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CHAIN CONDITIONS AND INTEGRAL EXTENSIONS

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ABSTRACT. Let R be a ring, integral of bounded degree n over a subring C of the center of R. Various theorems are proved relating chain conditions in R and C. When R is a semi-prime ring which is torsion free for the regular elements of C and which is $n!$-torsion free, then R is a Goldie ring when C is. If also, $n!$ is a unit in R and C is "integrally closed" then C is a Noetherian ring (has Krull dimension) if and only if R is a Noetherian ring (has Krull dimension). Finally, assuming only that $n!$ is a unit in R, any R module has Gabriel dimension exactly when it does as a C module, in which case the dimensions are equal.

In this paper we consider rings which are integral extensions of central subrings and investigate whether certain chain conditions on one ring of the pair transfer to the other. Specifically, we examine the situation when one of the rings is a Goldie ring, a Noetherian ring, or has Krull dimension, and also consider modules over the larger ring which have Gabriel dimension with respect to one of the rings. To show that these conditions transfer from one ring to the other, we need to assume that the degrees of integrality are bounded, and usually, that this bound is invertible. Examples are presented to show that these assumptions are necessary. For the case of Noetherian rings, or of rings with Krull dimension, we must also assume that the central subring is "integrally closed" in its quotient ring, although we do not know if this assumption is necessary.

The questions studied here were raised as a consequence of similar ones for rings with involution [**16** and **17**], where the situation of a ring quadratic over its center occurs as a special case which must be considered. The more specific subject of algebraic algebras of bounded degree has been studied in the past, but the work was concerned with problems of a different kind. Some results on the finite dimensionality of such algebras were obtained in [**12**] and [**15**], such as for division algebras over their centers [**12**; Theorem 7, p. 701], or for semi-simple algebras over infinite

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perfect fields [**12**; Corollary, p. 704]. Other work in these papers concerns commutativity theorems [**12**], topological representations of such algebras [**15**], and in both, the Kurosh problem, culminating in the result of Kaplansky [**15**; Theorem 6.1, p. 71] that algebraic algebras of bounded degree are locally finite. The more general situation of algebras whose nilpotent elements have bounded index was studied by Levitzki [**18**] who was concerned, primarily, with structural properties such as the existence of certain matrix ideals and the relation between the algebra and its primitive images. Studying chain conditions in integral extensions has a somewhat more commutative flavor than these earlier works. Although the relation between the prime ideals in a commutative ring and those of an integral extension is well known, and although there are counter-examples to the transfer of the ascending chain condition [**20**; Example 4, p. 207], the questions about chain conditions in the case of bounded degree do not seem to have been considered. In fact, the strong assumption of bounded degree appears difficult to use to advantage in the purely commutative setting.

For C a commutative ring, call a polynomial $p(x) = x^k + c_1x^{k-1} + \cdots + c_k$, with $c_i \in C$, a monic polynomial of $C[x]$. We call a ring R *integral* over a central subring C if for each $r \in R$ there exists a monic polynomial $p(x)$ of $C[X]$ so that $p(r) = 0$. If, in addition, it is possible to choose $p(x)$ of degree at most n, then R is *integral of bounded degree n* over C. Note that in this case of bounded degree n, there is for each $r \in R$, a monic polynomial of degree exactly n which is satisfied by r. An important result which follows from this is that when R is integral of bounded degree n over C, R satisfies a polynomial identity [**9**; Lemma 6.2, p. 155]. Also, we recall that if R is an algebraic algebra over a field F, then each non-nilpotent element of R has a multiple which is a nonzero idempotent. To see this, observe that from the minimal polynomial for $r \in R$ one obtains a relation of the form $r^k = r^{k+1}g(r) \neq 0$, and then use the proof of [**9**; Lemma 1.3.2, p. 22].

I. Goldie conditions. In this section we study when the Goldie chain conditions transfer from one of our rings to the other. It is easy to see that one must assume that R is a semi-prime ring to force any reasonable finiteness condition on R from C. For example, let C be a commutative ring, and set $R = C[x_1, x_2, \ldots]/I$, where I is the ideal generated by all x_ix_j. Each $r \in R$ can be written $r = c_0 + \sum c_ix_i$. Then $(r - c_0)^2 = 0$, so R is integral over C of bounded degree two but contains infinite direct sums of ideals. On the other hand, if R is a semi-prime Goldie ring, any central subring is also a semi-prime Goldie ring. Therefore, we shall consider the situation when R is a semi-prime ring and C is a Goldie ring. We begin with the special case when C is a field and prove an easy but useful lemma.

LEMMA 1. *Let R be an algebraic algebra over the field F so that $R \cong F_1 \oplus \cdots \oplus F_t$, for each F_i a separable field extension of F. If R is of bounded degree n over F, and either* char $F > n$ *or* char $F = 0$, *then* $\dim_F R \leqq n$.

PROOF. The separability of each F_i over F and the bound on the degrees of elements show that $\dim_F F_i \leqq n$. To prove the lemma by induction on t, it suffices to consider an algebra $B \oplus K$, algebraic of bounded degree n, over F, where B has an element of degree m, K is a separable extension field of F with $\dim_F K = k$, and to show that $B \oplus K$ contains an element of degree at least $m + k$ over F. If $h(x) \in F[x]$ is the minimal polynomial for $(b, t) \in B \oplus K$, then $h(x)$ is divisible by the corresponding polynomials for b and for t. Take $b \in B$ with minimal polynomial $g(x)$ of degree m and $t \in K$ so that $K = F(t)$. If $f(x)$ is the minimal polynomial over F for t, then $f(x)$ is irreducible of degree k, and for each $1 \leqq i \leqq n$, $f(x + i) \in F[x]$ is also irreducible over F and has degree k. Now $\{f(x + i) \mid 0 \leqq i \leqq n\}$ contains $n + 1$ distinct irreducible polynomials, by the assumption on char F, and since $\deg g(x) \leqq n$, there is some j with $f(x + j) \nmid g(x)$. Thus $(b, t - j)$ has degree at least $m + k$.

For any ring R, let R^* denote the semi-group of regular elements of R, and $Z(R)$, the center of R.

THEOREM 1. *Let R be a semi-prime ring and F a field contained in $Z(R)$. If R is integral of bounded degree n over F, then R is F^*-torsion free and has the same identity element as F. If also either* char $F = 0$ *or* char $F > n$, *then R is a finite dimensional algebra over F. In this case, writing $R = M_{n(1)}(D_1) \oplus \cdots \oplus M_{n(t)}(D_t)$ for D_i division algebras, with $Z(D_i) = Z_i$ and* $\dim_{Z_i} D_i = m_i^2$, *one has* $\sum n(i)m_i \dim_F Z_i \leqq n$, $\dim_F R \leqq n^2$, *and each Z_i is a separable extension of F.*

PROOF. If e is the identity element of F, then e is a central idempotent in R, and we may write, formally, $R = eR \oplus (1 - e)R$. Since $F \subset eR$, each element of $(1 - e)R$ annihilates every nonleading coefficient of its minimal polynomial in $F[x]$. Therefore, the ideal $(1 - e)R$ is nil of index n, so must equal zero [**10**; Lemma 1.1, p.1]. Hence e is the identity of R and the elements of F^* are units in R.

As we have just observed, no nonzero right ideal of R can be nil, so our earlier remark about algebraic algebras over fields implies that every right ideal of R contains an idempotent. If the right ideal eR, for $e^2 = e$, is not minimal, then eRe is algebraic over F and contains a proper idempotent. Thus $eR \supset e_1R + e_2R$ for e_1 and e_2 orthogonal idempotents.

Similarly, if $eR \subset T$, properly, then $T = eR + (1 - e)R \cap T$, so $eR + fR \subset T$ for e and f orthogonal idempotents. Consequently R will be Artinian if there is a finite bound on the cardinality of sets of orthogonal idempotents. We claim that n is such a bound. If $\{e_1, e_2, \ldots, e_{n+1}\}$ is a set of orthogonal idempotents, the assumption on char F enables one to choose $n + 1$ distinct elements $a_1, a_2, \ldots, a_{n+1} \in F$. Consider $g = a_1e_1 + \cdots + a_{n+1}e_{n+1}$ and note that any polynomial over F having g as a root must have each a_j as a root. This forces the degree of g over F to be $n + 1$. Therefore, R must be an Artinian algebra, so a finite direct sum of matrix algebras over algebraic division algebras, as claimed.

Each division algebra D_i is of bounded degree n over F, so each D_i is finite dimensional over its center [**12**; Theorem 7, p. 701]. In fact, the assumption on char F forces any subfield of D_i containing F to be a finite dimensional separable extension of F, which itself yields the finite dimensionality of D_i over Z_i [**9**; Theorem 4.2.1, p. 95]. Since $\dim_{Z_i}D_i = m_i^2$, D_i contains a maximal subfield K_i with $\dim_{Z_i}K_i = m_i$. For each i, the direct sum of $n(i)$ copies of K_i appears in R as diagonal matrices in $M_{n(i)}(D_i)$. Using Lemma 1 gives the inequality $\sum n(i)m_i\dim_F Z_i \leqq n$, from which it follows that $\dim_F R \leqq n^2$.

It is clear that the first inequality in Theorem 1 can be an equality, by taking a direct sum of n copies of F for example, and that the second equality can be obtained by taking $R = M_n(F)$. Also, if rather than assuming $F \subset R$, one simply assumes that R is an algebra over F, then the proof of Theorem 1, after the first paragraph, is still valid.

Before presenting the main result of this section, we give two examples which show the necessity of the characteristic assumption on F.

EXAMPLE 1. Let $F = GF(p)(X)$ for X an infinite set of indeterminates, and let K be an algebraic closure of F. Select $Y \subset K$ by $Y = \{x^{1/p} \mid x \in X\}$. Then $A = F[Y]$ is an infinite dimensional extension of F, yet $a^p \in F$ for each $a \in A$. Note that each $a \in A$ is a root of $x^{p+1} - bx$ for some $b \in F$, so that char $F \neq n$ does not suffice in Theorem 1.

The next example is in the same spirit and shows that A need not be a Goldie ring without the characteristic assumption on F.

EXAMPLE 2. Let $A = \prod_{i \in N}(\mathrm{GF}(p))_i$, the complete direct product of $\mathrm{GF}(p)$ over the natural numbers. Each element of A satisfies $x^p - x = 0$, but it is clear that A is not a Goldie ring. As in Example 1, if we consider each element of A to satisfy $x^{p+1} - x^2 = 0$, then char $F \neq n$ is not enough in Theorem 1. This observation will be relevant to Theorem 2.

Our next theorem does the general case of noncommutative semi-prime rings.

THEOREM 2. *Let R be a semi-prime ring and C a subring of $Z(R)$ so that C is a Goldie ring and R is C^*-torsion free. If R is integral over C of bounded degree n, and if R is $n!$-torsion free, then R is a left and right Goldie ring with quotient ring $S = R(C^*)^{-1}$. Furthermore, S is a finitely generated $Q(C) = C(C^*)^{-1}$ module, and the minimal primes of R intersected with C are the minimal primes of C.*

PROOF. Observe that $C \neq 0$, for otherwise R would be nil of bounded index n, contradicting the semi-primeness of R[**10**; Lemma 1.1, p. 1]. Set $S = R(C^*)^{-1}$, the localization of R at C^*, and note that R embeds in S because R is C^*-torsion free [**1**; Section 2]. Since C is a semi-prime Goldie ring, $C(C^*)^{-1} = Q(C) \subset S$ is a direct sum of fields $F_1 \oplus \cdots \oplus F_t$, and either char $F_i > n$ or char $F_i = 0$ by the assumption that R is $n!$-torsion free. If f_i is the identity of F_i, then f_i is a central idempotent in S, $S \cong f_1S \oplus \cdots \oplus f_tS$, and f_iS is a semi-prime algebra over F_i, algebraic of bounded degree n. By Theorem 1, f_iS is a semi-simple Artinian algebra, finite dimensional over F_i, Thus S is a semi-simple Artinian ring which is finitely generated as a $Q(C)$ module. The integrality of R over C, and the C^*-torsion freeness assumption imply that regular elements of R are invertible in S. Therefore, R is an order in S, so is a Goldie ring [**10**; Theorem 4.5, p. 70].

For the statement about minimal primes, we may assume that R is not a prime ring, for if it is, C is a domain. Let $P \neq 0$ be a minimal prime of R and set $B = \mathrm{ann}(P)$. Now $B(P \cap C) = 0$, so $P \cap C$ contains no element of C^*. Thus $P \cap C \subset M_1 \cup \cdots \cup M_s$, where $\{M_i\}$ are the maximal annihilator ideals of C. Consequently, $P \cap C \subset M_j$ for some j. Since $P \cap C$ is a prime ideal of C, $P \cap C = M_j$ is a minimal prime ideal of C.

When R is commutative, combining Theorem 2 with Theorem 1 gives

COROLLARY. *If R is a commutative semi-prime ring with* 1, *integral of bounded degree n over a subring $C = F_1 \oplus \cdots \oplus F_t$ for F_i fields with* char $F_i > n$ *or* char $F_i = 0$, *then R is a finite direct sum of at most nt fields, each a finite dimensional separable extension of some F_i.*

Example 2 shows that the condition that R is $n!$-torsion free in Theorem 2 cannot be replaced with the condition that R is n-torsion free. Our next example shows that Theorem 2 is false unless R is C^*-torsion free.

EXAMPLE 3. For each prime $p > 3$, let $F_p = \mathrm{GF}(p^2)$, and set $A = \oplus_p F_p$. Denote by R, the ring obtained from A by adjoining 1. That is, $R = A \times J$, for J the ring of integers, with component-wise addition, and multiplication given by $(a, n) \cdot (b, m) = (ab + am + bn, nm)$. Clearly, R is a commutative semi-prime algebra over $J \cong (0, J)$. If e_p is the identity element of F_p, then $Je_p \cong \mathrm{GF}(p)$ and $F_p \cong Je_p(y_p)$, where y_p is a root of an irreducible quadratic polynomial in $Je_p[x]$. Any element of R has a unique expression as $r = a_0 + \sum_p(a_p y_p + b_p e_p)$, where the summation ranges over a finite set of primes and all $a_i, b_j \in J$. Using the fact that the F_p annihilate one another in R, and that each y_p^2 is a J linear combination of y_p and e_p, one obtains $r^2 = c_0 + \sum_p(c_p y_p + d_p e_p)$, where $c_p y_p = 0$ if $a_p y_p = 0$. Since the primes p appearing are distinct, one can solve, simultaneously, the congruences $a_p t \equiv c_p (\text{mod}\, p)$, and it follows that $r^2 - tr = m + \sum k_p e_p$. If $k \in J$ is a simultaneous solution of $k \equiv k_p (\text{mod}\ p)$, then $(r^2 - tr - m)^2 = (\sum_p k_p e_p)^2 = \sum_p k_p^2 e_p = k(\sum_p k_p e_p) = k(r^2 - tr - m)$. Consequently, each element of R satisfies a monic polynomial of degree 4 over J and R is 4!-torsion free since each $p > 3$. Therefore, all of the hypotheses of Theorem 2 are satisfied by R and J, except that R is not J^*-torsion free. Although J is a commutative domain, R is not a Goldie ring because $\sum F_p$ is a direct sum of ideals of R.

One can obtain a noncommutative example by repeating the construction above with F_p replaced by $M_2(\mathrm{GF}(p))$, or more generally by $M_k(\mathrm{GF}(p))$ for k fixed and $p > 2k$.

As a final comment on the C^*-torsion free assumption in Theorem 2, note that although, as Example 3 shows, regular elements of C can be zero divisors on R, no nonzero element of R can annihilate C^*. To see this, let $Q(C) = F_1 \oplus \cdots \oplus F_t$ again for $\{F_i\}$ fields with $f_i c^{-1}$ the identity of F_i. For any $d \in C$ set $T = \{f_i | df_i = 0\}$. Then $d + \sum_T f_i \in C^*$ since as an element of $Q(C)$, it has a nonzero coordinate in each F_i. If $y \in R$ satisfies $yC^* = 0$, then in particular $yc = 0$, so $(yf_i)^2 = y^2 f_i^2 = y^2 f_i c = 0$. Now $H = \text{Ann}\, C^*$ is an ideal of R, and Hf_i is nil of index 2, so $Hf_i = 0$ by Levitzki's Theorem [**10**; Lemma 1.1, p. 1]. By adding suitable f_j to the coefficients of the monic polynomial in $C[x]$ satisfied by y, we may assume that this polynomial has coefficients in $C^* \cup \{1\}$. But $yC^* = 0$ forces $y^n = 0$. Using Levitzki's Theorem again gives $H = 0$.

II. Ascending chain condition and Krull dimension. In this section we study the relation between the lattices of right (left) ideals of R and of its central subring C. As consequences are results on the ascending chain condition and on Krull dimension. For the definitions and elementary properties of Krull dimension, see [7]. To show that chain conditions transfer from C to R, we want to improve upon Theorem 2 by showing that R embeds as a C submodule of a finitely generated C module. To do this requires an additional assumption on C. When C is a domain this assumption is simply that C be integrally closed.

DEFINITION. If C is a commutative ring with 1, call C an *MIC* ring if C/P is integrally closed in its quotient field for each minimal prime ideal P of C.

For the case of primary interest, when C is a semi-prime Goldie ring, let the total ring of quotients of C be $Q(C) = C(C^*)^{-1} \cong F_1 \oplus \cdots \oplus F_t$, with e_i the identity element of the field F_i. Note that $e_iC \cong C/P_i$ for P_i a minimal prime ideal of C. This follows from writing $e_i = d_ic^{-1}$, for $d_i \in C$, and observing that $P_i = \mathrm{Ann}(d_i)$. Conversely, every minimal prime ideal is one of the P_i above, for otherwise it would contain every d_i, and so the regular element $d_1 + \cdots + d_t$. Hence, our definition is equivalent to assuming that e_iC is integrally closed in F_i. If C were integrally closed in $Q(C)$, in the usual sense, then since e_i is integral over C, it would follow that $C = e_1C \oplus \cdots \oplus e_tC$, and the minimal primes of C would be co-maximal. Of course, any finite direct sum of integrally closed integral domains is an *MIC* ring. To see that an *MIC* ring need not be integrally closed in $Q(C)$, we present an easy example.

EXAMPLE 4. Let $C = F[x, y]/(xy)$ for any field F. Then C is a semi-prime Goldie ring with 1 and has no nontrivial idempotents, so cannot be a direct sum of ideals. Now (x) and (y) are the minimal prime ideals but are not co-maximal, so our remarks above show that C is not integrally closed in $Q(C)$. Since $C/(x) \cong F[y]$ and $C/(y) \cong F[x]$, it is clear that C is an *MIC* ring.

For simplicity in stating the results in this section, we shall say that R and C are *as usual* if R is a ring with 1 and C is a subring satisfying: (i) $1 \in C \subset Z(R)$, (ii) C is a semi-prime Goldie *MIC* ring, (iii) R is C^*-torsion free, (iv) R is integral of bounded degree n over C, and (v) $n!$ is a unit in R.

Our next result is the extension of Theorem 2 mentioned above. Thus any condition on lattices of submodules which holds for sublattices, finite direct sums, and homomorphic images, will transfer from C to R. The ascending chain condition and Krull dimension are two such conditions. Others would be the nonexistence of chains of submodules of a given infinite cardinal, or the nonexistence of a well-ordered chain of a given ordinal. For this particular theorem, we do not need the invertibility of $n!$, but only that R is $n!$-torsion free.

THEOREM 3. *If R and C are as usual, and if R is a semi-prime ring, then R embeds as a C submodule of a finitely generated C module. In particular, if C is a Noetherian ring, then R is Noetherian as a C module, or if C has Krull dimension t, then R has Krull dimension at most t as a C module.*

PROOF. By the assumptions on C, one obtains from Theorem 2 that R is a semi-prime Goldie ring. Therefore, R contains a finite collection of minimal prime ideals $P_1, \ldots, P_k$ with $\bigcap P_i = 0$ [**10**; p. 73–74]. It follows from the last statement of Theorem 2, that R/P_i and $(C + P_i)/P_i$ are as usual. Clearly, as a C module, R embeds in $R/P_1 + \cdots + R/P_k$, so it suffices to prove the Theorem for R a prime ring.

As we remarked earlier, the fact that R is integral of bounded degree over C means that R satisfies a polynomial identity. A theorem of Formanek [**4**; Theorem 1, p. 79] shows that R embeds as a $Z(R)$ submodule in a finitely generated free $Z(R)$ module. Since this is also a C module embedding, to prove the theorem, it suffice to show that $Z(R)$ embeds as a C submodule of a finitely generated C module. But $Z(R)$ is integral over C of bounded degree n and char $Z(R) = 0$ or char $Z(R) > n$, so the quotient field of $Z(R)$ is a finite dimensional separable extension of the quotient field of C. It follows [**21**; Theorem 7, p. 264] that $Z(R)$ embeds as a C submodule of a finitely generated C module, completing the proof of the theorem.

Our earlier examples show the necessity of some of the hypotheses in Theorem 3. If one assumes that R is a Goldie ring the C^*-torsion free assumption may not be needed, but we are unable to show this. Although we do not know if it is necessary to assume that C is an *MIC* ring in order to prove that R embeds in a finitely generated C module, there is no hope of proving Theorem 3 without assuming that R is of bounded degree over C, even when R is a domain of characteristic zero and C is a Noetherian ring. Our example is in the spirit of [**20**; Example 4, p. 207] and also shows that bounded degree is necessary for transferring chain conditions from R to C.

EXAMPLE 5. Let $X = \{x_1, x_2, \ldots\}$ be an infinite set of indeterminates over the rational numbers, Q, and consider $T = Q[X]$ and its subring $A = Q[x_1^2, x_2^2, \ldots]$. Note that, as rings, $A \cong T$ and both are integrally closed. For the prime ideals $(x_i^2) \subset A$, set $S = A - \bigcup (x_i^2)$. Then A_S is a *PID*, T_S is integral over A_S, T_S is a Noetherian ring since each of its nonzero prime ideals is maximal and principal, and T_S is still integrally closed. It follows that $A_S[y] \subset T_S[y]$ is an integral extension of Noetherian integrally closed domains. Set $A_S[y] = C$ and $R = C + C[x_1]y + C[x_1, x_2]y^2 + \cdots$. Clearly, R is integral over C but R is not Noetherian since $(x_1y) \subset (x_1y, x_2y^2) \subset \cdots$, so that R cannot be embedded in a finitely generated C module. Furthermore, an argument like that in [**7**; Proposition 9.1, p. 60] shows that R does not have Krull dimension. Therefore, without the assumption of bounded degree, chain conditions need not transfer from C to R. To see that they may not transfer in the other direction, simply consider the integral extension $R \subset T_S[y]$.

To prove that chain conditions on R are inherited by C, for integral extensions of bounded degree n, we must assume that R and C are as usual. As in Theorem 3, we do not know if all the assumptions on C are necessary. To see that the invertibility of $n!$ and the integrally closed condition on C cannot both be discarded, we present an example of

Chaung and Lee [3; Example, p. 17] which is based on a similar example of Nagarajan [19] in the characteristic two case.

EXAMPLE 6. Chaung and Lee [3; Example, p. 17] have constructed a commutative, Noetherian integrally closed domain A of characteristic zero with an automorphism T of order two, so that the T-fixed point ring B of A is not a Noetherian ring. Now 2 is not a unit in A, but A is integral of degree two over B, since any $y \in A$ satisfies $X^2 - (y + yT)X + (yT)y = 0$. Let $R = A[x]$ and extend T to R by setting $(x)T = x$. It is easy to see that $B[x]$ is the T-fixed point ring of R, so R is again integral of degree two over $B[x]$. Although R is a Noetherian ring, $B[x]$ fails to have Krull dimension [7; Proposition 9.1, p. 60].

Before stating the theorem about chain conditions, we require a result which shows that when R is a prime ring, it is a generator when considered as a C module.

THEOREM 4. *If R and C are as usual and R is a prime ring, then there exists a C module homomorphism T of R onto C.*

PROOF. Since $C \subset Z(R)$ and R is a prime ring, R embeds in $Q(R) = R(C^*)^{-1}$ [1; §2]. It is clear that $Q(R)$ is a prime ring integral of bounded degree n over F, the ring of fractions for C, that $Z(Q(R)) = K = Z(R)(C^*)^{-1}$ is a field algebraic over F, and that $\text{char}(Q(R)) = 0$ or $\text{char}(Q(R)) > n$. It follows from Theorem 3 that $Q(R)$ is finite dimensional over K, say $\dim_K Q(R) = m^2$. Let f be the C algebra isomorphism of R into $M_{m^2}(K)$ given by the right regular representation of $Q(R)$ onto itself with respect to some fixed K basis. Then $(r)V = \text{Trace}((r)f)$ is a C linear map from R into K. Of course, $(r)V$ is the negative of the sum of the characteristic values of $(r)f$, including multiplicity. Now the minimal polynomial $m_K(x)$ of r over K is the same as that for $(r)f$ over K, and $m_K(x)$ divides $m_F(x)$, the minimal polynomial for r over F. Since C is integrally closed in F, $m_F(x) \in C[x]$ [21; Theorem 4, p. 260]. It follows that in any splitting for $m_K(x)$ over K, the roots of $m_K(x)$ are integral over C. Therefore, the characteristic values of $(r)f$ are integral over C, which forces $(r)V$ to be integral over C.

The characteristic assumption on F and the fact that K is algebraic of bounded degree n over F force K to be a finite dimensional separable extension of F with $K = F(u)$ and $\dim_F K \leqq n$. If L is a normal closure of K over F, then L is a normal (Galois) extension of F and $\dim_F L = s$ divides $n!$, so s is a unit in C. Let G be the Galois group of L over F. For $g \in G$, $((r)V)g$ is integral over C, so $((r)V)S$ is integral over C, where S is the usual Galois trace from L to F. Therefore T: $R \to C$ given by $(r)T = ((r)V)S$ is a C module homomorphism and for $c \in C$, $(c)T = sm^2c$. Since s is a unit in C, T is onto if m is also a unit in C. Recall that $m^2 =$

$\dim_K Q(R)$, and write $Q(R) \cong M_u(D)$ for D a division algebra with $\dim_K D = q^2$, so that $m = uq$. There exist elements in $M_u(K) \subset Q(R)$ whose minimal polynomial over K has degree u. But every element in $Q(R)$ satisfies a polynomial over K of degree at most n, so $u \leqq n$ and u is a unit in C. Now q is the dimension over K of a maximal subfield N of D. But $N \subset Q(R)$, so is algebraic over K of bounded degree n, and is separable over K. Thus, $\dim_K N = q \leqq n$, q is a unit in C which shows that m is a unit in C, completing the proof of the theorem.

For any ring A, let $L(A)$ be the lattice of ideals of A. The homomorphism given in Theorem 4 enables us to prove that $L(C)$ embeds in $L(R)$ for our usual situation.

THEOREM 5. *If R and C are as usual and R is a semi-prime ring, $L(C)$ embeds in $L(R)$.*

PROOF. Exactly as in the proof of Theorem 3, one uses Theorem 2 to conclude that R is a Goldie ring and then obtain an embedding of R in the direct sum of $\{R/P_i\}$, for $\{P_i\}$ of minimal prime ideals of R. Also R/P_i and $(C + P_i)/P_i$ are as usual. Given $I \in L(C)$, it is clear that $IR \in L(R)$. To see that this association is an embedding it suffices to do so in each R/P_i, and so, to assume that R is a prime ring. Now let $I, J \in L(C)$ with $J \subseteqq I$ and suppose that $JR = IR$. If T is the C module homomorphism given in Theorem 4, then $(JR)T = J(RT) = JC = J$. Similarly $(IR)T = I$ so that $J = I$ results, proving that $I \to IR$ is an embedding of $L(C)$ into $L(R)$.

Using Theorem 3 together with Theorem 5 shows that any of the chain conditions mentioned before Theorem 3 hold for right ideals of R if and only if they hold for left ideals. In this regard, we note that since R satisfies a polynomial identity, the ascending chain condition on $L(R)$ implies that R is both left and right Noetherian by a result of Cauchon [**2**; Proposition 1.1, p. 101]. When R is a Noetherian ring or has Krull dimension then it must be a Goldie ring [**7**; Corollary 3.4, p. 20], but even in this case we do not know if the C^*-torsion free assumption can be eliminated. Because of the importance of these chain conditions, we state for them one consequence of Theorem 5, using Theorem 3. Recall that any Noetherian ring has Krull dimension [**7**; Proposition 1.3, p. 7].

THEOREM 6. *Let R and C be as usual. If R is a semi-prime ring which* (*is a right Noetherian ring and*) *has right Krull dimension t, then C* (*is a Noetherian ring and*) *has Krull dimension t. Furthermore, R* (*is a left Noetherian ring and*) *has left Krull dimension t.*

When R satisfies either chain condition in Theorem 6, it must satisfy the same condition as a C module by Theorem 3. Clearly, any finitely generated R module also has this chain condition as a C module. These observations enable us to obtain an extension of Theorem 6 to the case when R is not semi-prime. In what follows, let $N(R)$ denote the prime radical of R.

THEOREM 7. *Let R and C be as usual. Then*

(a) *if R has right Krull dimension t then C has Krull dimension t, and if in addition, $N(R)$ is a finitely generated R module then R as a C module has Krull dimension t so the left Krull dimension of R is t,*

(b) *if R is a right Noetherian ring then R is Noetherian as a C module, so C is a Noetherian ring, and R is also a left Noetherian ring.*

PROOF. We prove only (a) since the proof of (b) is similar. First note that the right Krull dimensions of R and of $R/N(R)$ are the same [**7**; Corollary 5.8, p. 36]. As a semi-prime ring, C must embed in $R/N(R)$. It is straightforward to check that $R/N(R)$ and $(C + N(R))/N(R)$ are as usual. From Theorem 6 applied to $R/N(R)$ we have that C has Krull dimension t, and so, as a C module $R/N(R)$ has Krull dimension t, by Theorem 3. If $N(R) = N$ is finitely generated as an R module, then N/N^2 is finitely generated as an R/N module so has Krull dimension as a C module at most t. Thus R/N^2 has Krull dimension at most t as a C module. Repeating this argument for N/N^3 and then higher powers of N, gives that R/N^k has Krull dimension at most t as a C module. But N is nilpotent [**7**; Theorem 5.1, p. 32] and C is a C submodule of R, so R must have Krull dimension t as a C module. In particular, the left Krull dimension of R, as an R module, is at most t. Since our argument is right-left symmetric, this left Krull dimension of R must be exactly t.

III. Gabriel dimension. In our study of Gabriel dimension, we can drop the assumptions that R is C^*-torsion free and that C is integrally closed, and can work with modules over R or C rather than with the rings themselves. Before getting to a brief discussion of Gabriel dimension, we require a result on nil rings of bounded index. In general, a nil ring of bounded index need not be nilpotent. A standard example is to let $A = \mathrm{GF}(2)[x_1, x_2, \ldots]/I$ for I the ideal generated by $\{x_i^2\}$ and then take R to be all "polynomials" in A with zero constant term. Clearly, $y^2 = 0$ for all $y \in R$, but $x_1x_2 \cdots x_n \neq 0$ for all n. However, if R is a nil algebra of bounded index n over a field F with either char $F = 0$ or char $F > n$ then a result of Higman [**11**; Theorem 1, p. 2] shows that R is nilpotent. A slight modification of Higman's argument works if one assumes that R is $n!$-torsion free rather than assuming that R is an algebra. Let $S_n(x_1, \ldots, x_n)$ denote the sum of the $n!$ products of $x_1, \ldots, x_n$ taken in all possible orders. Then $S_n(x_1, \ldots, x_n) = 0$ is the linearization of the relation $x^n = 0$, and $n!x^n = S_n(x, \ldots, x)$, so R is nil of index n exactly when $S_n(r_1, \ldots, r_n) = 0$ for all

$r_1, \ldots, r_n \in R$. Using only that R is $n!$-torsion free, Higman's argument shows that $S_{n-1}(y_1, \ldots, y_{n-1}) = 0$ for $y_i \in B$, the ideal of R generated by all $(n-1)-st$ powers. By induction on n, B is nilpotent, say of index k. Using Zorn's Lemma there is an ideal V of R maximal with the properties that $B \subset V$ and $V^k = 0$. It follows easily that R/V is $n!$-torsion free and has each element nilpotent of index $n-1$, so R/V is also nilpotent. Therefore, R is nilpotent. For future reference, we state this as a proposition.

PROPOSITION 1 (HIGMAN). *If R is a nil ring of bounded index n, and is $n!$-torsion free, then R is nilpotent.*

The concept of Gabriel dimension is a categorical one defined and explored in [**5**], [**7**], and [**8**]. Our needs are better served by a noncategorical approach, so we give an equivalent noncategorical definition of Gabriel dimension. All modules under consideration will be right unital modules, and $N \leqq_R M$ means that N is an R submodule of M. For proper submodules we write $N <_R M$. If it exists, the Gabriel dimension of an R module M is an ordinal number denoted by $\mathrm{Gdim}_R M$. The definition proceeds by induction, beginning with $\mathrm{Gdim}_R M = 0$ if and only if $M = 0$. Let α be a nonlimit ordinal and suppose that $\mathrm{Gdim}_R M = \beta$ has been defined for all $\beta < \alpha$. Call A an α-simple R module if for every $0 \neq N \leqq_R A$, both $\mathrm{Gdim}_R N \nless \alpha$ and $\mathrm{Gdim}_R(A/N) < \alpha$. Then $\mathrm{Gdim}_R M = \alpha$ if $\mathrm{Gdim}_R M \nless \alpha$ and if for each $N <_R M$, M/N contains a β-simple module for $\beta \leqq \alpha$. When α is a limit ordinal, the definition $\mathrm{Gdim}_R M = \alpha$ is the same, except that $\beta < \alpha$ must hold.

The following important property of Gabriel dimension follows by induction on α.

PROPOSITION 2. *For any* $N \leqq_R M$, $\mathrm{Gdim}_R M = \sup\{\mathrm{Gdim}_R N, \mathrm{Gdim}_R(M/N)\}$, *if either side exists.*

PROOF. [**8**; Lemma 1.3, p. 462] or [**17**; Proposition 1].

It is immediate from the definition that an α-simple R module has Gabriel dimension α. The only difficulty in proving Proposition 2 from our definition is in showing that nonzero submodules of α-simple modules are also α-simple. From the definition it is easy to see that if M is the direct sum of modules $\{M_i\}$ with $\mathrm{Gdim}_R M_i = \alpha_i$, then $\mathrm{Gdim}_R M = \sup_i\{\alpha_i\}$. Since any sum of submodules is a homorphic image of a direct sum of modules, we obtain a useful consequence of Proposition 2.

PROPOSITION 3. *For any R module M and ordinal number α,*

(i) *if $\{M_i\}$ are R submodules of M with* $\mathrm{Gdim}_R M_i \leqq \alpha$, *then* $\mathrm{Gdim}_R(\sum M_i) \leqq \alpha$, *and*

(ii) *M contains a unique maximal R submodule N_α such that*

$\mathrm{Gdim}_R N_\alpha \leqq \alpha$, *and* M/N_R *contains no nonzero submodule* H *with* $\mathrm{Gdim}_R H \leqq \alpha$.

When a ring R with 1 has Gabriel dimension as a module over itself, then any R module M satisfies $\mathrm{Gdim}_R M \leqq \mathrm{Gdim}_R R$, since M can be represented as a sum of cyclic submodules. However, if M is also a ring it need not have Gabriel dimension. For example, let $M = F[X]$, the polynomial ring over a field F in an infinite set of indeterminates X. Since $\mathrm{Gdim}_F F = 1$, $\mathrm{Gdim}_F M = 1$, but M, as an M module does not have Gabriel dimension [**8**; p. 472]. This example shows that Gabriel dimension viewed as a condition on the lattice of submodules is not inherited by sublattices, unlike the other chain conditions which we have considered.

For our usual situation of R integral over a central subring C, the next result is the crucial step in showing that an R module M with Gabriel dimension as a C module satisfies $\mathrm{Gdim}_R M \leqq \mathrm{Gdim}_C M$. By M_A we shall mean M considered as an A module.

THEOREM 8. *Let R be a ring with* 1, *C a subring of R, and α a nonlimit ordinal so that*: (i) $1 \in C \subset Z(R)$; (ii) *R is integral of bounded degree n over C*; (iii) *$n!$ is a unit in R*; *and* (iv) *for any R module H,* $\mathrm{Gdim}_C H < \alpha$ *implies that* $\mathrm{Gdim}_R H < \alpha$. *If $M_R = NR$ for N_C α-simple, then* $\mathrm{Gdim}_R M \leqq \alpha$.

PROOF. First observe that (i)-(iv) are inherited by any nonzero homomorphic image of R. Thus, if $A = \mathrm{Ann}_R M = \{r \in R \mid Mr = 0\}$, we may replace R with R/A, C with $(C + A)/A$, and so assume that M_R is faithful. For $c \in C - \{0\}$, right multiplication by c on N gives rise to the C module isomorphism $Nc \cong N/\mathrm{Ann}_N(c)$, where $\mathrm{Ann}_N(c) = \{y \in N \mid yc = 0\}$. But N is an α-simple C module and $Nc \leqq_C N$ so either $Nc = 0$ or $\mathrm{Ann}_N(c) = 0$. In the first case $0 = NcR = Mc$, so $c = 0$, a contradiction. Therefore, N is $C - \{0\}$ torsion free, which implies that C is a domain. Also, $yC \cong C$ for $y \in N - \{0\}$, so that C is an α-simple C module.

Using Proposition 3 and the fact that $\alpha - 1$ exists, let $T \leqq_R M$ be maximal with the property that $\mathrm{Gdim}_R T < \alpha$. Then M/T contains no nonzero R submodule with Gabriel dimension less than α, and if $\mathrm{Gdim}_R(M/T) \leqq \alpha$ then $\mathrm{Gdim}_R M \leqq \alpha$ by Proposition 2. Suppose that $m \in M - T$ and that $mc \in T$ for some $c \in C^*$. Then $mRc \leqq_R T$ and $H = (mR + T) \leqq_R (M/T)$. Now $H_C = \sum_{r \in R}(mr + T)\, C$, and $(mr + T)\, C \cong C/A(r)$ for $A(r) = \{c \in C \mid mrc \in T\}$. Since $A(r) \neq 0$ is an ideal of C, which is α-simple, $\mathrm{Gdim}_C(C/A(r)) < \alpha$, and so by Proposition 3 $\mathrm{Gdim}_C H < \alpha$ forcing $\mathrm{Gdim}_R H < \alpha$ by (iv), contradicting the choice of T. Therefore, unless $M = T$, M/T is C^*-torsion free. Furthermore, M/T is generated as an R module by $(N + T)/T$, and we may assume that $N \cap T = 0$, for otherwise $\mathrm{Gdim}_C(N/N \cap T) < \alpha$ and as above we can represent M/T as a sum of

C modules of Gabriel dimension less that α to obtain $\mathrm{Gdim}_R(M/T) < \alpha$. Consequently, replacing M with M/T leaves all of our hypotheses unchanged, and in addition we have that M is C^*-torsion free and that M_R has no nonzero submodule of Gabriel dimension less that α. Of course, we still have that C is a domain and is α-simple.

Next we show that we may assume that R is a semi-prime ring. Let $r \in R$ and $c \in C^*$ so that $rc = 0$. Then $Mrc = 0$, so the C^*-torsion freeness of M forces $r \in \mathrm{Ann}\, M$. Clearly, $C \cap \mathrm{Ann}\, M = 0$ so C embeds in $R/\mathrm{Ann}\, M$ and replacing R with this factor ring leaves us in the same situation as before, except that now R is C^*-torsion free. Since R is integral of bounded degree n over C, and is C^*-torsion free, the nil radical K of R is nil of bounded index n. But R is $n!$-torsion free, so by Proposition 1, K is nilpotent of some index t. Consider $M \supset MK \supset \cdots \supset MK^{t-1} \supset 0$. Each quotient $V_i = MK^i/MK^{i+1}$ is naturally an R/K module, $\mathrm{Gdim}_R V_i = \mathrm{Gdim}_{R/K} V_i$, and from Proposition 2 to show $\mathrm{Gdim}_R M \leqq \alpha$ it suffices to show $\mathrm{Gdim}_R V_i \leqq \alpha$. Observe that $C \cap K = 0$ so that C embeds in R/K and remains a domain which is α-simple. Clearly, hypotheses (i)-(iv) still hold for R/K and C, and it is easy to see that R/K is C^*-torsion free. Therefore, replacing R with R/K allows us to assume that R is a semi-prime ring, and it is surely enough to prove that $\mathrm{Gdim}_R M \leqq \alpha$ for any R module M. Our comment after Proposition 3 shows that we need only prove $\mathrm{Gdim}_R R \leqq \alpha$.

As a commutative domain, C is a Goldie ring, so Theorem 2 may be used to obtain that R is a Goldie ring. It follows that there exist prime ideals $P_1, \ldots, P_k$ of R so that $\bigcap P_i = 0$ and as an R module, R embeds in the direct sum of $\{R/P_i\}$. The proof will be finished if we show $\mathrm{Gdim}_R(R/P_i) \leqq \alpha$ for each i. Each R/P_i is a $(C + P_i)/P_i$ module and C is α-simple, so $\mathrm{Gdim}_C(R/P_i) \leqq \alpha$. If $\mathrm{Gdim}_C(R/P_i) < \alpha$, then from assumption (iv) $\mathrm{Gdim}_R(R/P_i) < \alpha$. On the other hand if $\mathrm{Gdim}_C(R/P_i) = \alpha$ then $(C + P_i)/P_i \cong C$ so $C \cap P_i = 0$. Thus C embeds in $A_i = R/P_i$ as an α-simple C module. The hypotheses (i)-(iv) hold for A_i and C, and A_i is a prime ring satisfying a polynomial identity. It follows [17; Theorem 4] that $\mathrm{Gdim}_R A_i = \mathrm{Gdim}_{A_i} A_i \leqq \alpha$, completing the proof of the theorem.

It is now easy to obtain one direction of the general result on Gabriel dimension.

THEOREM 9. *Let R be a ring with* 1 *and C a subring of R satisfying*: (i) $1 \in C \subset Z(R)$; (ii) *R is integral of bounded degree n over C*; *and* (iii) *$n!$ is a unit in R. For any R module M, if* $\mathrm{Gdim}_C M = \alpha$, *then* $\mathrm{Gdim}_R M \leqq \alpha$. *In particular, if* $\mathrm{Gdim}_C C = \alpha$, *then* $\mathrm{Gdim}_R R \leqq \alpha$.

PROOF. Proceed by induction on α, assuming the theorem holds for R modules M with $\mathrm{Gdim}_C M < \alpha$. Use Proposition 3 to obtain $H \leqq_R M$

maximal with $\mathrm{Gdim}_R H \leqq \alpha$. Then for $T \leqq_R M/H$ with $\mathrm{Gdim}_R T \leqq \alpha$, one must have $T = 0$. Assuming that $M/H \neq 0$, $\mathrm{Gdim}_C(M/H) \leqq \alpha$, so $(M/H)_C$ contains a β-simple module N. From Theorem 8 one may conclude that $\mathrm{Gdim}_R(NR) \leqq \beta \leqq \alpha$, contradicting the choice of H. Therefore $H = M$ and $\mathrm{Gdim}_R M \leqq \alpha$, proving the theorem.

That some assumption about n-torsion freeness is required in our last two theorems can be seen from Example 2. We present an example which shows that $n!$-torsion freeness is not sufficient to guarantee that M_R has Gabriel dimension when M_C does.

EXAMPLE 7. Let J be the ring of integers, $X = \{x_1, x_2, \ldots\}$ a set of indeterminates indexed by the positive integers, and set $R = J[X]$. Denote by I the ideal of R generated by x_i^2 and $2x_i$ for all $x_i \in X$. Then if $C = I + J$, it is easy to see that $p^2(x) \in C$ for every $p(x) \in R$, so R is integral of bounded degree two over C, and R is torsion free. Set $M = R/(I + 2R)$ and observe that M is a ring whose ideals are its R submodules. Also, M_C is in effect $M_{\mathrm{GF}(2)}$, so that $\mathrm{Gdim}_C M = 1$, and $M_R = (1_M \cdot C)R$ is generated as an R module by the 1-simple C module $1_M \cdot C \cong \mathrm{GF}(2)$. To prove that M_R fails to have Gabriel dimension, it suffices to show M_M fails to have it. Clearly, as a ring, $M \cong \mathrm{GF}(2)[X]/K$ for K the ideal generated by all x_i^2. If $\mathrm{Gdim}_M M$ exists, let N be a β-simple submodule. It is straightforward to show that N must contain a monomial, say $x_1 x_2 \cdots x_k$ by rearrangement of indices, so $x_1 \cdots x_k M$ is a β-simple M submodule of M. But as an M module $x_1 \cdots x_k M \cong M/\mathrm{Ann}_M(x_1 \cdots x_k) = M/(x_1 M + \cdots + x_k M) = T$, and the M submodules of T are exactly the ideals of $T \cong \mathrm{GF}(2)[x_{k+1}, x_{k+2}, \ldots]/L$ for L the ideal generated by $\{x_j^2 \mid j \geqq k + 1\}$. Evidently T and M are isomorphic rings, so their lattices of ideals are the same. Consequently, the lattice of ideals of M is the same as the lattice of M submodules of T_M, so M_M must be β-simple, implying that $\mathrm{Gdim}_M(x_1 \cdots x_k M) < \beta$, a contradiction. Therefore, M_R cannot have Gabriel dimension.

To show that R modules with Gabriel dimension have Gabriel dimension as C modules, we begin with a result about simple modules analogous to Theorem 8.

THEOREM 10. *Let R be a ring with* 1, *C a subring of R, and α a nonlimit ordinal so that*: (i) $1 \in C \subset Z(R)$; (ii) *R is integral of bounded degree n over C*; (iii) *$n!$ is a unit in R; and* (iv) *for any R module H,* $\mathrm{Gdim}_R H < \alpha$ *implies that* $\mathrm{Gdim}_C H < \alpha$. *If M_R is α-simple, then* $\mathrm{Gdim}_C M \leqq \alpha$.

PROOF. For any $c \in C - \{0\}$, $Mc \leqq_R M$ and $Mc \cong M/\mathrm{Ann}_M(c)$ as R modules. The α-simplicity of M_R forces either $Mc = 0$ or $\mathrm{Ann}_M(c) = 0$. As in the proof of Theorem 9, we may replace R with $R/\mathrm{Ann}\, M$ and C with

$(C + \mathrm{Ann}M)/\mathrm{Ann}\, M$, so we may assume that M_R is faithful and is C-$\{0\}$ torsion free. Thus, C is a domain and $C^* = C - \{0\}$.

Let $W \leqq_R M$, $A = \mathrm{Ann}_R W$, and $V = \{m \in M \mid mc \in W$ for some $c \in C^*\}$. Since C is a domain, $V \leqq_R M$. Should $V = M$, then for each $m \in M$ and some $c \in C^*$, $mc \in W$, which implies $mcA = mAc = 0$. The C^*-torsion freeness of M forces $mA = 0$, so $MA = 0$ and thus $A = 0$. This argument shows that if $A \neq 0$ then $V \neq M$. Now $W \subset V$, so if both $W \neq 0$ and $A \neq 0$, V must be a proper nonzero submodule of M. In this case M/V is C^*-torsion free, for if $mc \in V$, for $m \in M$ and $c \in C^*$, then $mcc_1 \in W$ for $c_1 \in C^*$ so by definition $m \in V$. But M_R is α-simple, so $\mathrm{Gdim}_R(M/V) < \alpha$ forcing $\mathrm{Gdim}_C(M/V) < \alpha$ by assumption (iv). Take $\bar{m} \in M/V - \{0\}$ and observe that $\bar{m}C \cong C$ as C modules, so $\mathrm{Gdim}_C C < \alpha$. Our comment after Proposition 3 gives $\mathrm{Gdim}_C M < \alpha$, proving the theorem. Therefore, we are finished unless each nonzero submodule of M_R is faithful. With this assumption, it is clear that R is a prime ring. By Theorem 2 R is a Goldie ring and has finite uniform dimension, say t.

Next, let $m \in M - \{0\}$ and set $A = \mathrm{Ann}_R(m)$. For $v \in A$, let $p(x) \in C[x]$ be the monic polynomial of least degree satisfied by v. Clearly, v is regular in R exactly when $p(0) \neq 0$. But $mv = 0$ implies that $0 = mp(v) = mp(0)$, so v must be a zero divisor in R using the C^*-torsion freeness of M. Consequently, no element of A is regular in R which means that A cannot be an essential right ideal of R. Let U be a uniform right ideal of R so that $A \cap U = 0$. Then as R modules, $mU \cong U$, and U_R is α-simple. Therefore, every uniform right ideal of R is an α-simple R module [**6**; Lemma 2.3, p. 594], and $W = U_1 \oplus \cdots \oplus U_t$ is essential in R for some choice $\{U_i\}$ of uniform right ideals of R. For $y \in W$, regular in R and satisfying the monic polynomial $q(x) \in C[x]$ of least degree, $q(0) \neq 0$ and $d = q(0) = q(0) - q(y) \in W \cap C^*$. Clearly, $dR \leqq_R W$ and gives an R module embedding of R into W. If $I \neq 0$ is an ideal of C, dIR is an ideal of R, so the primeness of R yields $dIR \cap U_i = W_i \neq 0$. Thus $\oplus W_i \leqq dIR \leqq dR \leqq W$. Since $W/\oplus W_i \cong \oplus(U_i/W_i)$ and each U_i is an α-simple R module, it follows that $\mathrm{Gdim}_R(W/\oplus W_i) < \alpha$, forcing $\mathrm{Gdim}_C(W/\oplus W_i) < \alpha$ by hypothesis. From Proposition 2 we have $\mathrm{Gdim}_C(dR/dIR) < \alpha$, and $dR/dIR \cong R/IR$ yields $\mathrm{Gdim}_C(R/IR) < \alpha$. Hence $\mathrm{Gdim}_C((C + IR)/IR) = \mathrm{Gdim}_C(C/(C \cap IR)) < \alpha$.

Our goal is to improve that last inequality to $\mathrm{Gdim}_C C/I < \alpha$; for then, if $\mathrm{Gdim}_C I < \alpha$ we would have $\mathrm{Gdim}_C C < \alpha$, and if $\mathrm{Gdim}_C I \nless \alpha$ for all I, then C_C is α-simple. In either case, $\mathrm{Gdim}_C C \leqq \alpha$, so $\mathrm{Gdim}_C M \leqq \alpha$ follows. Note that for $y \in I - \{0\}$, yC is a nonzero ideal of C contained in I, so to show $\mathrm{Gdim}_C C/I < \alpha$, it suffices to show that $\mathrm{Gdim}_C C/yC < \alpha$. Apply the inequality we have obtained to $I = yC$ and observe that $C \cap (yC)R = C \cap yR$, so $\mathrm{Gdim}_C C/(C \cap yR) < \alpha$. Choose $yr \in C \cap yR$. If $f(x) \in C[x]$ is the monic polynomial of least degree, say k, satisfied by

r, then $0 = y^k f(r) = (yr)^k + yc_1(yr)^{k-1} + \cdots + y^k c_k$. But $yr \in C$ implies that $(yr)^j \in C$ and it follows that $(yr)^k \in yC$. Thus, in C/yC the ideal $(C \cap yR)/yC$ is nil of bounded index n. Hypothesis (iii) gives us that C/yC is $n!$-torsion free, so Proposition 1 enables us to conclude that $(C \cap yR)^s \subset yC$, for some s. Consider the chain of C modules $C \supset yR \cap C \supset (yR \cap C)^2 \supset \cdots \supset (yR \cap C)^s$. Each quotient $V_i = (yR \cap C)^i/(yR \cap C)^{i+1}$ is a $C/(yR \cap C)$ module and $\mathrm{Gdim}_C(C/(yR \cap C)) < \alpha$ implies that $\mathrm{Gdim}_C V_i < \alpha$. Using Proposition 2 yields $\mathrm{Gdim}_C(C/(yR \cap C)^s) < \alpha$. However, $yC \supset (yR \cap C)^s$, so $\mathrm{Gdim}_C C/yC < \alpha$ as desired, completing the proof of the theorem.

Using Theorem 10, the result for arbitrary R modules follows fairly easily.

THEOREM 11. *Let R be a ring with* 1 *and C a subring of R satisfying*: (i) $1 \in C \subset Z(R)$; (ii) *R is integral of bounded degree n over C*; *and* (iii) *$n!$ is a unit in R. For any R module M, if* $\mathrm{Gdim}_R M = \alpha$, *then* $\mathrm{Gdim}_C M \leqq \alpha$. *In particular, if* $\mathrm{Gdim}_R R = \alpha$, *then* $\mathrm{Gdim}_C C \leqq \alpha$.

PROOF. Proceed by induction on α. Given M_R with $\mathrm{Gdim}_R M = \alpha$, then by definition, M contains a β-simple submodule for $\beta \leqq \alpha$. Consequently, either the induction assumption, or Theorem 10 guarantees that M contains a nonzero R submodule N with $\mathrm{Gdim}_C N \leqq \alpha$. Using Proposition 3, there exists a nonzero, maximal such R submodule H. If $M/H \neq 0$, then a repetition of the argument shows that M/H contain a nonzero R submodule T/H with $\mathrm{Gdim}_C T/H \leqq \alpha$. But then $\mathrm{Gdim}_C T \leqq \alpha$, contradicting the choice of H. Consequently, $H = M$ proving the theorem.

The following theorem is an immediate corollary of Theorem 9 and Theorem 11.

THEOREM 12. *Let R be a ring with* 1 *and C a subring of R satisfying*: (i) $1 \in C \subset Z(R)$; (ii) *R is integral of bounded degree n over C*; *and* (iii) *$n!$ is a unit in R. For any R module M*, $\mathrm{Gdim}_R M = \mathrm{Gdim}_C M$, *if either exists. In particular* $\mathrm{Gdim}_R R = \mathrm{Gdim}_C C$ *if either exists.*

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