is invertible.

Suppose r is not invertible; write $r = a/b$, $s = f_1/g_1$, $t = f_2/g_2$ with the usual conditions on the numerators and denominators. We have

$$af_1 \cdot (1 + x^2) = bg_1 \tag{1}$$

$$af_2 \cdot (1 + x^2) = bg_2x. \tag{2}$$

Since neither b nor g_1 has real roots, neither do a nor f_1. Thus deg a and deg f_1 are even, and since

$$\deg a + \deg f_1 + 2 = \deg b + \deg g_1$$

but deg a is strictly less than deg b (as r is not invertible) and deg $f_1 \leqq$ deg g_1, we deduce that deg a = deg $b - 2$. Thus we can write a/b as $u \cdot 1/(1 + x^2)$, where u is a unit in D. Putting this in (2), we obtain $uf_2/g_2 = x$, a contradiction since x is not bounded.

Now one can check that D is $E[\mathrm{I}, \mathrm{II}_0]$ a quadratic integral extension of the polynomial ring $E[\mathrm{I}]$, so the residue fields of D must be E-algebraic, but also by 2.10(ii), must be formally real; so all the residue fields are just the natural images of E.

Now let us study the total orderings and extremal states. As E is a subfield of $\mathbf{R}$, the Archimedean total orderings (if any) of $E(x)$ correspond to the points of $\mathbf{R} - E$ (the E-transcendentals in $\mathbf{R}$). To each non-Archimedean total ordering, we associate the maximal ideal M as in 2.12(b); this is the kernel of a unique extremal state, as D decomposes as an E-vector space, $D = E \oplus M$. So the extremal states of $(D, 1)$ correspond to the points of $\mathbf{R} \cup \infty$, the faithful ones identified with the points of $\mathbf{R} - E$. If h is any bounded rational function in one real variable, one sees immediately that both $\lim_{t\to\infty} h(t)$ and $\lim_{t\to-\infty} h(t)$ exist and are equal. Hence any such function h extends to the one point compactification of the reals (if z is the point at infinity, define $h(z)$ to be the limit as t approaches infinity). It is routine to check that the point-open topology on $\mathbf{R} \cup \infty$ (viewed as extremal states of D) is just the usual one-point compactification, so the extremal state space is homeomorphic to the circle; in particular, it is connected, in contrast to the result of 2.8.

To obtain disconnected extremal state spaces from a formally real field that is a real algebra over the reals, form the field of fractions of the quadratic extension $\mathbf{R}[x, y]/(y^2 - f)$, where f is a square-free polynomial in x; if we call this field E, the extremal state space of E_b consists of disjoint circles, corresponding to the zeroes on the real line (the regions where f is positive correspond to the primes in $E_b \cap \mathbf{R}(x)$, and these split; the primes corresponding to the zeroes of f remain inert).

The extremal state space, or what amounts to the maximal ideal space, of the bounded subring of $E = \mathbf{R}(x, y)$, the rational function field of two variables, is however extremely rich. Not all bounded rational functions are continuous even on the (real) plane, but their discontinuities are always at finite sets (for example $xy/(x^2 + y^2)$ is bounded, and is continuous everywhere except at the origin and ∞). The points on the plane, therefore, do not correspond to multiplicative states. However, we may associate to each point p in $S^2 = \mathbf{R}^2 \cup \infty$, a huge collection of states, as follows.

Let $(x(t), y(t))$ be a path through the point p, such that the coordinate functions are real analytic, $(x(0), y(0)) = p$, and 0 is the only point in some neighbourhood of 0 where the path runs through p (i.e., the path locally hits p just once). Then if f is a bounded rational function of two variables, $f(x(t), y(t))$ is a bounded meromorphic function of one real variable and it is easy to check this is actually analytic in the neighbourhood of 0. Thus we may define the extremal state $t_{p,P}$ depending on p and the path P through p,

$$t_{p,P}(f) = \operatorname*{Lim}_{t \to 0} f(x(t), y(t)).$$

If F is a polynomial in two variables, whose zero set is a non-trivial curve containing p, then each branch of its zero set has a locally analytic parametrization. If G, F are two distinct irreducible polynomials that define (real) curves containing p, then the bounded rational function $f(x, y) = (F^2 - G^2)/(F^2 + G^2)$ distinguishes the states corresponding to branches of F from those of G. However, it is not clear that two distinct analytically parametrized paths can be distinguished by bounded rational functions. Discussions with M.D. Choi revealed that there were extremal states of $\mathbf{R}(x, y)_b$ not arising out of paths.

After this article had been accepted for publication, there appeared a paper by Heinz-Warner Schülting (*Uber die Erzeugendenanzahl Invertierbarer Ideale in Prüferringen*, Comm. in Algebra 7(13), 1331–1349) in which $\mathbf{R}(x, y)_b$ is shown to be a Prüfer domain with a 3-generated ideal that is not 2-generated—destroying a long-held conjecture.

Note that in the case that more than one transcendence degree occurs in the affine field E, the bounded subring E_b is not necessarily Noetherian, so that even though $\mathbf{R}(x, y)_b$ lies between a Noetherian finitely generated domain and its field of fractions, it is not Noetherian.

LEMMA 2.14A. *Let E be a formally real field, that is an algebra over the* *real numbers, and let X denote the extremal state space of E_b. If E_b is Noetherian, then for each r in $\mathbf{R}$, for each e in E_b, the set $V_{r,e} = \{x \in X \mid e(x) = r\}$ must be finite.*

PROOF. If E_b were Noetherian, it would be a Dedekind domain, hence every element of E_b would be contained in only finitely many maximal ideals. However, because E_b is a real algebra, ker x is a maximal ideal for every point x of X, and $V_{r,e}$ is just the set of maximal ideals containing the element $r - e$ of E_n, so must be finite.

It appears likely that if E is a formally real affine (finitely generated, as a field) field over the reals, and E_b is Noetherian, then E is finite-dimensional over a purely transcendental extension in one variable over the reals, i.e., there exists X in E such that $[E: \mathbf{R}(X)]$ is finite.

EXAMPLE 2.15. This example will show a formally real field E such that $J(E_b) \neq (0)$, but $J*(E) = (0)$ (c.f. 1.7); in fact E_b is local, and E has an Archimedean ordering.

Let K denote the real closure of the rationals, and let G be the rational function field in one variable over K, $G = K(x)$. Let $a: G \to \mathbf{R}$ be the homomorphism assigning x to a fixed transcendental, say e. Let $b: G \to K((x))$ be the inclusion, assigning x to x. View G as a subring of the direct product, $\mathbf{R} \times K((x))$, via the map $c = (a, b)$. Let E be a maximal subfield of $\mathbf{R} \times K((x))$, containing the image of c, with respect to being algebraic over $c(G)$—this requires a simple Zorn's Lemma argument. If an element t of E is a square in both $\mathbf{R}$ and $K((x))$ (E obviously embeds in both fields), then t must be a square in E: write $t = (r, s)$; if $z^2 = r$, $y^2 = s$, for z in $\mathbf{R}$ and y in $K((x))$, set $v = (z, y)$. If t is not a square in E, then $E[v]$ is a field and a quadratic extension contradicting the maximality of E. Since there is an obvious embedding of E in the reals, E has an Archimedean ordering, so in particular, $J*(E) = (0)$.

Next, $9 - a(x)^2$ and $9 - b(x)^2$ are squares in their respective fields, so $c(x)$ lies in E_b. We will show that every ideal of E_b is generated by $c(x^n)$ for some n. Pick v in $E_b - \{0\}$; then, if P_2 is the projection onto $K((x))$ restricted to E, $p_2(v)$ is bounded, so lies in $K[[x]]$, as we have seen in the computation just prior to 2.14. So we may write $p_2(v) = x^m u$, u a unit in $K[[x]]$. Set $w = vx^{-m}$ in E (we will drop the $c(-)$). If p_1 is the restricted projection to the reals, we see that there exists an integer M so that $M - p_1(w^2)$ and $M - p_1(w^{-2})$ are both squares in the reals. Now, if we write $p_2(w) = u_0 + \sum u_i x^i$, then $u_0 \neq 0$, and $p_2(w^{-1}) = u_0^{-1} + \sum u_i' x^i$. So if N is any integer greater than both u_0^2 and u_0^{-2}, then $N - p_2(w^2)$ and $N - p_2(w^{-2})$ are both squares. Hence, if $P = \text{Max}\,(M, N)$, $P - w^2$ and $P - w^{-2}$ are both squares in E, whence both w and w^{-1} lie in E_b. Hence $v = wx^m$ and $x^m = vw^{-1}$, with all the terms in E_b, whence $vE_b = x^m E_b$.

It easily follows that all ideals of E_b are of the form (x^m), whence E_b is a local principal ideal domain, with maximal ideal (x).

The process of forming the "intersection" of the two fields $\mathbf{R}$ and $K((x))$ as above, can be generalized, and used to localize (in a sense) at finite sets of total orderings. For instance, if $\{R_i\}_{i=1}^n$ is a collection of inequivalent real closures of a formally real field G, one can enlarge G to a field H, so that H is algebraic over G, and the inclusions of G in R_i can be extended to H (so the total orderings are extended), and an element h is a square if and only if its image in every R_i is a square (in other words, the n total orderings on H determine the natural ordering). I have not been able to establish that these are the only total orderings on H, but this seems plausible.

EXAMPLE 2.16. This example will show a formally real field E and a quadratic extension F such that the Galois involution is positive definite, but the paradoxical properties $E_b \neq E$ and $F_b = F$ hold.

We first require a lemma describing when the Galois automorphism is positive definite. The proof is quite elementary, so we omit it.

LEMMA 2.17. *Let E be a formally real field, and d a non-square in E.*

(a) *The Galois automorphism on $F = E[\sqrt{d}]$ is positive definite if and only if d is not a sum of squares in E.*

(b) *The field $F = E[\sqrt{d}]$ is formally real if and only if $-d$ is not a sum of squares in E.*

Let E be the formally real example of 2.15. Consider the element $z = 1/x^2 - 1$ of E. Then z is negative in the real embedding, and is a square in $K((x))$ $(x^2z = 1 - x^2)$; in particular z is not a square in E, and we may from the quadratic extension $F = E[y]$, where $y^2 = z$. By 2.17, the Galois automorphism is positive definite. Since $y* = -y$, we have

$$yy* + (1/x)*(1/x) = 1.$$

Thus $1/x$ lies in F_b. Since trivially, E_b is contained in F_b, we have $E = E_b[1/x] \subsetneqq F_b$, whence $F = E(y] \subset F_b$. so $F = F_b$.

3. **Division Rings.** Let $(D, *)$ be a division ring with (positive definite) involution. We first show that only rarely can D be generated by its unitaries if it is an algebra over the reals, and we construct a division algebra that is not generated over its centre by its unitaries. This answers a question of Maurice Chacron, but Professor Cohn has constructed examples of characteristic 2 division algebras with $*$ (necessarily not positive definite) in which 1 is the only unitary [**4**]. We next consider one of the order properties of D^s that holds if D is commutative, Riesz decomposition; if D^s satisfies this in the noncommutative case, D is usually at

worst quaternionic over its centre. Finally, we enlarge a recent theorem of Holland [12, Theorem 2], on monotone sigma-complete division rings. Explicitly, we show that if D^s is monotone sigma-complete (in its natural ordering), then D is one of the reals, complexes, or real quaternions.

The proof of the first result (Corollary 3.5) is so elementary in view of the results of §1 that we go into a more general class of rings.

A ring R with involutory anti-automorphism $*$, is *$*$-regular* if it is von Neumann regular, and satisfies the condition that $xx* = 0$ implies $x = 0$ (the latter is a much weakened form of positive definiteness); equivalently, for all r in R, there exists a projection p such that $rR = pR$.

PROPOSITION 3.1. [**18**; Lemma 5, Theorem 1]. *Let $(R, *)$ be a ring with (positive definite) involution.*

(a) *If p is nonzero projection of R_b, then $\|p\| = 1$.*

(b) *If R_b is $*$-regular, then $J*(R) = (0)$.*

Recall that an element d is *normal* if $dd* = d*d$.

PROPOSITION 3.2. *Let $(R, *)$ be a ring with involution, and suppose R_b is $*$-regular.*

(a) *If R_b is a complex $*$-algebra, then for all d in R_b, the subset of the complexes,* spec d, *defined by*

$$\text{spec } d = \{b \in \mathbf{C} |\ d - b \textit{ is not invertible in } R_b\}$$

is not empty. If additionally, d is normal, then spec d *is finite, and there exists a finite orthogonal set of projections $\{p_i\}$ in R_b such that $\sum_i p_i = 1$, and $d = \sum_i b_i p_i$ for some complex numbers, b_i.*

(b) *If R_b is a real algebra, the same results apply for symmetric elements d.*

PROOF. By 3.1, $\|\ \ \|$ is a norm, so R_b can be completed to a Banach algebra E (1.9). Any element of a regular ring is either invertible or a zero divisor, so spec d equals the usual spectrum, as computed as an element of E, denoted $\text{spec}_E\, d$.

(a). Here E is a complex Banach algebra, so $\text{spec}_E\, d$ is nonempty, establishing the first statement. For the normal element d, define the $*$-subalgebra T of R_b to be the bicommutant (in R_b) of $\{d, d*\}$ (T is the collection of elements of R_b commuting with all those elements in R_b that commute with d and $d*$). By [**11**; Lemma 8.2], T is $*$-regular, and T is obviously commutative, and consists of normal elements. It is immediate that spec t $= \text{spec}_T\, t$ for all t in T. We may write $dT = (1 - p)T$ for some projection p of T. If a is a nonzero element of spec d, then $T(d - a) = T(1 - q)$ for some nonzero projection $q \equiv q(a)$ of T (*nonzero* since $d - a$ cannot be invertible in T, and T is $*$-regular). Since a is not zero, $pT \cap qT = (0)$, so $pq = qp = 0$. Let d' denote the relative inverse of d (that is, the element d' such that $d'dd' = d'$, $dd'd = d$, $dd' = 1 - p$; [**13**; Lemma 4]). Then

$(d + p)(d' + p) = 1$, whence 0 does not lie in spec $(d + p)$. But for nonzero a in spec d, we see that a also lies in spec $(d + p)$, for $(d + p)q = dq = aq$, so $d + p - a$ is a zero-divisor. Thus $0 \in \text{Spec}\, d - \text{Spec}(d + p)$ and spec $d - \{0\} \subset \text{spec}(d + p)$, so as spectra are always compact, 0 is never a limit point in spec d. Applying this to $d - b$ for any b in spec d, we see that spec d contains no limit points, so must be finite (being compact).

Let spec $d = \{a_1, a_2, \ldots, a_n\}$, and $q_i \equiv q(a_i)$ be projections in T corresponding to the a_i, in the sense that $T(d - a_i) = T(1 - q_i)$, so $dq_i = a_i q_i$. As above, we see that $\{q_i\}$ is an orthogonal set, and $d(\sum q_i) = \sum a_i q_i$. We thus need only show $\sum q_i = 1$.

Set $q = \sum_i q_i$. If $d(1 - q) = 0$, then $1 - q$ will be contained in whichever (if any) of the q_i that correspond to the element 0 of spec d, a contradiction unless $q = 1$. Suppose that $t = d(1 - q)$ is not zero. Then t belongs to $(1 - q)R_b(1 - q)$, but it is routine to check that this equals $((1 - q)R(1 - q))_b$; it follows that $\text{spec}_{(1-q)T}t$ is nonempty. If $rT \neq (1 - q)T$, say $tT = eT$, e a projection, then $1 - q - e$ is a nonzero projection, and $d(1 - q - e) = 0$, so $1 - q - e$ would be contained in one of the q_i, again a contradiction, since $1 - q - e$ is orthogonal to q. Thus $tT = (1 - q)T$, whence 0 is not in $\text{spec}_{(1-q)T}t$. Replacing t by $t - b(1 - q)$ for any complex number b, we see that the same process shows b is not in $\text{spec}_{(1-q)T}t$, whence the latter is empty, a contradiction. Thus $d(1 - q) = 0$, so $q = 1$.

(b). In a real $C*$ algebra (as E is, by 1.9), symmetric elements have real spectra, so the methods of (a) can be applied.

Thus result has some connections with well-known problems about algebraic algebras. If R is an algebraic $*$-algebra such that $xx* = 0$ implies $x = 0$ (in 3.2, $*$ was positive definite, R at least a real algebra), then R is $*$-regular: every symmetric element generates a finite dimensional algebra, the condition on the anti-automorphism guarantees that there are no nilpotent ideals, so the finite dimensional algebra is regular; if $xx*yxx* = xx*$, then the condition on $*$ yields $x(x*y)x = x$; whence R is regular. When $*$ is positive definite, it is a matter of routine to check that $R = R_b$ as well. We saw above that if R were $*$-regular, a complex $*$-algebra, and $*$ is positive definite, then all normal elements (including all unitaries, symmetrics, skew-symmetrics) are algebraic. Does this imply all elements of the algebra are algebraic? There is also the question (well-known in another formulation) of whether algebraic regular algebras are locally finite dimensional. A less commonly posed (but also interesting, because the structure of locally finite dimensional semisimple algebras over an algebraically closed field is reasonably well understood) question is whether regular locally finite dimensional algebras are locally semisimple. It is easy to see that a locally finite dimensional $*$-algebra with $xx* = 0$ implying $x = 0$ is locally semisimple—the condition on $*$ guarantees that any $*$-subalgebra has no nilpotent ideals.

THEOREM 3.4. *Let $(D, *)$ be a real algebra with (positive definite) involution, and suppose $D = D_b$. If D is Artinian, then D is finite dimensional over the reals.*

PROOF. As $xx* = 0$ implies $x = 0$, the radical of D is zero, so D is semisimple. Also, we note that any central idempotent is a projection; we may thus assume D is simple. Let eD be a minimal right ideal; because D is regular, D is $*$-regular, so we may assume e is a projection. Set $H = eDe$, with the induced involution; H is a division $*$-algebra over the reals. Pick $h = h*$ in H; by 3.2(b), we may write h as a real linear combination of orthogonal projections in H, whence h can only be a scalar. Hence the symmetric part of H is one dimensional, and it easily follows (e.g., [3]) that H is either $\mathbf{R}$, $\mathbf{C}$, or the real quaternions. As $D = M_n H$ (for some n), D is finite dimensional over the reals.

COROLLARY 3.5. *Let $(D, *)$ be a real division $*$-algebra, with positive definite involution. If D is generated by its unitaries over the reals, then D is one of the reals, complexes or real quaternions.*

PROOF. All unitaries are bounded, so we have $D = D_b$, and 3.4 applies.

Maurice Chacron has asked if it is possible to construct a division ring with involution that is not generated over the centre by its unitaries. If the involution is positive definite, we need only take an example which is not finite dimensional over the reals, but has the reals or complexes as centre. For example, let F be either the reals or complexes equipped with conjugation, and form the Weyl algebra $F[X, Y]/(XY - YX - 1)$. Let D be the division algebra of quotients of the Weyl algebra. Then it is well known that the centre of D is F. Define an involutory anti-automorphism on the Weyl algebra by defining $X\# = Y$, $Y\# = X$, and $r\# = \bar{r}$ if r belongs to F—that this is well-defined is straightforward, and routine computations with the associated graded algebra show that $\#$ is positive definite. It follows that the extension of $\#$ to D is also positive definite. Thus $(D, \#)$ is an infinite dimensional division algebra with centre either $\mathbf{R}$ or $\mathbf{C}$, so by 3.5, cannot be generated by its unitaries.

In another vein, Professor Cohn has constructed characteristic 2 division $*$-algebras (necessarily $*$ is not positive definite) in which 1 is the only unitary.

In the commutative case, we have seen that with practically any ordering invariant under inverses, the Riesz decomposition property holds. We will now show that this is an essentially commutative property; for example, if $(D, *)$ is a division ring with positive definite involution, such that D^s satisfies the decomposition property, and additionally D is algebraic over the rationals (this has the effect that $D = D_b$), then D is either commutative or a quaternionic algebra over its centre, i.e., 4-dimensional over its centre (2.7).

We will require some results about real $C*$ algebras that are well-known for complex ones, and can be proved by exactly the same techniques.

(1) Any closed two-sided ideal is closed under the involution. (One constructs a positive approximate unit, as in the complex situation.)

(2) Any quotient by a closed two-sided ideal is itself a (real) $C*$ algebra, and the intersection of the primitive ideals is zero.

(3) The spectrum of a symmetric element is a compact subset of $\mathbf{R}$.

THEOREM 3.6. *Let $(S, *)$ be a ring with positive definite involution, such that all terms of the form $1 + \sum x_i x_i*$ are invertible. Then, if S^s satisfies the Riesz decomposition property, $S_b/J*(S)$ is $*$-embeddable in a product of copies of the real quaternions; in particular, if $J*(S) = (0)$, then S has no nilpotents and satisfies a polynomial identity of degree* 2.

PROOF. Let S_B be as in 1.9 and 1.10. Then S_B is the completion of $S_b/J*(S)$, $(S_B)_b = S_B$, and by 1.16, the state space of $(S_B^s, 1)$ is affinely homeomorphic to that of $(S_b^s, 1)$. By [**8**; I.2.5] this state space is a Choquet simplex, and by 1.9, S_B is a (real) $C*$ algebra.

Now, it is well-known in the theory of (complex) $C*$ algebras that a $C*$ algebra whose symmetric part has Riesz decomposition must be commutative; the following proof that S_B^s is commutative also yields the real version of this folkloric result.

Let c be an element of S_B^s and let $C*(c)$ denote the real $C*$ algebra inside S_B^s generated by 1 and c. Then it is clear that $C*(c)$ is commutative and isometrically isomorphic to $C(\text{spec } c, \mathbf{R})$ (for the latter result, use the 2×2 matrix trick involved in the proof of 1.9 to show $C*(c) \otimes_{\mathbf{R}} \mathbf{C}$ is a complex $C*$ algebra generated by c, so must be $C(\text{spec } c, \mathbf{C})$; then observe that $C*(c)$ can have only real images, since all elements of the form $1 + a^2$ are invertible in S_B and thus in $C*(c)$). We will show the induced subgroup ordering on $C*(c)$ agrees with its own natural ordering.

If, for d in $C*(c)$, x_i in S_B, $d = \sum x_i x_i*$, we must show $d = a^2$ for some a in $C*(c)$. Since $C*(c) \cong C(X, \mathbf{R})$, there exist D, E in $C*(c)$ such that $DE = 0$, $D + E = d$, $D = a^2$ and $E = -b^2$ with a, b in $C*(c)$. Then $ab = ba = 0$. and

$$b(\textstyle\sum x_i x_i*)b* = -(bb*)^2 = -b^4$$

(of course, $b = b*$); thus $\sum(bx_i)(bx_i)* + (bb*)* = 0$; by positive definiteness (1.9), $bb* = 0$, and so $b = 0$, whence $d = a^2$. Thus $C*(c)^+ = (S_B^s)^+ \cap C*(c)$.

By [**7**; Theorem 4.2], every state on $(C*(c), 1)$ can be extended to a state of $(S_B^s, 1)$. The states of $(C*(c), 1)$ determine its ordering (in the sense that x is positive if and only if its image at all states is positive), and it follows immediately that the states of $(S_B^s, 1)$ determine its ordering, in other words that S_B^s is an Archimedean group. Being norm complete, by [1; II.I.8], S_B^s is order-isomorphic to Aff(K), where $K = S(S_B^s, 1)$ is the state space.

We determined earlier that K is a Choquet simplex. Let M be a maximal right ideal of S_B. Then M is closed (since $\|1 - x\| < 1$ implies c is invertible), and so $M_{sa} = M \cap S_B^s$ is closed. We will show M_{sa} is a directed and convex subgroup of S_B^s.

Select $c = c*$ in M_{sa}. Then form $C*(c)$ as previously, and identify $C*(c)$ with $C(X, \mathbf{R})$ via the Gelfand mapping $\hat{}$. Then $J = M_{sa} \cap C*(c)$ is a proper closed ideal of $C*(c)$, so there exists a compact subset V of X such that $\hat{J} = \{f \in C(X, \mathbf{R}) \mid f/V = 0\}$. Since c lies in M_{sa}, $\hat{c}/V = 0$. Then $d = |\hat{c}|$ is positive and $d/V = 0$. As d is positive, we may find e in $C*(c)$ so that $e = xx*$ (some x in $C*(c)$) and $\hat{e} = d$; in particular, e belongs to M_{sa}, and as both $d - \hat{c}$ and d are positive in $C(X)$, so are $e - c$ and e in $C*(c)$, and both lie in M_{sa}. As $c = e - (e - c)$, M_{sa} is directed.

Now suppose for some a in S_B^s and b in M_{sa} that $0 \leqq a \leqq b$. Since a is positive, we may find via the Gelfand mapping on $C*(a)$ an element $x = x$ in $(S_B^s)^+$ with $a = x^2$. It suffices now to show x belongs to M_{sa} (for then a would also belong and hence M_{sa} would be convex).

If $x \notin M_{sa}$, there would exist r in S_B and m in M so that $xr = 1 - m$. If $\|r\| < K$, with $K > 1$, then

$$Kx^2 = kxx* \geqq (xr)xr)*.$$

From the identity,

$$(1) \qquad (t - u)(t - u)* + (t + u)(t + u)* + 2(tt* + uu*),$$

we deduce $(t = 1 - m, u = m)$,

$$1/2K \leqq (xr)(xr)*/K + mm*/K \leqq xx* + mm* \leqq b + mm*.$$

Thus $b + mm*$ is invertible, a contradiction since b, m belong to M. Hence M_{sa} is convex.

On the other hand, if I is a proper convex directed subgroup of S_B^s, then the subset of S_B, $J = \{r \in S_B| \ rr* \in I\}$ is a right ideal. For a, b in J, $aa* + bb*$ belongs to I, and from the identity (1), we have $(a + b)(a + b)* \leqq 2(aa* + bb*)$, so convexity of I assures J is closed under addition. If a lies in J and r in S_B, then $rr* \leqq n$ for some integer n, whence $(ar)(ar*) = a(rr*)a* \leqq naa*$; thus ar belongs to J.

Hence if I is a convex directed subgroup of S_B^s containing M_{sa}, then the right ideal $\{r \in S_B| \ rr* \in I\}$ would strictly contain M. In particular, M_{sa} is a maximal directed convex subgroup of $S_B^s \cong$ Aff(K).

Now K is a simplex so by [1; II.5.19], if f is any extremal state of S_B^s, ker f must be a directed convex subgroup. Now the quotient group $(S_B^s/M_{sa}, \bar{1})$ has an order unit, so possesses a state, and by the Krein Milman Theorem, has an extremal state $\bar{f}$; this induces an extremal state f on B_B^s such that $M_{sa} \subseteqq \ker f$; by maximality, $M_{sa} = \ker f$.

Since M is a right ideal, if a belongs to M_{sa}, so does a^2. From $f([b - f(b)]^2) = 0$ for all b in S_B^s, we deduce that $f(b^2) = (f(b))^2$. Applying this to $b = c + d$ (c, d in S_B^s), we obtain that $f(cd + dc) = 2f(c)f(d)$; in other words, f is a Jordan algebra homomorphism from S_B^s to the reals. Since the intersection of the maximal right ideals of S_B is zero, we obtain an embedding of Jordan algebras

$$(S_B^s, 1) \to (C(\beta X, \mathbf{R}), 1),$$

where X is the set of extremal states of S_B^s endowed with the discrete topology, and the Jordan operation on the right is just the usual multiplication. Hence the Jordan product $a \circ b = ab + ba$ on S_B^s is associative. It follows immediately that for all a, b, c in S_B^s, $(ab - ba)c = c(ab - ba)$.

Let B be the Banach algebra generated by S_B^s, within S_B. It is easy to check that B is a Banach $*$-algebra, and in fact is a (real) $C*$ algebra, since all terms of the form $(1 + \sum a_i a_i*)^{-1}$ belong to B. Then for any a, b in S_B^s, $ab - ba$ belongs to the centre of B, denoted $Z(B)$. If $ab - ba$ is nonzero for some such a, b, since $Z(B)$ is itself a (real) $C*$ algebra, there exists a maximal ideal N of $Z(B)$ with $ab - ba$ not in N.

Now NB is a proper two-sided ideal of B (if $1 = \sum n_i b_i$, then

$$n_i n_i* + \sum_{i<j} (n_i n_j* + n_j n_i*)$$

is greater than a positive rational, and

$$\sum_{i<j} (n_i n_j* + n_j n_i*) \leqq \sum n_i n_i* \cdot \lambda$$

for a sufficiently large integer λ; hence $\sum n_i n_i*$ would be invertible, hence would be invertible in $Z(B)$, a contradiction). Let T be a primitive ideal containing NB. Then T is a closed proper two-sided $*$-ideal of B, with $T \cap Z(B) = N$ (by the maximality of the latter). In the Banach algebra B/T, $\overline{ab} - \overline{ba}$ belongs to the image of the centre, but this is a field, so is either the reals or the complexes. Hence $\overline{ab} - \overline{ba}$ is a nonzero scalar. As is well-known (compare the spectra of $\overline{ab}$ and $\overline{ba}$), this is impossible. Hence all the elements of S_B^s commute with each other.

Let P be a primitive $*$-ideal of S_B; by the comments just preceding this theorem, the intersection of all such P is zero. Since every symmetric in $C = B_B/P$ is the image of a symmetric, all of the symmetric elements in C commute as well. If $r^2 = 0$ and r lies in C, then from $(t + r*)rr* = rr*(r + r*)$, we deduce $(rr*)^2 = 0$, and so $r = 0$. Hence C is a prime ring with no nilpotents; a one-line computation shows C can have no divisors of zero. Thus if c is a symmetric element of C, spec c can consist only of one point, that is $C^s = \mathbf{R}$. It easily follows that C is a division ring, so being a Banach algebra must be one of $\mathbf{R}$, $\mathbf{C}$, or the real quaternions.

Thus B_B and hence $S_b/J*(S)$ must be embeddable in a product of copies of the quaternions. If $J*(S)$ is zero, then S_b is embeddable in such a product, so satisfies the standard polynomial identity of degree 2. As S_b is an order in S, so does S.

When $J*(S)$ is not zero, a little more information can be deduced. Pick an element u in S_b^s and let T be the convex subgroup of S_B^s generated by u. Then the unitless ring U generated by T can be dealt with as S_b has been (if for example, S is a division ring), and we deduce $U/J*(U)$ satisfies a polynomial identity. I was not able to prove that this forced the division ring to be quaternionic over its centre.

COROLLARY 3.7. *Let $(D, *)$ be a division ring with positive definite involution, such that D^s satisfies Riesz decomposition. Suppose either of the following two conditions hold*:

(a) $J*(D) = (0)$, *or*

(b) *D is algebraic over the rationals*;

then D is a quaternionic division algebra over its centre, or D is commutative.

PROOF. Any commutative algebraic extension of the rationals is totally Archimedean, and it easily follows that (b) implies $D = D_b$, so that in particular, (b) implies (a). When (a) holds, by 3.6, D_b satisfies a polynomial of degree 2; by 1.2, so does D, and it is well-known that for division rings, this implies D is 4-dimensional over its centre (hence D is quaternionic), or else D is commutative.

Before establishing the results concerning monotone σ-complete division rings, we require a clarification of the notion of Archimedean orderings. If G is a partially ordered group, it is said to be *Archimedean* if $nz \leqq y$ for all positive integers n implies $z \leqq 0$. This is distinct from the weaker notion of Archimedean that is sometimes, confusingly, used, namely $z \geqq 0$, and $nz \leqq y$ for all positive integers n implies $z = 0$, although when the ordering is total, or more generally is a lattice, the two notions agree. In **[15]**, the latter property is referred to as weakly Archimedean. In case G possesses an order unit, then being Archimedean is equivalent to the states determining the ordering, that is

$$(+) \qquad f(a) \geqq f(b) \text{ for all states } f \text{ implies } \quad a \geqq b.$$

In the course of the proof of 3.6, we showed that $(+)$ holds for the symmetric part of a (real) $C*$ algebra. However, $(+)$ need not hold for its dense subalgebra $S_b/J*(S)$, as Example 2.9 demonstrates.

A directed partially ordered abelian group G is *monotone σ-complete* if every countable increasing chain $x_1 \leqq x_2 \leqq x_3 \leqq \ldots$ that is bounded above, has a least upper bound (this will be denoted $\sup x_i$). In [**12**; 3 Corollary], Holland shows that if $(D, *)$ is a division ring with involution such that D^s possesses a total ordering whose positive cone contains all elements of the form dd^*, and is invariant under the operation $v \to xvx*$, and D^s is monotone sigma complete, then D must be one of the three finite dimensional real algebras. We will show that if $(D, *)$ is merely Artinian and D^s is monotone σ-complete in the natural ordering (or in many other possible orderings), then D is finite-dimensional over the reals. The methods of proof are again suggested by $C*$ techniques. The presence of the hypothesis that D^s admits a total ordering in Holland's result seems to be not very natural, especially if the result is to be applied to the problem of Baer $*$ rings of type I_∞ (as was the motivation in [**12**]).

When (G, u) is a monotone σ-complete group with order unit, it is straightforward to check that G has the weaker form of Archimedeanness referred to as weakly Archimedean, but does not generally possess the full Archimedean property. Fortunately, when G is a rational vector space and G^+ is invariant under multiplication by $1/n$ (n in $\mathbf{N}$), the group is Archimedean; this is due to D. Bruncker, as is the proof below.

PROPOSITION 3.8. (D. Bruncker). *A directed partially ordered rational vector space that is monotone σ-complete, is Archimedean.*

PROOF. Suppose $na = b$ for all positive integers n. As G is directed, we may assume b is positive, and we have $a - b/n$ for all positive n. Then $c - \operatorname{Inf}(b/n)$ exists (define it as $b \leqq \sup (b - (b/n))$, and is of course positive or zero. Since $c \leqq b/2n$ for all n, $2c \leqq b/n$ for all n, so $2c \leqq c$, whence $c \leqq 0$. Since $c \geqq 0$, we have $c = 0$. But $a \leqq b/n$, so $a \leqq c = 0$, whence G is Archimedean.

The following is an elementary adaptation of the proof on page 119 of [**15**]; the difference is due to the fact that $\sum_{k=n}^{\infty} 1/k^2$ is never rational, whereas $\sum_{k=n}^{\infty} 1/2^k$ always is.

PROPOSITION 3.9. *Let (G, u) be a monotone σ-complete rational vector space with order unit. Then with respect to the norm $\|a\|_1 = \sup\{|f(a)| \mid f$ a state of $(G, u)\}$, G is complete.*

PROOF. Because G is Archimedean, $\|a\|_1 = 0$ implies $a = 0$, and it easily follows from the pointwise ordering induced by the states that $\| \quad \|_1$ is a rational vector space norm. Let us identify the elements of (G, u) with functions on the state space; then u is identified with the constant function 1. Let $\{x_n\}$ be a Cauchy sequence of elements of G; we may suppose $\|x_n\| \leqq 1$ for all n, $x_0 = 0$, and

$$\|x_n - x_{n+1}\|_1 \leqq 2^{-n}$$

for all n. This translates to

$$(1) \qquad -1^{-n} \leqq x_{n+1} - x_n \leqq 2^{-n}$$

(2^{-n} represents the constant function with value 2^{-n}). Hence

$$0 \leqq 2^{-n} + (x_{n+1} - x_n) \leqq 2 \cdot 2^{-n},$$

so $\sum_{n=0}^{l}(2^{-n} + x_{n+1} - x_n) \leqq 4$, for all inetgers l. There thus exists an element y in G such that

$$y = \sup_l\{\sum_{n=0}^{l}(2^{-n} + x_{n+1} - x_n)\};$$

that is, set y_l to be the l-th sum, observe that $y_1 \leqq y_2 \leqq y_3 \leqq \cdots \leqq 4$, and $y = \sup y_l$. Now

$$0 \leqq y - y_l \leqq \sup_{p \geqq l}\{\sum_{k=l+1}^{p}(2^{-k} + (x_{k+1} - x_k))\} \leqq \sum_{j \geqq l} 2 \cdot 2^{-j} \leqq 2^{-(l-2)}.$$

Thus at any state, f, $|(y - y_l)f| \leqq 2^{-(l-2)}$, so $\{y_l\}$ converges to y.

Set $x = y - 2$. Then

$$\begin{aligned} \|x - x_l\| &= \|y - 2 - x_l\| \leqq \|y_{l-1} - 2 - x_l\| + 2^{-(l-3)} \\ &= \|1 + 2^{-1} + 2^{-2} + \cdots + 2^{-(l-1)} - 2 + x_l - x_l\| + 2^{-(l-3)} \\ &= 2^{-(l-1)} + 2^{-(l-3)}. \end{aligned}$$

Hence $\{x_l\}$ converges uniformly to the element x of G, and so G is complete.

THEOREM 3.10. *Let $(D, *)$ be an Artinian ring with positive definite involution, and suppose that D^s is monotone σ-complete in the natural ordering. Then D is a finite dimensional algebea over the reals.*

PROOF. As S_b^s is a convex subgroup of D_s, D_b^s is also monotone σ-complete. By 1.14, $\| \quad \|_1$ on D_b agrees with the natural norm, so D_b^s is norm complete in that norm (though as yet we have no information on the completeness of D_b itself). For c in D_b^s, form $C*(c)$ as in the proof of 3.6, and observe that completeness within D_b^s is sufficient to establish that $C*(c)$ is isometrically isomorphic with $C(\text{spec } c, \mathbf{R})$. If spec c were infinite, we could find an infinite nonzero set of elements $\{r_i\}$ indise $C*(c)$ such that $r_j r_i = r_i r_j = 0$ for $i \neq j$. This is impossible inside an Artinian ring, since the chain condition on annihilators is inherited. Hence spec c is finite.

We may thus find a finite orthogonal set of projections adding to 1, $\{p_i\}$, and a corresponding set of real numbers, $\{r_i\}$, such that $c = \sum r_i p_i$. It easily follows that for all $c = c*$ in D_b, there exists b in D_b with $cbc = c$.

From $xx* = 0$ implying $x = 0$, we deduce that D_b is regular and hence is $*$-regular. As D_b is an order in D, and nonzero divisors are invertible in regular rings, $D = D_b$. Now 3.4 applies.

Holland [**12**] established his monotone σ-complete result by showing that the presence of a single Archimedean total ordering on D^s implies D is embeddable in the real quaternions (assuming D is a division ring). The obvious question then occurs. If D^s is Archimedean in the natural ordering, is D algebraic over its centre? This appears unlikely; the obvious candidate for a counter-example is the Weyl algebra of this section.

Observe that in the proof of 3.10, the use of the natural ordering was really not necessary; only some of its weaker properties are required.

4. **Baer $*$ rings.** In this section we consider, for two classes of rings, when the ring of n by n matrices is Baer $*$ (with respect to $\# = *$-transpose). The first class includes semiprime PI Goldie rings (Theorem 4.5), and the second consists of the integral closure in a quadratic extension field of a Prüfer domain (Theorem 4.6)—of course the latter class is contained in the first, but the criteria are stated in terms of the quadratic extension. We then investigate the unitary equivalence of equidimensional subspaces of suitable inner product spaces over a field; this is equivalent to the matrix ring satisfying the $LP \sim RP$ property of Baer $*$ rings.

We drop our convention that involutions be positive definite, throughout this section.

A ring R with involution $*$ is said to be *matricially Baer* $*$ (with respect to $*$) if for all n, M_nR is Baer $*$, with respect to $\# = *$-transpose.

It has been asked, albeit by the author, [**9**; p. 247], what conditions are required of an Ore domain with involution that it be matricially Baer $*$. It was suggested in [**9**] that the techniques there should apply to domains satisfying a polynomial identity. Unfortunately, localization was very heavily relied upon, and this does not seem to work for PI rings. Instead, using very different ideas, we prove the PI result, and in so doing provide quick proofs of some other results of [**9**]. We are conforming, however, to the spirit of the bounded subring.

LEMMA 4.1. *Let D be any ring with involution $*$. Suppose that for all x in D, $1 + xx*$ is invertible in D. Then for all maximal two-sided ideals P of D, $P \neq P*$.*

PROOF. If $P \neq P*$, the map $D/(P \cap P*) \to D/P \times D/P*$ is an isomorphism. There thus exists a in D so that $a - 1$ belongs to P, but $a + 1$ lies in $P*$. Then $1 + aa* = a(a* + 1) - (a - 1)$ belongs to P, a contradiction.

LEMMA 4.2. *Let A be any ring with positive definite involution, and let*

Q be its maximal right quotient ring. Suppose the involution extends to Q. Then the involution is positive definite in Q, Q is matricially Baer $*$, and A is matricially Baer $*$ if and only if Q_b is contained in A.

PROOF. Implicit in [**11**; §7, 8], [**10**, 3.4(c)], and Lemma 1.4.

PROPOSITION 4.3. *Let R be a semiprime Goldie ring whose primitive images are von Neumann regular, and suppose R possesses an involution $*$ so that*

(a) *$P = P*$ for all primitive ideals,*

(b) *the involution induced on R/P is positive definite, for all primitive ideals P, and*

(c) *R is semihereditary.*

Then M_nR is Baer $*$ with respect to $*$-transpose for all n.

PROOF. Since M_nR is an order in a semisimple ring, the right annihilator of any subset is the annihilator of a one-element set in M_nR; as R is semihereditary, the annihilator is of the form EM_nR for some $E = E^2$ in M_nR. By [**14**; Theorem 26], it suffices to show $1 + XX\#$ is invertible in M_nR for all X in M_nR.

Let M_nP be a primitive ideal of M_nR (the primitive ideals of M_nR are precisely those of the form M_nP, P a primitive ideal of R). Since the induced involution is positive definite, the induced involution on $M_n(R/P) = (M_nR)/(M_nP)$ is also positive definite, so $1 + XX\#$ is not a divisor of zero modulo M_nP. Since $M_n(R/P)$ is regular, $1 + XX\#$ is invertible modulo M_nP. Any element of any ring that is invertible modulo every primitive ideal is invertible (a standard Zorn's Lemma argument), so $1 + XX\#$ is invertible.

PROPOSITION 4.4. *Let R be a semiprime Goldie ring with involution $*$, such that R/P is regular for all maximal ideals, and suppose M_nR is Baer $*$ with respect to $*$-transpose, for all n. Then.*

(a) *$P = P*$ for all maximal (two-sided) ideals P of R,*

(b) *for all maximal ideals P, the induced involution of R/P is positive definite, and*

(c) *R is semihereditary.*

PROOF. Let Q denote the (semisimple Artinian) complete ring of fractions. Then $*$ extends to Q, and by 4.2, the M_nQ are also Baer $*$; again by 4.2, Q_b is contained in R. Thus for x in R, $1 + xx*$ is invertible in R, and (a) follows from 4.1.

Next, the regular ring R/P is $*$-regular, by [**14**; Theorem 26], so $xx*$ belongs to P implies x belongs to P. But all the hypotheses apply to $M_n(R/P)$, so $XX\#$ belongs to M_nP implies X belongs to M_nP. It follows immediately that $\sum x_ix_i* \in P$ implies all the x_i belong to P, so (b) holds.

Finally, all the matrix rings over R are $p.p.$ rings, so R is semihereditary.

THEOREM 4.5. *Let R be a semiprime PI Goldie ring with involution $*$. Then M_nR is Baer $*$ with respect to $*$-transpose for all n, if and only if*

(a) $P = P*$ *for all primitive ideals of* R,

(b) *the induced involution on* R/P *is positive definite for all primitive ideals* P, *and*

(c) R *is semihereditary.*

PROOF. Observe that in a PI ring, all primitive ideals are maximal, and the residue rings are semisimple Artinian, hence regular. Now 4.3, 4.4 apply.

Now if A is a commutative domain, necessary and sufficient conditions were determined in [**9**; Theorem 2.3] so that A be matricially Baer $*$ (they also follow from 4.5). In the case that $*$ is the identity, these reduce to

(i) A is a Prüfer domain, and

(ii) all residue fields of A are formally real.

Let E be the quotient field of a domain A satisfying (i) and (ii); then E is formally real. Let F be a quadratic extension field of E, so $F = E[\sqrt{d}]$, and suppose the Galois automorphism is positive definite on F (2.17). Set B to be the integral closure of A in F. We propose to determine when B is matricially Baer $*$; this is the first question on p. 246 of [**9**]. Equivalently, knowing $E_b \subset A$, we determine when $F_b \subset B$.

It is possible that B not be matricially Baer $*$; for instance, if $E_b \neq E$, but $F_b = F$ (as in Example 2.16), set $A = E_b$; then B is not a field, so cannot contain F_b. Aside from this peculiarity, a necessary (but not sufficient) condition is that all the primes in A must not split.

Let us define an equivalence relation on the elements of $A - \{0\}$; $[a] = [a']$ if a/a' is a square in E, in other words, $E(\sqrt{a}) = E[\sqrt{a'}]$. If d is an element of E, we may always find a in A so that $E[\sqrt{d}] = E[\sqrt{a}]$; we will always assume the element d of E that defines F is an element of A.

LEMMA 4.6. *Let L be a finite dimensional field extension of K, and let D, E be unital orders in K, L respectively, with D contained in E, and E integral over D. Suppose M is a maximal ideal of D, and the localization of D at M is a valuation domain. Then the dimension of E/ME as a D/M vector space is less than or equal to* $[L: K]$.

PROOF. Localize at M, to obtain D_M and $E_M = (D - M)^{-1}E$. Clearly, $D_M/MD_M = D/M$ and $E_M/ME_M = E/ME$, so we may assume $D = D_M$, so D is a valuation domain.

Let $\{x_i + M\}_{i=1}^n$ be part of a basis for E/ME over D/M. If $n > [L: M]$, there exists a subset $\{a_i\}_{i=1}^n$ of K, not all zero with $\sum x_i a_i = 0$. We may obviously assume $\{a_i\} \subset D$. Since D is a valuation domain, the ideals $\{a_i D\}$ are totally ordered. Hence after relabelling, there exists a subset

$\{z_i\} \subset D$ with $a_i = z_i a_1$. We thus obtain $x_1 + \sum_2^n x_i z_i = 0$, a contradiction to the linear independence of $\{x_i + M\}$.

THEOREM 4.7. *Let A be a matricially Baer $*$ domain with respect to the identity, and let E be its quotient field. For an element d in A that is not a square in E, set $F = E[\sqrt{d}]$, and set B to be the integral closure of A in F. Then B, equipped with the Galois automorphism, is matricially Baer $*$ if and only if*

(I) *for every maximal ideal M of A, $[d]$ contains no nontrivial sums of squares modulo M.*

PROOF. Suppose B is matricially Baer $*$, and (I) is violated. Localize at the maximal ideal M, so as to create the valuation domain, $T = S_M^{-1}A$. Let U be the integral closure of T in F, and observe that $S_M^{-1}B$ is contained in U. By, for example, [**9**; Proposition 1.1], U is also matricially Baer $*$. Now (M) is the maximal ideal of T; let N be a maximal ideal of U. Then by 4.6, $[U/N{:}T/(M)] = 1$ or 2.

If $[d]$ contains a nontrivial sum of squares modulo M, we may find d'' in $[d]$, and $\{e_i\}$ in D, but not all in M so that $d'' - \sum e_i^2 = m \in M$. Since $d'' \notin M$ and $\sqrt{d''}$ generates F, the image x of d'' generates U/N over $T/(M)$ and the extension is quadratic. Setting $E_i = e_i + N$ in U/N, and observing that $xx* = -d'' + M$, we have $xx* + \sum E_i E_i* = 0$. By [**9**; 2.3(b)], $x = 0$, a contradiction.

Now suppose (I) holds. We first show that if N is a maximal ideal of B, then $N = N*$. Set $M = N \cap A$; M is clearly prime, and the field B/N is algebraic over the domain A/M, so M is maximal, and by 4.6, $[B/N{:}\ A/M] = 2$ or 1. If $[B/N{:}\ A/M] = 2$, then assuming $N \neq N*$, we see that $M \subset N \cap N*$ and $B/(N \cap N*) \cong B/N \times B/N*$, hence is of dimension at least 4 over D/M, a contradiction.

If on the other hand $[B/N{:}\ A/M] = 1$, but $N \neq N*$, then by 4.6, we must have $MB = N \cap N*$, and $[B/(N \cap N*){:}\ A/M] = 2$. We may thus find x in $B/(N \cap N*)$ so that $B/N \cap N* = A/M[x]$; let X be a preimage in B. Write $X = a + b\sqrt{d}$, a, b in E. Then $a^2 - b^2 d = XX* \in N \cap N* \cap A = M$. Set $d' = b^2 d$. Since $2a = \text{trace}(X)$ is integral, we have $2a$ belongs to A, and since $2 \geqq 1$ in the natural ordering, $1/2 \in E_b \subset A$, and so a belongs to A, and so d' belongs to A, and thus to $[d]$. But $d' - a^2 \in M$, contradicting a weak form of (I). (We use the full strength of (I) to show the induced involution is positive definite). Thus $N = N*$ in all cases.

If $B/N = A/M$, the quotient fields are formally real, the induced involution is the identity and is thus positive definite. If $[B/N{:}\ A/M] = 2$, find X in B such that its image in B/N, x, is not 0 but squares to an element of A/M, and thus generates B/N over A/M. Writing $X = a + b\sqrt{d}$, as above, we see that a is integral over A and belongs to E, so a lies in A. Then $z = x - (a + N)$ also generates B/N over A/M, and satisfies a quadratic equation, $z^2 = b^2d + M$. The induced involution on B/N sends z to $-z$, and by 2.17(a), this involution is positive definite if and only if z^2 is not a sum of squares in A/M, and this easily translates to condition (I) on A. Hence the induced involution on B/N is positive definite in all cases. Since the integral closure of a Prüfer domain in an algebraic extension is always Prüfer, we have verified conditions (i, ii iii) of 2.10(A), and so $F_b \subset B$, and 4.2 applies to yield that B is matricially Baer $*$.

Observe that as a not obvious consequence, (I) implies d is not a sum of squares in E. If one considers the identity involution on F, instead of the Galois involution, the criterion (I) must be changed appropriately.

In case A is a principal ideal domain, or more generally, if d can be chosen as a square-free product of principal primes or a unit, then the criterion of 4.7 can be simplified. In this case, one can show that $B = A[\sqrt{d}]$; if d is a unit, B is matricially Baer $*$ if and only if d is not a sum of squares modulo every maximal ideal; if (d) is a product of precisely the primes $P_1, P_2, \ldots, P_k$, (taken once each), then B is matricially Baer $*$ if and only if d is not a sum of squares modulo all other maximal ideals. In particular, if A is local and principal, and d is a generator of the maximal ideal, there is no condition whatever on d.

It is clear from the proof that none of the primes of A will split if and only if $[d]$ contains no nontrivial squares modulo any maximal ideal: however, this condition is not sufficient for B to be matricially Baer $*$ (a subring of $\mathbf{Q}(x)_b$ can be constructed for this purpose).

If R is a Rickart $*$-ring (see §1), for all r in R, there are projections $1 - p$ and $1 - q$ so that the right annihilator or r is $(1 - p)R$, and the left annihilator of r is $R(1 - q)$. Then $rp = r$, $qr = r$, and these are the minimal projections with these properties. We refer to p as the right projection of r, $RP(r) = p$, and q is the left projection of r, $LP(r) = q$. There is a notion of $*$-equivalence between projections: p is $*$-equivalent to $q(p \overset{*}{\sim} q)$ if there exists u in R such that $uu* = p$, $u*u = q$. The usual notion of equivalence between idempotents, $pR \cong qR$, is denoted $p \sim q$. If for all r in R, $LP(r) \overset{*}{\sim} RP(r)$, then R satisfies $LP \sim RP$ [**2**].

Let $(F, *)$ be a field with a positive definite involution. Let V be an n-dimensional vector space with basis $\{e_i\}$ and inner product generated sesquilinearly by the relation

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The question, whether equidimensional subspaces are unitarily equivalent, referred to in the introduction, is precisely the same as whether for all projections p, q in $(M_nF, \sharp)$, $p \sim q$ implies there exists a unitary u such that $upu* = q$. This latter condition is equivalent to both $p \overset{*}{\sim} q$ and $1 - p$

$\underset{\sim}{*}$ $1 - q$ together; however, since in M_nF, $p \sim q$ implies $1 - p \sim 1 - q$, and $rR = (LP(r))R$, the question reduces to, does M_nF satisfy $LP \sim RP$?

A formally real field is *Pythagorean* if every sum of squares is a square. If $(F, *)$ is a field with positive definite involution, then it is *$*$-Pythagorean* if for every finite subset $\{r_i\}$ of F, there exists s so that $ss* = \sum r_i r_i*$.

PROPOSITION 4.8. [**10**; Theorem 4.5]. *Let $(F, *)$ be a division ring with a positive definite involution. Then, if M_nF satisfies $LP \sim RP$ (with respect to $\sharp = *$-transpose), for some n greater than* 1, *all matrix rings over F also satisfy $LP \sim RP$.*

THEOREM 4.9. *Let F be a field with a positive definite involution $*$. Then $(M_nF, \sharp)$ satisfies $LP \sim RP$ for some (and hence every) n greater than* 1 *if and only if F is $*$-Pythagorean.*

PROOF. Suppose F is $*$-Pythagorean. With $n = 2$, a rank one projection P must have the form

$$P = \begin{bmatrix} a & b \\ b* & 1 - a \end{bmatrix},$$

where $a = a*$ and $a(1 - a) = bb*$; thus $a = bb* + a^2 = bb* + aa*$. By hypothesis, there exists c so that $cc* = a$; set $d = (c*)^{-1}b$, and define the matrix,

$$X = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

Then $X\sharp X = P$, and since $d*c = b*c*^{-1}c^{-1}b = b*a^{-1}b = 1 - a$, we obtain that

$$XX\sharp = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $*$-equivalence is transitive, all rank 1 projections are $*$-equivalent to each other, so M_2F satisfies $LP \sim RP$, hence by 4.8, all matrix rings over F do. Conversely, if $LP \sim RP$ holds in M_nF, then [**16**; Lemma 1] applies, to yield the result.

In the above proof, the only property of fields (as opposed to division rings) used, is that $aa* = a*a$ for all a. All quaternionic division rings with the natural involution have this property, so we obtain the following corollary from the proof.

COROLLARY 4.10. *If $(D, *)$ is a quaternionic division algebra with the natural involution, and this is positive definite, then all matrix rings over D satisfy $LP \sim RP$ if and only if M_2D does, and this occurs if and only if D is $*$-Pythagorean.*

These results lead to some apparently paradoxical examples. For example, if E is the rationals, or any finite dimensional field extension thereof, E cannot be $*$-Pythagorean for any involution; yet if D is the rational quaternions (of type $-1, -1$), so is only 4-dimensional over the rationals, D is $*$-Pythagorean since all sums of rational squares are sums of four squares, hence M_2D satisfies $LP \sim RP$. On the other hand $M_2(D \otimes D) = M_8\mathbf{Q}$ does not.

Somewhat more startling is that $(F, *)$ can be $*$-Pythagorean, while E, the fixed subfield, is not Pythagorean. Choose for E any formally real field in which every sum of squares is a sum of two squares, but not every sum of squares is a square (for example, $E = \mathbf{R}(x)$), and set $F = E[i]$. The norms of elements of F are the sums of two squares of E.

THEOREM 4.11. *Let $(A, *)$ be a matricially Baer $*$ commutative domain, with quotient field $(F, *)$. Then M_nA satisfies $LP \sim RP$ if and only if M_nF does, and this is equivalent to all sums of norms in A being norms in A. When this occurs, every finitely generated ideal of A is principal.*

PROOF. By [**9**; Theorem 2.3, final statement], all $*$-equivalences implemented in M_nF can be implemented in M_nA, and all projections of M_nF lie in M_nA. The first if and only if thus follows.

Suppose F is ($*$–) Pythagorean. Set $s = \sum x_i x_i*$, with x_i in A. By hypothesis, there exists t in F with $tt* = s$. It suffices to show that t belongs to A. If t did not so belong, by [**9**; Proposition 2.1(c)], there would exist a maximal ideal M of A, and an element d in M such that dt belongs to D but not to M. As $M = M*$, neither does $t*d*$ belong, so the product, $dsd* = (dt)(dt)*$ cannot lie in M either; but this contradicts d belonging to M and s to A.

Conversely, if A is ($*$–) Pythagorean, it is very easy to check that F is as well.

Finally, because A is a Prüfer domain, to show every finitely generated ideal is principal, we need only show that 2-generated projectives are free. Let P be a two-generated nonzero module, that is not free on two generators. There exists a projection p in M_2A such that $P \cong p(A \oplus A)$; necessarily rank $p = 1$. Since p is equivalent to $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ within M_2F, they are $*$-equivalent, and since $*$-equivalences of M_2F may be implemented within the bounded subring, p is $*$-equivalent to $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ within M_2A. Hence

$$P \cong p(A \oplus A) \cong \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}(A \oplus A) = A,$$

so P is free.

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