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ALMOST PERIODIC FUNCTIONS ON SEMITOPOLOGICAL SEMIGROUPS

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Let S be a semitopological semigroup with identity e. One of the ways of defining almost periodicity of a function $f \in C(S)$ is to say that, $(*)$ for each $\varepsilon > 0$, there is a finite subset $A \subset S$ such that the set of left translates $\{L_s f \mid s \in S\}$ is contained in $\{h \in C(S) \mid \|L_s f - h\| < \varepsilon$ for some $s \in A\}$. By allowing the subset A to be totally bounded (suitably defined) or compact in this and related definitions, one obtains a large number of function classes. Recently T. Kayano defined some of these classes (none involving A compact) and gave some relationships among them. In the present note, we tighten up some of his results and show in particular that $f \in C(S)$ is almost periodic if f satisfies the condition $(*)$ above with totally bounded sets A and also $\|L_s f - f\| \to 0$ whenever $s \to e$. We also present some examples and a theorem showing some classes can be different; for example, functions satisfying $(*)$ with sets A compact or totally bounded need not be almost periodic.

Preliminaries and first results. A semigroup S with identity e that is also a topological space is called a *semitopological semigroup* if the maps $s \to st$ and $s \to ts$ are continuous from S into S for all $t \in S$; if, as well, S admits inverses, i.e., if S is a group, it is called a *semitopological group*. We denote by $C(S)$ the space of bounded continuous complex-valued functions on S furnished with the supremum norm $\|f\| = \sup_{s \in S}|f(s)|$. The *left translate* $L_t f$ of $f \in C(S)$ by $t \in S$ is defined by $L_t f(s) = f(ts)$ for all $s \in S$, and a subset A of S is called *right* [*left*] *totally bounded* if, given any neighbourhood V of e, there exist a natural number N and $t_1, \ldots, t_N \in A$ such that

$$A \subset \bigcup \{Vt_i \mid 1 \leqq i \leqq n\} \; [A \subset \bigcup \{t_i V \mid 1 \leqq i \leqq N\}].$$

We now wish to define some classes of functions in $C(S)$, classes involving a family $\mathscr{A}$ of subsets of S. We shall say $f \in C(S)$ satisfies conditions

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I, II, III or IV with respect to $\mathscr{A}$ if, for every $\varepsilon < 0$, there is an $A \in \mathscr{A}$ such that $\{s \in A \mid \|L_{rs}f - L_{rt}f\| < \varepsilon \text{ for all } r \in S\} \neq \emptyset$ for all $t \in S$, $\{s \in A \mid \|L_s f - L_t f\| < \varepsilon\} \neq \emptyset$ for all $t \in S$, $\{s \in At \mid \|L_s f - f\| < \varepsilon\} \neq \emptyset$ for all $t \in S$, or $A\{s \in S \mid \|L_s f - f\| < \varepsilon\} = S$, respectively. We define a function $f \in C(S)$ to belong to class I_F, I_T or I_C if there is a family $\mathscr{A}$ of finite subsets, right totally bounded subsets, or compact subsets, respectively, of S such that f satisfies condition I with respect to $\mathscr{A}$. Eight more classes II_F through III_C and IV_F and IV_C are analogously defined. Class IV_T is defined also analogously, but in terms of sets that are left totally bounded.

REMARKS. 1. A function $f \in C(S)$ is *almost periodic*, written $f \in AP(S)$, if $\{L_s f \mid s \in S\}$ is relatively norm compact in $C(S)$. Thus $f \in AP(S)$ is equivalent to $f \in \mathrm{II}_F$. See [**1**, §III.9], for example.

2. In case S is a topological group, the class IV_C coincides with the class of functions left almost periodic in the sense of Bohr; see [**2**, **3**] (and note that, in [**3**], "uniformly almost periodic" means "almost periodic in the sense of Bohr").

3. This paper was inspired by [**4**] and the definitions above that involve conditions I-III and finite subsets or totally bounded subsets appear there. (Our notation differs from that in [**4**].) It is clear that many more definitions can be made by replacing "left" with "right" some or all of the time. We mention that condition I is in fact symmetrical, since the set of its definition can be written as $\{s \in A \mid |f(rsu) - f(rtu)| < \varepsilon \text{ for all } r, u \in S\}$. We mention as well that, if condition III is altered to condition III′ by replacing At with tA, then $\mathrm{III}'_F = \mathrm{III}_F$ for all semitopological groups G, and $\mathrm{III}'_T = \mathrm{III}_T$ and $\mathrm{III}'_C = \mathrm{III}_C$ if G also has continuous inversion. These assertions follow because the set $\{s \in G \mid \|L_s f - f\| < \varepsilon\}$ is symmetric.

THEOREM 1. *Let S be a semitopological semigroup. Then the following assertions hold.*

(i) $\mathrm{I}_F = \mathrm{II}_F = AP(S)$, $\mathrm{I}_T \subset \mathrm{II}_T$, $\mathrm{I}_C \subset \mathrm{II}_C$ *and, of course,* $\mathrm{I}_F \subset \mathrm{I}_T$, $\mathrm{I}_F \subset \mathrm{I}_C$, *etc.*

(ii) *If S is a semitopological group, we also have* $\mathrm{II}_F = \mathrm{III}_F = \mathrm{IV}_F$, $\mathrm{II}_T = \mathrm{III}_T$, *and* $\mathrm{II}_C = \mathrm{III}_C$.

(iii) *If S also has continuous inversion, we get as well* $\mathrm{III}_T = \mathrm{IV}_T$, $\mathrm{III}_C = \mathrm{IV}_C$.

(iv) *If S is a topological group, we get the further equality* $\mathrm{I}_C = \mathrm{II}_F$.

(v) *Finally, if S is a locally compact topological group, we get further equalities* $\mathrm{I}_T = \mathrm{I}_C$, $\mathrm{II}_T = \mathrm{II}_C$, $\mathrm{III}_T = \mathrm{III}_C$, and $\mathrm{IV}_T = \mathrm{IV}_C$.

PROOF. Much of this is obvious and well known. We note first that, if S is a semitopological group, $A \subset S$ and $f \in C(S)$, then $\{s \in A \mid \|L_s f - L_t f\| < \varepsilon\} \neq \emptyset$ if and only if $\{s \in At^{-1} \mid \|L_s f - f\| < \varepsilon\} \neq \emptyset$, and either of these inequalities holds for all $t \in S$ if and only if $A^{-1}\{s \in S \mid \|L_s f - f\| < \varepsilon\} = S$. (The appearance of A^{-1} in the last line is the reason that left totally bounded sets were used in the definition of class IV_T and that continuous inversion is needed to get the equalities of (iii) here.) The proof of (iv) is in [**3**, Chapter 4].

Theorem 1 is not the same as the analogous result, Theorem 4 in [**4**], partly because the sets of function classes considered here and in [**4**] are not identical and because some containments are omitted in [**4**] (e.g., $\mathrm{I}_F \subset \mathrm{II}_F \subset AP(S)$ always, which is part of (i) above), but also because Theorem 4 in [**4**] involves an unnecessary condition on the topology of S. Theorem 3 in [**4**] also involves this condition unnecessarily and our next theorem here (which should be viewed as the generalization to semitopological semigroups of part of Theorem 4.61 in [**3**]) gets the conclusion of Theorem 3 in [**4**] from hypotheses that are as weak as possible. We need first some definitions.

If S is a semitopological semigroup, a function $f \in C(S)$ is called *left* [*right*] *uniformly continuous*, LUC [RUC] for short, at s if

$$\|L_{s_\alpha} f - L_s f\| \to 0 \ \ [\|R_{s_\alpha} f - R_s f\| \to 0]$$

whenever $s_\alpha \to s$ in S ($R_s f$ being the *right translate*, $R_s f(t) = f(ts)$); f is called LUC (RUC) if it is LUC (RUC) at every point of S. The almost periodic functions are LUC and RUC; see [**1**]. In case S is a semitopological group, we have, for example, that $f \in C(S)$ is LUC if and only if, given $\varepsilon > 0$, we can find a neighbourhood V of e such that $|f(s) - f(t)| < \varepsilon$ whenever $st^{-1} \in V$; also, if f is LUC at one point, f is LUC (at every point). This last assertion fails for many semigroups; see Example 3 and Theorem 3 ahead. We remark that the notation of this paragraph is similar to that in [**1**, **7**], but opposite to that in [**3**, **8**].

THEOREM 2. *Let S be a semitopological semigroup. Suppose f satisfies* II_T *and is* LUC *at e. Then $f \in AP(S)$.*

PROOF. Suppose $\varepsilon > 0$ is given. Let $A \in \mathscr{A}$ be such that $\{s \in A \mid \|L_s f - L_t f\| < \varepsilon/2\} \neq \varnothing$ for all $t \in S$. Let V be a neighbourhood of e such that $\|L_r f - f\| < \varepsilon/2$ for all $r \in V$. Since A is right totally bounded, there exist a natural number N and $t_1, \ldots, t_N \in A$ such that $A \subset \bigcup \{Vt_i \mid 1 \leqq i \leqq N\}$. Then, if $t \in S$ and $s \in A$ satisfies $\|L_s f - L_t f\| < \varepsilon/2$, we have $s = rt_i$ for some $r \in V$ and some i, $1 \leqq i \leqq N$, and

$$\begin{aligned}\|L_t f - L_{t_i} f\| &\leqq \|L_t f - L_s f\| + \|L_s f - L_{t_i} f\| \\ &= \|L_t f - L_s f\| + \|L_{t_i}(L_r f - f)\| < \varepsilon.\end{aligned}$$

Hence $f \in AP(S)$.

Examples and further results. We now present three examples which show that the classes of functions of the previous section do not always all coincide. An example like the first one here was given by Wu [8]; see also [**6**].

EXAMPLES. 1. Let S be the semidirect product of the circle group and the plane, $S = T \times C$ with multiplication $(w, z)(w', z') = (ww', z + wz')$. ($S$ is the Euclidean group of the plane.) Then one verifies readily that the function f defined by $f(w, z) = f(w, x + iy) = e^{ix}$ is in IV_C (which equals each of III_C, II_C, IV_T, III_T and II_T), but is not LUC, hence is not in $AP(S)$ (which equals each of I_F, II_F, III_F, IV_F, I_T and I_C).

2. Let S be the rational numbers modulo 1. Then every $f \in C(S)$ is in I_T (which equals each of II_T, III_T and IV_T), but precisely the uniformly continuous functions in $C(S)$ are in all the other classes as well.

3. Let $S = R \cup \{\infty\}$ be the one-point compactification of the usual additive real numbers with ∞ acting as a zero for S, i.e., $\infty s = s\infty = \infty\infty = \infty$. Then every $f \in C(S)$ is in I_C (which equals each of II_C, III_C and IV_C), but the constant functions are the only ones that are in all the other classes as well. This semigroup illustrates two problems with applying group-theoretic methods to semigroups: although S is compact, it is not (left or right) totally bounded as defined above; also, every $f \in C(S)$ is LUC (equivalently, RUC) at e, but only the constant functions are LUC at infinity.

Amplifying on the ideas involved in the two comments just made about the last example, we present our last theorem, which also shows that $f \in \mathrm{II}_C$ does not always imply f is RUC for functions on semitopological semigroups as it does for functions on topological groups [**3**, Theorem 4.58]. (Example 2 shows $f \in \mathrm{II}_T$ need not imply f is RUC.)

THEOREM 3. *Let S be a compact semitopological semigroup. Then every $f \in C(S)$ is* LUC *and* RUC *at e. Also, the following three assertions about S are pairwise equivalent.*

(a) *S is a topological semigroup, i.e., multiplication is a continuous function from the product space $S \times S$ into S.*

(b) $\mathrm{II}_T = \mathrm{II}_C$.

(c) *Every $f \in C(S)$ is* LUC (*or* RUC).

Furthermore, each of (a), (b) *and* (c) *is implied by*

(d) *S is right* (*or left*) *totally bounded* (*as defined above*).

PROOF. The first assertion is proved using a result of Lawson [**5**, Proposition G.1] and then the method of proof of a result of Mitchell [**7**, Theorem 7].

Let S be a semitopological semigroup. It is clear that (d) implies (b). Suppose (b) holds. Then $C(S) = \mathrm{II}_C = \mathrm{II}_T$, and Theorem 2, coupled with the first assertion of this theorem implies $C(S) = AP(S)$. Hence (a) holds [**1**; especially §III.9]. If $f \in C(S)$ and f is LUC, then the map $s \to L_s f$: $S \to C(S)$ is continuous, hence has compact image, i.e., $f \in AP(S)$. Thus (c) implies (a). If (a) holds, then $C(S) = AP(S)$, hence $\mathrm{II}_F = AP(S) = \mathrm{II}_T = \mathrm{II}_C$ and (b) holds; also, (c) holds [**1**, III.14.5 (ii)].

REMARK. To see that condition (a) need not imply condition (d), take any infinite compact topological semigroup and adjoin an identity as a discrete point.

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