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OPEN COMPACT MAPPINGS, MOORE SPACES AND ORTHOCOMPACTNESS

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ABSTRACT. Two examples are given to show that an open compact map between zero dimensional Moore spaces need not preserve quasi-metrizability even if the domain space is separable or metacompact.

Perfect maps preserve quasi-metric spaces as well as non-Archimedean quasi-metric spaces and γ-spaces [**14**], [**12**]. The same is true for arbitrary closed maps with the first countable images [**16**], [**13**]. This paper is concerned with open and pseudo-open maps of quasi-metric spaces.

It was observed in [**5**] that quasi-metric spaces and γ-spaces are preserved under open finite-to-one maps; the corresponding result holds for non-Archimedean quasi-metric spaces. In answer to a question raised by R. F. Gittings [**6**], we show that a further generalization of these results is false; open compact maps do not preserve quasi-metrizability.

While the open compact images of metric spaces are the metacompact Moore spaces, which are very nice non-Archimedean quasimetric spaces [**4**], we show that one more application of an open compact map may yield a Moore space which is not quasimetrizable (Example 2). Hence there are non-quasi-metrizable spaces in MOBI, the smallest class containing all metric spaces and closed under open compact maps [**1**]. Example 2 answers a question asked by H. R. Bennett [**2**]. Example 1 shows that open compact maps do not preserve quasi-metrizability in the class of separable Moore spaces.

In both examples the domain spaces are non-Archimedean quasi-metric while the image is not quasi-metrizable and hence not γ, since developable γ-spaces are quasi-metrizable [**8**].

Since a developable space is orthocompact if and only if it is non-Archimedean quasi-metrizable [**4**], examples show that orthocompactness is not preserved under open compact maps. A Moore space may fail to be orthocompact even if it is an open compact image of a metacompact Moore

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space or an orthocompact separable Moore space. Thus Example 2 answers Question 2 of [**6**], and improves upon the examples of J. Chaber [**3**] of nonmetacompact Moore spaces in MOBI.

A distance function d is called *quasi-metric* if $d(x, z) \leqq d(x, y) + d(y, z)$, *non-Archimedean quasi-metric* if $d(x, z) \leqq \max\{d(x, y), d(y, z)\}$ and *γ-metric* if $d(x, z_n) \to 0$ whenever $d(x, y_n) \to 0$ and $d(y_n, z_n) \to 0$. A non-Archimedean quasi-metric space is a quasi-metric space, and a quasi-metric space is a γ-space, i.e., has a γ-metric. A space is non-Archimedean quasi-metrizable if and only if it has a σ-interior preserving base ($= \sigma$-Q-base) [**4**], [**11**]. A collection of open sets is *interior preserving* if the intersection of any subcollection is open. A collection of sets is *point countable* if any point belongs to no more than countably many members. Every space in MOBI has a point countable base [**1**]. A map ($=$ continuous mapping) is *open* if the image of each open set is open, and it is *compact* if the pre-images of the points of the range are compact sets in the domain.

EXAMPLE 1. The first example is an open compact map of a separable zero-dimensional non-Archimedean quasi-metrizable Moore space onto a separable zero-dimensional non-quasi-metrizable Moore space.

The range space X is similar to a space defined independently in [**7**] and [**11**]. The underlying set of X is a subset $A \cup B$ of the plane, $\mathbf{R}^2$, where $A = \{\langle x, 0\rangle \mid x \text{ is irrational}\}$ and $B = \{\langle x, y\rangle \mid x, y \text{ are rational}, y > 0\}$. The topology of X is defined as follows. For $a \in A$ and $n \in N$, let $T(a, 1/n)$ denote the set of all points in B that belong to the interior of the isosceles right triangle above $\mathbf{R} \times \{0\}$ having vertex a and hypotenuse of length $2/n$ parallel to $\mathbf{R} \times \{0\}$. The sets $U_n\{a\} = \{a\} \cup T(a, 1/n)$ form a neighbourhood base for a. For $b \in B$ and $n \in N$, let $C(b, 1/n)$ denote the intersection with B of the circle of radius $1/n$ and center b. The sets $U_n\{b\} = C(b, 1/n)$ form a neighbourhood base for b. The author showed in [**11**] that there is a continuous semi-metric on X (hence X is developable) and that X is not quasimetrizable. It is easy to see that X is zero dimensional.

Let us show that X is an image under an open compact map of a quasi-metric Moore space X_0. The underlying set of X_0 is $A \times \{0\} \cup B \times I$, where $I = [0, 1]$. The set $B \times I$ is open in X_0, and has the usual topology of a subspace of $\mathbf{R}^3$; for each $p \in B \times I$ and $n \in N$ let $\tilde{U}_n(p)$ denote the intersection of $B \times I$ with the sphere of radius $1/n$ and center p. For each $p \in A \times \{0\}$, $p = (x_0, 0, 0)$ and $n \in N$ let $\tilde{T}(p, 1/n)$ denote the set of all points in $B \times I$ that belong to the interior of the solid bounded by the cones

$$C_1\colon (x - x_0)^2 + (1 + 1/n)\, z^2 = y^2,$$
$$C_2\colon (x - x_0)^2 + (1 - 1/n)\, z^2 = y^2,$$

and the plane $y = 1/n$, and let $\tilde{U}_n\{p\} = \{p\} \cup \tilde{T}(p, 1/n)$. For each $p \in X_0$ the sets $\tilde{U}_n\{p\}$ form a neighbourhood base for p. The projection

$f(\langle x, y, z\rangle) = \langle x, y\rangle$ is an open map of X_0 onto X, since $f(\tilde{U}_n\{\langle x, y, z\rangle\}) = U\{\langle x, y\rangle\}$ and the pre-images of the points of X are compact sets.

We now show that X_0 is developable and has a σ-interior preserving base so that X_0 is non-Archimedeanly quasi-metrizable. Since $\{\{\tilde{U}_n(p)| p \in X_0\}|n \in N\}$ is a development, the proof may be completed by showing that for each $n \in N$, $\{\tilde{U}_n\{p\}\,|\,p \in X_0\}$ has an interior preserving refinement. The subspace $B \times I$ is metrizable so that for each $n \in N$, $\{U_n\{p\}\,|\,p \in B \times I\}$ has an open interior preserving (even a locally finite) refinement γ_n.

Let $n \in N$. For each $p \in A \times \{0\}$ and $q \in \tilde{U}_{n+1}(p) - \{p\}$, let $k(p, q)$ denote the least $k \in N$ such that $\mathrm{st}(q, \gamma_k) \subset \tilde{U}_n(p)$ and choose $G(p, q) \in \gamma_{k(p, q)}$ such that $q \in G(p, q)$. Set

$$V_n(p) = \{p\} \cup (\bigcup\{G(p, q)|q \in \tilde{U}_{n+1}(p) - \{p\}\}).$$

Then $\tilde{U}_{n+1}(p) \subset V_n(p) \subset \tilde{U}_n(p)$. Let us show that $\{V_n(p)|p \in A \times \{0\}\}$ is interior preserving. Let $x \in \bigcap_{i=1}^{\infty} V_n(p_i)$. Then for each $i \in N$, there exists $q_i \in \tilde{U}_{n+1}(p_i) - \{p_i\}$ such that $x \in G(p_i, q_i)$. If $\langle k(p_i, q_i)\rangle$ is bounded, then $\bigcap_{i=1}^{\infty} G(p_i, q_i)$ is open and $\bigcap_{j=1}^{\infty} V_n(p_i)$ is a neighborhood of x. If $\langle k(p_i, q_i)\rangle$ is not bounded, we may suppose that $\langle k(p_i, q_i)\rangle \to \infty$. For each $i \in N\{x, q_i\} \subset G(p_i, q_i) \in \gamma_{k(p_i, q_i)}$, and so $\langle q_i\rangle \to x$. Since $x \in B \times (0, 1)$, there exists $\varepsilon > 0$ such that for all $i \in N$, $q_i \in B \times [\varepsilon, 1]$. The Euclidean distance δ between the sets $\tilde{U}_{n+1}(p) \cap B \times [\varepsilon, 1]$ and $B \times I - \tilde{U}_n(p)$ does not depend on $p \in A \times \{0\}$, and δ is positive. Choose $m \in N$ such that $1/m < \delta$. Then for each $i \in N$, $\mathrm{st}(q_i, \gamma_m) \subset U_n(p_i)$ so that $k(p_i, q_i) \leqq m$; this contradicts $\langle k(p_i, q_i)\rangle \to \infty$.

We have proved that $\{V_n\{p\}\,|\,p \in A \times \{0\}\}$ is interior preserving. Hence $\beta_n = \gamma_n \cup \{V_n\{p\}\,|\,p \in A \times \{0\}\}$ is an open interior perserving refinement of $\{\tilde{U}_n\{p\}\,|\,p \in X_0\}$, and $\langle\beta_n\rangle$ is a development. Hence $\bigcup\beta_n$ is an σ-interior preserving base for X_0.

Although X_0 is not zero-dimensional, there is a zero dimensional subspace X_{00} of X_0 such that $f(X_{00}) = X$ and $f|X_{00}\colon X_{00} \to X$ is open and compact; $X_{00} = A \times \{0\} \cup B \times J$, $J \subset I$, $J = \{0, x_1, x_2, \ldots\}$, $x_n \to 0$ and x_n are close enough to one another so that $f(\tilde{U}_n\langle x, 0, 0\rangle\} \cap X_{00}) = U_n(\langle x, 0\rangle)$.

The non-quasi-metrizable space X of the Example 1 is separable and, while it is an open compact image of a non-Archimedean quasi-metric Moore space, it is not in MOBI. Moreover, if has no point countable base, since a separable space with a point countable base is second countable. The following example provides a non-quasi-metrizable Moore space in MOBI.

EXAMPLE 2. The second example is an open compact map of a metacompact, zero-dimensional Moore space onto a non-quasi-metrizable zero-dimensional Moore space.

Using ideas of F. D. Tall [15] Wicke and J.M. Worrell [17], as combined by J. Chaber [3], we shall modify the space X of Example 1 to obtain a non-quasi-metrizable space Y with a point countable base. Let A, B, $T(a, 1/n)$ and $C(b, 1/n)$ have the same meaning as in Example 1. Let $\mathfrak{A}$ be the collection of all countable infinite subsets of A and let $Y = A \cup (B \times \mathfrak{A})$. A basic neighbourhood of $a \in A$ is $V_n\{a\} = \{a\} \cup T(a, 1/n) \times \mathfrak{A}(a)$, where $\mathfrak{A}(a) = \{\alpha \in \mathfrak{A} \mid a \in \alpha\}$. A basic neighbourhood of $\langle b, \alpha\rangle \in B \times \mathfrak{A}$ is $V_n\{\langle b, \alpha\rangle\} = C(b, 1/n) \times \{\alpha\}$.

Obviously Y is zero-dimensional and $\{\{V_n(p) \mid p \in Y\} \mid n = 1, 2, \ldots\}$ is a development for Y. Moreover Y, like the space X of Example 1, has a continuous semi-metric. To see that Y is not quasi-metrizable, suppose that there exists a quasi-metric d for Y with spheres $S(x, r)$. For each $m, n \in N$, let

$$\alpha(m, n) = \{a \in A \mid V_n\{a\} \subset S(a, 1/m) \subset S(a, 2/m) \subset V_1\{a\}\}.$$

Since $A = \bigcup \alpha(m, n)$ by the Baire Category Theorem some $\alpha(m, n)$ is dense in an open interval I of $\mathbf{R} \times \{0\}$ in the Euclidean topology. We assume without loss of generality that the length of I is $\leqq 1/n$. Pick a point $b \in B$ inside the isosceles right triangle $T \subset \mathbf{R}^2$ that lies above I and has I as its hypotenuse. Now pick some countable $\alpha \subset \alpha(m, n) \cap I$ dense in I in the Euclidean topology. Obviously $\alpha \in \mathfrak{A}$. Let $Y_\alpha = B \times \{\alpha\}$. Consider the set $S(\langle b, \alpha\rangle, 1/m) \cap Y_\alpha$. There exists $k \in N$ such that $C(b, 1/k) \times \{a\} \subset S(\langle b, \alpha\rangle, 1/m) \cap Y_\alpha$. Since α is dense in I in the Euclidean topology and the length of the hypotenuse of T is $\leqq 1/n$, one can find $a \in \alpha$ such that $b \in T(a, 1/n)$ and such that the side of $T(a, 1/n)$ is so close to the center b of $C(b, 1/k)$ that $C(b, 1/k) \not\subset T(a, 1)$. Pick $c \in C(b, 1/k) \backslash T(a, 1/n)$. Since $a \in \alpha \in \mathfrak{A} \cap \alpha(m, n)$, we have $T(a, 1/n) \times \{\alpha\} = V_n\{a\} \cap Y_\alpha \subset S(a, 1/m) \cap Y_\alpha \subset S(a, 2/m) \cap Y_\alpha \subset V_1\{a\} \cap Y_\alpha = T(a, 1) \times \{a\}$. We have $d(a, \langle b, \alpha\rangle) < 1/m$ since $\langle b, \alpha\rangle \in T(a, 1/n) \times \{\alpha\} \subset S(a, 1/m)$, and $d(\langle b, \alpha\rangle, \langle c, \alpha\rangle) < 1/m$ since $\langle c, \alpha\rangle \in C(b, 1/k) \times \{\alpha\} \subset S(\langle b, \alpha\rangle, 1/m)$, while $d(a, \langle c, \alpha\rangle) \geqq 2/m$ since $S(a, 2/m) \cap Y_\alpha \subset T(a, 1) \times \{\alpha\}$ and $\langle c, \alpha\rangle \notin T(a, 1) \times \{\alpha\}$. Hence d is not a quasi-metric.

In order to show that Y is in MOBI, we construct a metacompact Moore space Y_0 and show that Y is the image of Y_0 under an open compact map. Consider the set $Y_\alpha = B \times \{\alpha\}$, $\alpha \in \mathfrak{A}$. Since α is countable, we have $\alpha = \{a_1, a_2, \ldots\} \subset A$. Add one point to α, say a^*, and let the set $\alpha \cup \{a^*\}$ be denoted by α^*. Set $Y_\alpha^* = Y_\alpha \times \alpha^*$, and $Y_0 = A \cup (\bigcup_{\alpha \in \mathfrak{A}} Y_\alpha^*)$. Define a topology on Y_0 as follows. A basic neighbourhood of $a \in A$ is

$$\tilde{V}_n\{a\} = \{a\} \cup (\textstyle\bigcup_{\alpha \in \mathfrak{A}(a)} T(a, 1/n) \times \{\alpha\} \times \{a\}).$$

A basic neighbourhood of $\langle b, \alpha, a\rangle \in Y_\alpha^*$, $a \in \alpha$, is $\tilde{V}_n\{\langle b, \alpha, a\rangle\} = C(b, 1/n) \times \{\alpha\} \times \{a\}$. A basic neighbourhood of $\langle b, \alpha, a^*\rangle \in Y_\alpha^*$ is

$$\tilde{V}_n\{\langle b, \alpha, a^*\rangle\} = \{a^*\} \cup (\bigcup_{m \geq n} C(b, 1/m) \times \{\alpha\} \times \{a_m\}).$$

Obviously Y_0 is a zero-dimensional Moore space, like Y. Since $Y_0 - A$ is a topological sum of open zero-dimensional countable subspaces, it is an open metrizable subspace of Y_0. The metacompactness of Y_0 follows since $\tilde{V}_n(a) \cap \tilde{V}_n(b) = \emptyset$ whenever $a \neq b$. Let us define a map f from Y_0 onto Y. For $a \in A$ we set $f(a) = a$, and for $\langle b, \alpha, a\rangle \in Y_\alpha^*$, $a \in \alpha^*$, we set $f(\langle b, \alpha, a\rangle) = \langle b, \alpha\rangle \in Y_\alpha$. Obviously f is open and the pre-images of the points of Y are compact.

REMARK. The theorems on closed and open mappings mentioned in the beginning cannot be generalized in another direction. Pseudo-open mappings do not preserve quasi-metrics even if they are two-to-one.

Let us give an outline of a counter-example. Take the quasi-metric space of Example 1 of **[11]**. The underlying set is the plane, a basic neighbourhood is a point together with a circle above it. Now take a similar space on the set with the circles below the points. The intersection of two topologies gives the semi-metric space of the Example 1 or **[9, 10]**; a basic neighbourhood is a point along with two circles-above and below the point. This will be the range space X, while the topological sum of the quasi-metric spaces is the domain. The obvious quotient map is pseudo-open since the range is first countable **[1]**.

X is not γ, since a semi-metric γ-space is developable **[14]**, and hence has a σ-discrete network, while X has none **[9, 10]**. A slight modification (triangles in place of circles) will make the domain space even non-Archimedean quasi-metrizable.

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