ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 3, Summer 1982

DIFFERENTIATION ON THE DUAL OF A GROUP: AN INTRODUCTION

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Introduction. This paper is the first of a series wherein we investigate a theory of differentiation on the dual of a group together with its applications to and its interrelationships with other aspects of the theory of groups and their representations. One of the major objectives in the study of locally compact group G is the computation of $\hat{G}$, the set of all (unitary equivalence classes of) continuous, unitary, irreducible representations of G equipped with a natural topology, the Fell topology, cf. 3.4, 18.1 [**8**]. There are immediate problems; first of all it is usually very difficult to compute all the elements of $\hat{G}$. Secondly, $\hat{G}$ with its natural topology is rarely Hausdorff; and elementary examples yield spaces $\hat{G}$ in which not all points are even closed. Finally $\hat{G}$, in general, seems to lack any natural elementary algebraic structure. For these reasons the study of a "differentiable structure" or of any other structure of $\hat{G}$ in any generality is very difficult; to see any structure at all one usually is forced to investigate specific classes of groups—or, indeed, specific groups. Of course, a great deal of study directed at specific groups and specific classes of groups has been carried out with immense success over the last several decades; and we shall directly benefit from these studies.

The seeming intractibility of $\hat{G}$ leads us to consider other structures closely related to $\hat{G}$, namely, the space of all continuous, unitary representations of G, denoted $\mathrm{Rep}_u(G)$. This space can be refined even further to $P(G)_1$, the space of diagonal coefficients (of norm one) of the elements of $\mathrm{Rep}_u(G)$, i.e., $P(G)_1$ is the collection of all continuous functions of positive type (of norm one) on G. The space $P(G)$ is often called the space of positive definite functions. (There are strong indications that even larger spaces can be usefully employed in the study of G and $\hat{G}$, but we leave this possibility totally untouched for now.)

The space $P(G)_1$ is a convex, Hausdorff, topological semigroup. To recover $\hat{G}$ one needs "only" to find the extreme points of $P(G)_1$ and then perform a canonical construction. Thus a thorough understanding of

*This research has been supported by the Alfred P. Sloan Foundation and the National Science Foundation.

Received by the editors on February 17, 1981, and in revised form on June 3, 1981.

$P(G)_1$ leads to a thorough understanding of $\hat{G}$. Now $P(G)_1$ is not a locally compact, differentiable, manifold, Nevertheless, it does have a certain amount of natural differentiable structure. In this paper we shall investigate, almost exclusively, the differentiable structure that $P(G)_1$ has at its identity, **1**. We shall start very naively with the classical (but very fundamental) notion of "limit of difference quotients"; and we shall see where this leads us. It turns out that $P(G)_1$ always has some differentiable structure at **1**, even if G is discrete! We were led, using elementary considerations from the theory of operator algebras, i.e., C^* algebras, to a natural characterization of "derivatives" at **1** in $P(G)_1$. Geometrically a "derivative" at **1** will be called a *semitangent vector at* **1**, the collection of all such vectors is denoted by $N_0(G)$. Algebraically these semitangents have a characterization as the class of (normalized) continuous functions of negative type on G.

With these characterizations of $N_0(G)$ in hand, a literature search showed that our semitangents appeared throughout all mathematical theories related to groups. Semitangents, though not called such, appear in physics, probability theory, imbedding problems, cohomology of group actions, and so on. To be specific the following five objects or classes of objects are all examples of semitangent (or sets of semitangent) vectors! Though four examples are in the relatively elementary context of $\mathbf{R}^n$, n-dimensional Euclidean space, all five examples generalize completely.

(1) $D = d^2/dx^2$ on $\mathbf{R}^1$; $D(f\bar{f}) \geqq (D\bar{f})f + \bar{f}(Df)$, pointwise inequality of functions. This is a simple example of a completely dissipative operator from physics. The operator D is a semitangent vector concretely realized by what we shall call a *semiderivation.*

(2) The Lévy-Khinchin formula on $\mathbf{R}^n$:

$$-\psi(y) = c + i\ell(y) + q(y)$$
$$+ \int_{\mathbf{R}^n-\{0\}} \left[1 - \exp(-i(x|y)) - \frac{i(x|y)}{1+\|x\|^2}\right]\left[\frac{1+\|x\|^2}{\|x\|^2}\right] d\mu(x)$$

where $x, y \in \mathbf{R}^n$, $c \geqq 0$, ℓ is a continuous linear form, q is a continuous, nonnegative quadratic form and μ is a non-negative bounded measure on $\mathbf{R}^n - \{0\}$ such that the above integral converges.

(3) A function ψ on $\mathbf{R}^1$ that satisfies $\psi(e) \leqq 0$, $\psi(-x) = \overline{\psi(x)}$, and

$$\int_{\mathbf{R}^1} \psi(x)\left(\frac{d\varphi}{dx}\right)^{\#} * \frac{d\varphi}{dx}(x)dx \geqq 0 \text{ for all } \varphi \in C_c^\infty(\mathbf{R}^1),$$

where $\varphi^{\#}(x) = \overline{\varphi(-x)}$, the bar denoting the complex conjugate, $*$ denoting convolution, and φ an infinitely differentiable function with compact support.

(4) $H^1(G, H(\pi)) = Z^1(G, H(\pi))/B^1(G, H(\pi))$, the first cohomology group of continuous, unitary representation π of G.

(5) The "screw functions" of J. von Neumann and I.J. Schoenberg

which allow isometric imbeddings of $\mathbf{R}^n$ into Hilbert space. (A corollary: ℓ^p, $1 \leqq p \leqq \infty$, $p \neq 2$, cannot be isometrically imbedded in any Hilbert space.)

The main point of this paper is that a unified treatment of seemingly diverse phenomena (which seem even more diverse in the context of a general locally compact group action) can be achieved by means of a thorough study of the fundamental classical notion of differentiation on, albeit a non classical space, the dual of a group. We have tried to motivate thoroughly and explain for the non specialist every aspect of the development. It would be helpful but not absolutely necessary if the reader were familiar with §13, [**8**]. We have additional results in almost all of the areas touched upon in this paper which we have not included in the hopes of keeping this paper a comfortably readable size. Any one of the topics discussed could easily have an entire volume devoted to it. We also mention that strictly speaking we are dealing in this paper with (possibly) unbounded, semidifferential operators that arise from the space $\hat{G}$, or $P(G)_1$. There is a growing body of research on derivations associated with function spaces on $\hat{G}$: e.g., $C^*(G)$. We do not address this latter topic in any direct way at this time.

It is our hope that this program of study of $\hat{G}$ will allow us to understand the structure of $\hat{G}$ in terms of the algebraic and geometric structure of G—even if we are not able to concretely compute every element in $\hat{G}$.

At this time the author would like to acknowledge the generous support he received as an Alfred P. Sloan fellow. Without this support and that of the National Science Foundation this and related papers probably would not have been written for many years, if ever. The author would like to kindly thank Professors E.G. Effros, R.V. Kadison, L. Pukansky, J. Rosenberg, and their colleagues at the University of Pennsylvania for their hospitality while part of this paper was being written. Also, I would like to thank Professors C.C. Moore, M. Rieffel, M. Takesaki, W. Arveson, C. Akemann and their colleagues in the University of California system for their hospitality while another part of this paper was being written. In particular, E.C. Gootman, a contemporaneous visitor at the University of Pennsylvania, showed me an elegant proof to the first part of Proposition 1. We also had conversations with P. Muhly, a contemporaneous vistor at U.C. Berkely and S. Wassermann a visitor at the University of Pennsylvania. Professor C.C. Moore introduced me to Proposition 9 by proving it in a seminar. I would also like to thank my student, Eric Larsen, for taking an interest in and making a contribution to this subject.

1. Translation on the Dual of G. If G is a locally compact, abelian group there is a group $\hat{G}$, the group (under pointwise multiplication) of continuous homomorphisms of G into the complex numbers of norm one, i.e., characters, which is again (with the compact-open topology) a locally compact abelian group called the *dual group of* G [**31**]. A *translation of* $\hat{G}$ *by* $x_0 \in \hat{G}$ is defined as a map of the form $x \in \hat{G} \mapsto x_0 x \in \hat{G}$. If we look at the transpose of this map, T_{x_0}, on a function space like $C_0(\hat{G})$, the complex algebra of continuous functions which vanish at infinity on $\hat{G}$, we gain a new perspective and a new tool for studying translation maps.

The importance of the latter perspective becomes more apparent in the case of non-commutative locally compact groups. In this more general setting a dual group is not to be found [**37**]; and the correct choice of the analogue of $\hat{G}$, or the "underlying dual space" is not overwhelmingly obvious. In fact various researchers in the field have experimented with different choices for the dual space of G with varying success. Strangely enough, however, there is good agreement as to what the "natural" analogue of $C_0(\hat{G})$ should be; it is $C^*(G)$, the universal enveloping C^*—algebra of $L^1(G)$. Note that $L^1(G)$ is the Banach *—algebra (with convolution and the usual adjoint) of equivalence classes of Haar integrable functions on G. See [**8**] for a definition and discussion of $C^*(G)$.

The basis for this analogy is as follows. When G is abelian $C^*(G)$ and $C_0(\hat{G})$ are isometrically *-isomorphic, and $C_0(\hat{G})$ may be characterized as the (sup-norm) closure in $L^\infty(\hat{G})$ of $L^1(G)^\wedge = \{\hat{f} : f \in L^1(G)\}$, where $\hat{f}$ is the Fourier transform of f. Note that we view the elements of $L^\infty(\hat{G})$ as multiplication operators on $L^2(\hat{G})$, the Hilbert space of functions square-integrable with respect to Haar measure on $\hat{G}$; and thus it is a *-subalgebra of $\mathscr{L}(L^2(\hat{G}))$, all bounded linear operators on $L^2(\hat{G})$. A weak analogue of the Fourier transform for nonabelian G is ω, see [**35**], the *universal representation of* G, or of $L^1(G)$, on the Hilbert space H_ω. The norm closure in $\mathscr{L}(H_\omega)$, the space of all bounded operators on H_ω, of $\omega(L^1(G))$ is isometrically *-isomorphic to $C^*(G)$. Thus $C^*(G)$ is an analogue of $C_0(\hat{G})$. One might object and say that λ, the left regular representation of either G or $L^1(G)$ on $L^2(G)$ is the "correct" non-abelian analogue of the Fourier transform on abelian groups since λ is unitarily equivalent to the Fourier transform for abelian G. Thus the norm closure of $\lambda(L^1(G))$ in $\mathscr{L}(L^2(G))$ might be considered the "correct" analogue of $C_0(\hat{G})$. This closure of $\lambda(L^1(G))$ is called $C^*_\lambda(G)$, the reduced C^*-algebra of G. The two algebras $C^*(G)$ and $C^*_\lambda(G)$ are the same precisely when G is amenable, cf. §18.3 [**8**], in particular, if G is abelian. In the non-amenable case $C^*_\lambda(G)$ is a nontrivial quotient of $C^*(G)$. Since $C^*_\lambda(G)$ can be recovered as a quotient of $C^*(G)$ we will conventrate our study on $C^*(G)$.

We thus have in hand an analogue for $C_0(\hat{G})$, namely, $C^*(G)$, even though we have no underlying group when G is not commutative. What, then, are the analogues of translations? A rather satisfying theory can be constructed if we take, as the underlying space of $C^*(G)$, the space $P(G)_1$ of continuous functions of positive type which are one at e, the identity of G. See [37], [8], [12]. The treatment of generalized translations which we will now give can be further generalized and made more abstract and perhaps more useful. However, in this paper we wish to remain as concrete as possible and push the classical point of view of translation and differentiation as far as it will go.

Now $P(G)_1$ is a convex set whose extreme points, ext $P(G)_1$, identify with $\hat{G}$ if G is abelian. Thus ext $P(G)_1$, often called the pure positive type functions of norm one on G, is an abelian group if G is abelian; but it is not even a semigroup in general. Nevertheless, $P(G)_1$ is always a semigroup under pointwise multiplication; and we have for general locally compact G the following definition.

DEFINITION 1. The (generalized) translation of $P(G)_1$ by $p_0 \in P(G)_1$ is the map $p \in P(G)_1 \mapsto p_0 p \in P(G)_1$, which is defined since $P(G)_1$ is a semigroup.

Translations as just defined are rarely invertible (only when p_0 is a character of G) and are thus decidedly different from translations in groups as classically understood. A better feeling for this new notion of translation is obtained by seeing what it means in the abelian case. Here the semigroup $P(G)_1$ can be identified (via the Fourier-Stieltjes transform and Bochner's Theorem) with the convolution semigroup of probability measures on $\hat{G}$, i.e., the positive linear functionals of norm one on $C_0(\hat{G})$. Under this correspondence ext $P(G)_1$ identifies with the point masses on $\hat{G}$, viz., $\hat{G}$ itself.

The above notion of translation on $P(G)_1$ induces (via transposition) translations of $C^*(G)$.

DEFINITION 2. The translate of $a \in C^*(G)$ by $p \in P(G)_1$, denoted a_p, or $T_p a$, is that unique element in $C^*(G)$ which satisfies $\langle a_p, q\rangle = \langle a, pq\rangle$ for all $q \in P(G)_1$. The brackets denote the linear space duality between $P(G)_1 \subset B(G)$ (the Fourier-Stieltjes algebra of G) $\cong C^*(G)'$ and $C^*(G)$, [12].

REMARK. Assuming T_p is well defined on $C^*(G)$, as we will show below, $\langle T_p a, b\rangle = \langle a, pb\rangle$ for each $b \in B(G)$. To see this, express b as a linear combination of positive type functions. We can also extend the notion of translation to include "translation by any element b_0 in $B(G)$". Thus $T_{b_0}a$ for $a \in C^*(G)$ is that unique element of $C^*(G)$ that satisfies $\langle T_{b_0} a, b\rangle = \langle a, b_0 b\rangle$ for any $b \in B(G)$. To see that this extension is well defined let $b_0 = \sum_{k=0}^{3} i^k \alpha_k p_k$, where $i = \sqrt{-1}$, $\alpha_k \geqq 0$ and $p_k \in P(G)_1$, $k = 0, 1, 2, 3$. Then

$$T_{b_0} = \sum_{k=0}^{3} i^k \alpha_k T_{p_k}$$

is defined and easily verified to be independent of the decomposition chosen for b_0. Alternatively, the following proof can be repeated almost word for word for T_{b_0}.

A proof that a_p exists in $C^*(G)$ and is unique is in order. Uniqueness is clear since $P(G)_1$ is a total set in $B(G) \cong C^*(G)'$. Identifying as we will often do $C^*(G)$ with its universal representation $\omega(C^*(G))$, we see first that $T_p\omega(f) = \omega(pf) \in C^*(G)$ for all $f \in L^1(G)$, since $\langle \omega(pf), q\rangle = \int_G (pf)(x)q(x)\,dx = \int_G f(x)(pq)(x)\,dx = \langle \omega(f), pq\rangle = \langle T_p\omega(f), q\rangle$ for all $q \in P(G)_1$, cf., (2.9) [12]. Thus T_p is defined on a norm dense subspace of $C^*(G)$ and is norm decreasing there, since $\|\omega(pf)\|_{C^*(G)} = \sup\{|\langle \omega(pf), b\rangle| : b \in B(G), \|b\| \leqq 1\} = \sup\{|\langle \omega(f), pb\rangle| : b \in B(G), \|b\| \leqq 1\} \leqq \sup\{\|\omega(f)\|_{C^*(G)}\|p\|_{B(G)}\|b\|_{B(G)} : b \in B(G), \|b\| \leqq 1\} \leqq \|\omega(f)\|_{C^*(G)}$. By the usual Cauchy sequence argument T_p extends uniquely to all $C^*(G)$. We note in passing that in the abelian case, identifying $C^*(G)$ with $C_0(\hat{G})$, we get $T_p\hat{f} = \mu * \hat{f}$, for $f \in L^1(G) \subset C^*(G)$, where p is the inverse Fourier-Stieltjes transform of probability measure μ on $\hat{G}$.

It is clear that if $a \in C^*(G)$ and $a \geqq 0$ then $T_pa \geqq 0$, i.e., T_p is a positivity preserving linear map on $C^*(G)$. We can say much more, however, since T_p has the more delicate property of being completely positive.

PROPOSITION 1. *The translation operator T_p: $C^*(G) \to C^*(G)$ for $p \in P(G)_1$ is a completely positive, norm decreasing linear map. There is a unique extension of T_p to $W^*(G)$, also denoted T_p, which is identity preserving, completely positive, norm decreasing and σ-weakly continuous.*

REMARK. As in the remark following Definition 2, T_b: $W^*(G) \to W^*(G)$ is a well-defined linear map for each $b \in B(G)$. Note that $W^*(G)$ is defined in [35].

PROOF. We have observed above that T_p on $C^*(G)$ is a norm decreasing, linear map. We can extend this map to all of $W^*(G)$, e.g., by using the double transpose, or by σ-weak continuity, such that $\langle T_px, b\rangle = \langle x, pb\rangle$ for all $x \in W^*(G)$, $p \in P(G)_1$, $b \in B(G)$. If, abusing notation, $e = \omega(e)$ is the identity of $W^*(G)$ then $T_pe = p(e)e = e$. By definition, cf. [4], [34], T_p on $W^*(G)$ is completely positive if $T_p \otimes I_n$: $W^*(G) \otimes M_n \to W^*(G) \otimes M_n$ is positive for each natural number n. Note that M_n is the C^*-algebra of complex $n \times n$ matrices and $W^*(G) \otimes M_n$ is the (unique) C^*-tensor product. The map $T_p \otimes I_n$ is defined by $T_p \otimes I_n(x \otimes e_{ij}) = T_p(x) \otimes e_{ij}$ where e_{ij} $i, j = 1, \ldots, n$ are the usual matrix units. The complete positivity of T_p on $W^*(G)$ can be established by means of a direct calculation. However, a far more elegant and informative proof can be had using Theorem 1.1.1. of [4], due originally to Stinespring. I wish to thank Elliott C. Gootman for bringing this proof to my attention.

THEOREM (STINESPRING). *Let A be a C^*-algebra with identity and let H be a Hilbert space. Then a linear map φ of A into $\mathscr{L}(H)$ has the form $\varphi(x) = \mathrm{v}^*\pi(x)\mathrm{v}$, where π is a representation of A on some Hilbert space K and* v *is a bounded operator from H to K if and only if φ is completely positive.*

We actually only need the only if part of this theorem which is easy to verify. Let $C^*(G)$ be identified with its universal representation $\omega(C^*(G))$ on H_ω. Let (H_p, ξ_p, π_p) be the Gelfand-Naimark-Segal triple specified up to unitary equivalence by p, viz., $p(g) = (\pi_p(g)\xi_p|\xi_p)$, cf. 13.4.5 [**8**]. We claim that $\omega(pf) = Q(\omega \otimes \pi_p)(f)Q^*$ where $f \in L^1(G)$, and $\omega \otimes \pi_p$ is the representation of $L^1(G)$ on $H_\omega \otimes H_{\pi_p}$ obtained by integrating the $\omega \otimes \pi_p$ representation of G, cf. 13.3.1 [**8**]. The operator Q: $H_\omega \otimes H_{\pi_p} \mapsto H_\omega \otimes \mathbf{C}\xi_p \cong H_\omega$ is the orthogonal projection onto $H_\omega \otimes \xi_p$ followed by the identification $x \otimes \xi_p \in H_\omega \to x \in H_\omega$. Since $T_p\omega(f) = \omega(pf)$ we will have that T_p is completely positive since both sides of the equation $\langle T_p\omega(f), b\rangle = \langle Q(\omega \otimes \pi_p)(f)Q^*, b\rangle$ are continuous in f. The f here is viewed as varying over $L^1(G)$, a dense subset of $W^*(G)$ with the σ-weak topology. Note that we used $\mathbf{C}$ above for the complex numbers.

Thus if $\varphi, \eta \in H_\omega$, then

$$\begin{aligned}(\omega(pf)\varphi|\eta) &= \int_G (\omega(g)\varphi|\eta)p(g)f(g)\,dg \\ &= \int_G (\omega(g)\varphi|\eta)(\pi_p(g)\xi_p|\xi_p)f(g)\,dg \\ &= ((\omega \otimes \pi_p)(f)\varphi \otimes \xi_p|\eta \otimes \xi_p) \\ &= (Q(\omega \otimes \pi_p)(f)Q^*\varphi|\eta).\end{aligned}$$

It will be important to us later to observe that it follows from Stinespring's theorem that an identity preserving, completely positive map φ on a C^*-algebra A satisfies the *Kadison-Cauchy-Schwarz* inequality:

$$\varphi(x^*x) \geqq \varphi(x)^*\varphi(x) \text{ for } x \in A,$$

where $\geqq$ refers to the order on A induced by its positive elements.

We have now mentioned the important properties of the translation operators $\{T_p: p \in P(G)_1\}$ that we will need. Nevertheless, we will end this section with a partial investigation of the more obvious questions that might arise due to the non-invertibility of these translations. In particular, what sort of kernels can the operators T_p have?

First of all it is clear that the map $p \in P(G)_1 \mapsto T_p \in \mathscr{L}(W^*(G))$, the bounded linear operators on $W^*(G)$, is a (multiplicative) semigroup homomorphism, i.e., $T_{p_1p_2} = T_{p_1}T_{p_2}$ for $p_1, p_2 \in P(G)_1$ and that the semigroup of translations $\{T_p: p \in P(G)_1\}$ is commutative. Also if $\mathbf{1}$ is the identity of $P(G)_1$, $T_1x = x$ for all $x \in W^*(G)$. If $p \neq \mathbf{1}$ in $P(G)_1$, then $T_p \neq T_1$ as can be seen from the equation $T_pf = pf$ for $f \in L^1(G)$. More generally the map $p \mapsto T_p$ is one-to-one, since if $p_1 \neq p_2$ in $P(G)_1$ it is easy to find an $f \in L^1(G)$ such that $p_1f \neq p_2f$; hence $T_{p_1} \neq T_{p_2}$. The homomorphism $p \mapsto T_p$ is also easily seen to be continuous in the following sense: if $p_\alpha \to p_0$ uniformly on compact subsets of G then for $x \in C^*(G)$, $\|T_{p_\alpha}x - T_{p_0}x\|_{C^*(G)} \to 0$. We remark in passing that for each $x \in W^*(G)$, the map $p \in P(G)_1 \mapsto x_p \in W^*(G)$ is affine, i.e., $x_{\lambda p+(1-\lambda)p} = \lambda x_p + (1-\lambda)x_q$ for $\lambda \in [0, 1]$, $p, q \in P(G)_1$.

It is clear that $T_p \neq 0$ for any $p \in P(G)_1$, since any $f \in L^1(G)$ which is non-zero in a neighborhood of $e \in G$ satisfies $T_pf = pf \not\equiv 0$. It is equally clear that the kernel of T_p in $L^1(G) \subset C^*(G)$ is much larger than $\{0\}$ for many p. In particular if the support of p is contained in a compact neighborhood K of e, cf. (3.2) **[12]**, then any $f \in L^1(G)$ with support disjoint from K satisfies $T_pf = pf \equiv 0$. Along this same line of thought we have the following result.

PROPOSITION 2. *Let G be a locally compact group. If $p \in P(G)_1$, then T_p uniquely induces a well-defined norm decreasing, completely positive map, also called T_p, on $C^*_\lambda(G)$. If p is never zero on G then $\{a \in C^*_\lambda(G)\colon T_pa = 0\} = \{0\}$. This result is also true if $C^*_\lambda(G)$ is replaced by $W^*_\lambda(G)$ (defined below).*

REMARK. The reduced C^*-algebra $C^*_\lambda(G) \cong C^*(G)/\ker_{C^*(G)}\lambda$, where λ is the left regular representation, is isomorphic with $C^*(G)$ if and only if G is amenable. The W^*-algebra $W_\lambda(G)$ is defined to be the quotient algebra $W^*(G)/\ker_{W^*(G)}\lambda$. Note that $\ker_{W^*(G)}\lambda = [A(G)]^\perp$ in the $W^*(G)$, $B(G)$ duality, and hence $W^*_\lambda(G) = W^*(G)/[A(G)]^\perp \cong A(G)'$, the dual of $A(G)$ which is none other than the left-ring of G, i.e., the von Neumann algebra generated by $\{\lambda(g)\colon g \in G\}$, cf. **[12]**.

REMARK. If $p = e^\psi$ where ψ is a complex-valued function, then p is certainly never zero. We will soon encounter many positive type functions of this form.

PROOF OF PROPOSITION 2. We will first verify that for any $p \in P(G)_1$ that $T_p\colon C^*_\lambda(G) \to C^*_\lambda(G)$ is well-defined. This will follow if $x \in C^*(G)$, $\lambda(x) = 0$ implies $\lambda(T_px) = \lambda(x_p) = 0$. But for $x \in C^*(G)$ $\lambda(x) = 0$ is equivalent to $0 = \langle x, \omega_{\xi,\eta}\rangle$ for all $\xi, \eta \in L^2(G)$, i.e., $\langle x, a\rangle = 0$ for all $a \in A(G)$, the Fourier algebra of G, cf. (3.11) **[12]**. But then $\langle x_p, a\rangle = \langle x, pa\rangle = 0$ for all $a \in A(G)$ since $A(G)$ is an ideal in $B(G)$, and $p \in B(G)$. Thus $\lambda(T_px) = 0$. Note that the ideal property of $A(G) = A_\lambda(G) = \{$all coefficients of representation $\lambda\}$ was all that was needed to make T_p: $C^*_\lambda(G) \to C^*_\lambda(G)$ well defined. In particular, we can define $T_p\colon W^*_\lambda(G) \to W^*_\lambda(G)$ by the same argument used above. Recalling the definition of the quotient norm it is easy to show that T_p is norm decreasing. That T_p is completely positive follows from the definition of complete positivity and the fact that $T_p \otimes I_n : C_\lambda^*(G) \otimes M_n \to C_\lambda^*(G) \otimes M_n$ is the "quotient" of $T_p \otimes I_n : C^*(G) \otimes M_n \to C^*(G) \otimes M_n$.

Suppose now that $T_p\dot{x} = 0$ where $\dot{x} = x + \ker_{C^*(G)} \lambda \in C_\lambda^*(G)$, with $x \in C^*(G)$. Thus, $T_p\dot{x} = T_px + \ker_{C^*(G)} \lambda = \ker_{C^*(G)} \lambda$, i.e., $0 = \langle x_p, a\rangle = \langle x, pa\rangle$ for all $a \in A(G)$. Now $pA(G)$ is an ideal in $A(G)$ which does not vanish at any point of G since p does not. By the Tauberian property of $A(G)$, 3.38 **[12]**, this ideal is norm dense in $A(G)$, in particular $\langle x, a\rangle = 0$ for all $a \in A(G)$. Thus $\lambda(x) = 0$ and $\dot{x} = 0$. The same argument works for $W_\lambda^*(G)$.

There is a general statement contained in the proof of Proposition 2. Suppose π is a continuous unitary representation of G (hence also a $*$-representation of $W^*(G)$) on Hilbert space H_π. Let $z[\pi]$ be the (central projection) support of π in $W^*(G)$, viz., π is quasi equivalent to $x \in W^*(G) \mapsto z[\pi]\omega(x) \in z[\pi]W^*(G)$, cf. **[35]**. Let $A_\pi = z[\pi].B(G)$ be the subspace of $B(G)$ consisting of the coefficients of π, cf. **[35]**.

PROPOSITION 3. *The subspace A_π is an ideal in $B(G)$ if and only if $z[\pi]T_qz[\pi]\omega(x) = z[\pi]T_q\omega(x)$ for all $q \in P(G)_1$ and all x in a total subset of $W^*(G)$. In this case, the map $T_q\pi(f) = \pi(qf)$ for all $f \in L^1(G)$ and all $q \in P(G)_1$ is well defined.*

PROOF. We note first that A_π is an ideal in $B(G)$ if and only if $(pq).z[\pi] = pq$ for all $q \in P(G)_1$ and all $p \in z[\pi].P(G)_1$. Now in general we have that $\langle\omega(x), (pq).z[\pi]\rangle = \langle z[\pi]\omega(x), pq\rangle = \langle T_qz[\pi]\omega(x), p\rangle$ for x in $W^*(G)$. Also $\langle\omega(x), pq\rangle = \langle T_q\omega(x), p\rangle = \langle z[\pi]T_q\omega(x), p\rangle$, p in $z[\pi].P(G)_1$. Now if A_π is an ideal then $(pq).z[\pi] = pq$ for $p \in z[\pi].P(G)_1$ and $q \in P(G)_1$ and it follows that $\langle T_qz[\pi]\omega(x), p\rangle = \langle z[\pi]T_q\omega(x), p\rangle$. Thus $z[\pi]T_qz[\pi]\omega(x) = z[\pi]T_q\omega(x)$. Conversely, if $\langle T_qz[\pi]\omega(x), p\rangle = \langle z[\pi]T_q\omega(x), p\rangle$ then $\langle\omega(x), (pq).z[\pi]\rangle = \langle\omega(x), pq\rangle$ for a total set of $\omega(x)$; hence $(pq).z[\pi] = pq$ for all $q \in P(G)_1$ and A_π is an ideal. The fact that A_π is an ideal in $B(G)$ implies that $T_qx \in \ker_{W^*(G)} \pi$ for all $x \in \ker_{W^*(G)} \pi$. This fact together with the identification of $\omega(x) + \ker_{W^*(G)} \pi$ with $z[\pi]\omega(x)$ (via $z[\pi]W^*(G) \cong A_\pi' \cong W^*(G)/A_\pi^\perp \cong W^*(G)/\ker_{W^*(G)} \pi$), and the identification of $z[\pi]\omega(x)$ with $\pi(x)$ for $x \in W^*(G)$ implies that $T_p\pi(f) = \pi(pf)$ for $f \in L^1(G)$ is well defined.

A case where we have essentially that $T_p\pi(f) = 0 \neq \pi(pf)$ for some $f \in L^1(G)$ and certain π and p is as follows. Let G be a non-compact group. Let π be the representation determined by the complement of $A(G)$ in $B(G)$, cf. **[35]**. That is, we have $A_\pi \oplus A(G) = B(G)$. In particular A_π contains the coefficients of any finite dimensional unitary representation of G; A_π contains the constant functions at least. If $0 \neq p \in A(G)$ then $T_pz[\pi]\omega(x)$

satisfies $\langle T_p z[\pi]\omega(x), b\rangle = \langle z[\pi]\omega(x), pb\rangle = \langle z[\pi]\omega(x), (1 - z[\pi]).(pb)\rangle = 0$ for all b in $B(G)$, $x \in W^*(G)$, i.e., $T_p z[\pi]\omega(x) = 0$. Thus identifying $z[\pi]\omega(f)$ with $\pi(f)$ for $f \in L^1(G)$, we might say that $T_p\pi(f) = 0$ for all $f \in L^1(G)$. However, π is not the zero representation of $W^*(G)$, so $\pi(pf) \neq 0$ for some $f \in L^1(G)$. It is interesting to note, cf. **[18]**, that for $G = \mathbf{R}$ there exists a positive type function p such that $z[\pi].p = p$ and $p^2 \in A(G)$. Thus for our π above $T_p\pi(f) \neq 0$ for some $f \in L^1(G)$ but $(T_p)^2\pi(f) = 0$ for all $f \in L^1(G)$.

As a last remark we note that to each $x \in W_\lambda^*(G)$ there is a closed subset of G, which is non empty if $x \neq 0$, called the support of x. The support of x, denoted supp(x), may be defined to be the smallest closed subset F of G such that $\langle x, a\rangle = 0$ for all $a \in A(G) \cap C_c(G)$ which vanish in a neighborhood of F. Note that $C_c(G)$ is the set of continuous complex-valued functions with compact support of G. From (4.8) **[12]** we see that a necessary condition for $T_p x = 0$ for $x \in W_\lambda^*(G)$ is that $p(g) = 0$ for all $g \in$ supp(x). A sufficient condition that $T_p x = 0$ is that p vanish in a neighborhood of supp(x).

2. Differentiation at 1 in $P(G)_1$.

We now turn to the concept which unifies what may seem to be several diverse notions, examples of which were given in the introduction. Very simply we try to "differentiate at **1** in $P(G)_1$" as though it were a Lie group. Though $P(G)_1$ does not resemble a Lie group near **1**, it is nevertheless a convex subset of linear space $B(G)$. It therefore makes perfectly good sense to ask whether the geometric model of a derivative at **1**, namely a tangent vector to $P(G)_1$ at **1**, exists. Looking at this geometric model of a tangent as a limit of a difference quotient we quickly see that such a tangent can be represented by a function on the group G. Limiting ourselves (in this paper) to complex-valued limits we have the following notion.

Definition 3. A semitangent vector at **1** to $P(G)_1$ is any continuous, complex-valued function ψ on G satisfying $\psi(g) = \lim_{n\to\infty} n(p_n(g) - 1)$ for each $g \in G$ and some $\{p_n\} \subset P(G)_1$, $\{n\}$ the natural numbers. Note that we will often discuss groups G_d, where G_d represents the group G with the discrete topology.

Remark. It will be clear from the second corollary of Theorem 1 that Definition 3 could have been stated in a more general, yet equivalent, form. For example, $\{n\}$ could be replaced by any subsequence $\{n_j\}$ of the natural numbers such that $\lim_{j\to\infty} n_j(p_{n_j}(g) - 1)$ exists for all $g \in G$.

Remark. Note that $\lim_{n\to\infty} n(p_n(g) - 1) = \lim_{n\to\infty}(p_n(g) - 1)/(1/n - 0)$ represents a "derivative with respect to the parameter $1/n$" in the classical sense of a limit of quotients of differences.

We will denote by $N_0(G_d)$ the collection of all semitangent vectors at **1** in $P(G_d)_1$. Note that the subscript 0 in $N_0(G_d)$ is to remind one that if $\psi \in N_0(G_d)$, then $\psi(e) = 0$. We will denote by $N_0(G)$ the collection of all continuous functions on G which are in $N_0(G_d)$. If both ψ and $-\psi$ are in $N_0(G_d)$ we shall call ψ a tangent vector at $\mathbf{1}$ to $P(G_d)_1$. It will be clear that $N_0(G)$ is the set of semitangents at $\mathbf{1}$ to $P(G)_1$, cf. Theorem 1 and its corollaries. A $\psi \in N_0(G_d)$ is not ncessarily continuous; but if $\psi(g) = \lim_{n\to\infty} n(p_n(g) - 1)$, p_n necessarily converges to 1 at least pointwise on G. We will soon see that there are semitangents which are not tangents. We will also see that tangents generate one-parameter groups in $P(G_d)_1$ while semitangents generate one-parameter semigroups. This then is the justification of the term semitangent.

We will shortly address questions regarding existence of and methods of construction of elements on $N_0(G)$. For now let us quickly note that $p - \mathbf{1}$ is in $N_0(G)$ for each $p \in P(G)_1$. To see this observe that $p_t(\cdot) = \exp(t(p(\cdot) - \mathbf{1}))$ is in $P(G)_1$ for all $t \geqq 0$. To see this note that $\exp(tp(\cdot)) = \sum_{n=0}^{\infty} t^n p^n(\cdot)/n!$ is of positive type (and continuous by uniform convergence and the continuity of p) since the sum and product of positive type functions is again of positive type. The $e^{-t} > 0$ can be viewed as a normalization factor, i.e., $p_t(g) = e^{-t}e^{tp(g)}$ is 1 for $g = e$. Thus $\{p_t\}_{t\geqq 0}$ is a one-parameter semigroup in $P(G)_1$, continuous in the sense that if $t \to t_0$, then $p_t \to p_{t_0}$ uniformly on G. Now observe that $\lim_{n\to\infty} n(p_{1/n}(g) - 1) = \lim_{n\to\infty} n(\exp((1/n)(p(g) - 1)) = p(g) - 1$. Thus $p - \mathbf{1}$ is a semitangent at $\mathbf{1}$ on $P(G)_1$.

Let us now proceed to derive some necessary and sufficient conditions that a complex function ψ on G be a semitangent at $\mathbf{1}$. The following theorem is a natural consequence of the point of view we have thus far developed.

THEOREM 1. *Let ψ be a continuous, complex-valued function on G. Then ψ is a semitangent at $\mathbf{1}$ to $P(G)_1$, i.e., $\psi \in N_0(G)$, if and only if $\psi(e) = 0$, $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$, and for each choice of natural number n and each choice of n elements $g_1, g_2, \ldots, g_n$ from G the $n \times n$ matrix $\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i))$ is positive hermitian, i.e.,*

$$(1) \qquad \sum_{i,j=1}^{n} \{\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i)\}\lambda_i\bar{\lambda}_j \geqq 0$$

for any choice of complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

COROLLARY. *A function ψ is in $N_0(G_d)$ if and only if $\psi(e) = 0$, $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$ and for each natural number n and each choice of $g_1, \ldots, g_n$ in G we have that the $n \times n$ matrix $(\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i))$ is positive hermitian.*

REMARK. In the interest of clean typography we prove Theorem 1 using the "clean" definition of semitangent vector, namely Definition 3. The proof below will work, however, if a more general definition of semitangent vector is given, cf. the first remark following Definition 3.

PROOF OF THEOREM 1. By definition, if $\psi \in N_0(G)$ then ψ is continuous and for all $g \in G$, $\psi(g) = \lim_{n\to\infty} n(p_n(g) - 1)$, as in the definition of semitangent vectors. Since $p_n(e) = 1$ for all n, $\psi(e) = 0$; and since $p_n(g^{-1}) = \overline{p_n(g)}$ for all n and all $g \in G$, $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$. Now consider the set of completely positive maps $\{T_{p_n}\}_{n=1}^{\infty}$ of $W^*(G)$ into $W^*(G)$. By the Kadison-Cauchy-Schwarz inequality we have that $T_{p_n}(x^*x) \geqq (T_{p_n}x^*)(T_{p_n}x)$ for all $x \in W^*(G)$. In other words

$$n\{T_{p_n}(x^*x) - (T_{p_n}x^*)(T_{p_n}x)\} \geqq 0$$

for all $x \in W^*(G)$ and all n. Let us see what this expression yields in the limit as $n \to \infty$ for $x = \sum_{k=1}^{n} \lambda_k g_k$, where $g_1, \ldots, g_n$ are in G and $\lambda_1, \ldots, \lambda_n$ are complex numbers. [Note that we are identifying g in G with $\omega(\varepsilon_g)$ in $\omega(W^*(G))$ where ε_g is the unit point mass at g.] To simplify notation we will use x for $\sum_{k=1}^{n} \lambda_k g_k$ until the final step of the computation. Thus

$$\begin{aligned}
&\lim_{n\to\infty} n\{T_{p_n}(x^*x) - T_{p_n}x^*T_{p_n}x\} \\
&= \lim_{n\to\infty} n\{T_{p_n}(x^*x) - x^*x + x^*x - (T_{p_n}x^*)x + (T_{p_n}x^*)x - (T_{p_n}x^*)(T_{p_n}x)\} \\
&= \lim_{n\to\infty} n(T_{p_n} - T_1)(x^*x) + \lim_{n\to\infty} n(T_1 - T_{p_n})x^*x + \lim_{n\to\infty} T_{p_n}x^*n(T_1 - T_{p_n})x \\
&= \lim_{n\to\infty} T_{n(p_n-1)}(x^*x) - \lim_{n\to\infty} (T_{n(p_n-1)}x)^*x - \lim_{n\to\infty} (T_{p_n}xT_{n(p_n-1)}x).
\end{aligned}$$

Now for our choice of $x = \sum_{k=1}^{n} \lambda_k g_k$ we see that

$$\begin{aligned}
\lim_{n\to\infty} T_{n(p_n-1)}\Big(\sum_{i,k=1}^{n} \bar{\lambda}_k\lambda_i g_k^{-1}g_k\Big) &= \lim_{n\to\infty} \sum_{i,k=1}^{n} \bar{\lambda}_k\lambda_i n(p_n(g_k^{-1}g_i) - 1)g_k^{-1}g_i \\
&= \sum_{i,k=1}^{n} \bar{\lambda}_k\lambda_i\psi(g_k^{-1}g_i)g_k^{-1}g_i.
\end{aligned}$$

In a similar fashion

$$\lim_{n\leftarrow\infty} \Big(T_{n(p_n-1)}\Big(\sum_{k=1}^{n} \lambda_k g_k\Big)\Big)^*\Big(\sum_{i=1}^{n} \lambda_i g_i\Big) = \sum_{i,k=1}^{n} \bar{\lambda}_k\lambda_i\psi(g_k^{-1})g_k^{-1}g_i;$$

$$\lim_{n\to\infty} \Big(T_{p_n}\sum_{k=1}^{n} \bar{\lambda}_k g_k^{-1}\Big)\Big(T_{n(p_n-1)}\Big(\sum_{i=1}^{n} \lambda_i g_i\Big) = \sum_{i,k=1} \bar{\lambda}_k\lambda_i\psi(g_i)g_k^{-1}g_i.$$

We observe that $\lim_{n\to\infty} p_n(g) = 1$ since $\lim_{n\to\infty} n(p_n(g) - 1)$ exists and is finite by assumption. Also for $x = \sum_{k=1}^{n} \lambda_k g_k$ the above limits can be viewed as being taken with respect to the norm in $W^*(G)$, e.g.,

$$\|T_{n(p_n-1)}g - \psi(g)g\|_{W^*(G)} = \|g\|_{W^*(G)}|n(p_n(g) - 1) - \psi(g)| \to 0$$

as $n \to \infty$, for $g \in G$.

Now the norm limit of positive elements in $W^*(G)$ is positive; hence we have

$$\sum_{i,k=1}^{n} \bar{\lambda}_k \lambda_i \{\psi(g_k^{-1}g_i) - \psi(g_k^{-1}) - \psi(g_i)\} g_k^{-1} g_i \geqq 0.$$

Since $\mathbf{1} \in P(G)_1$ we have that

$$\langle \sum_{i,k=1}^{n} \bar{\lambda}_k \lambda_i \{\psi(g_k^{-1}g_i) - \psi(g_k^{-1}) - \psi(g_i)\} g_1^{-1} g_i, \mathbf{1} \rangle \geqq 0,$$

thus

$$\sum_{i,k=1}^{n} \bar{\lambda}_k \lambda_i \{\psi(g_k^{-1}g_i) - \psi(g_k^{-1}) - \psi(g_i)\} \geqq 0$$

for each n, each choice of $g_1, \ldots, g_n$ in G and each choice of complex numbers $\lambda_1, \ldots, \lambda_n$.

We now turn to proving the converse. If the conditions on ψ of Theorem 1 really mean that ψ is a semitangent at $\mathbf{1}$ in $P(G)_1$, then in analogy with the theory of Lie groups it should not be unexpected that $\{e^{t\psi}\}_{t \geqq 0}$ is a one-parameter semigroup in $P(G)_1$. If we could indeed establish this then ψ must be a semitangent, since $\psi = \lim_{t \to 0}(1/t)(e^{t\psi} - \mathbf{1})$ and $e^{t\psi} \in P(G)_1$ for all $t \geqq 0$.

We will thus show that a continuous function ψ satisfying equation (1) of Theorem 1 and $\psi(g^{-1}) = \psi(g)$ for all $g \in G$, determines a one-parameter semigroup of continuous functions of positive type, $\{e^{t\psi}\}_{t \geqq 0}$. If $\psi(e) = 0$, then this semigroup is in $P(G)_1$. The proof is straightforward. We merely must verify that for $t \geqq 0$ that $e^{t\psi}$ is indeed of positive type, i.e.,

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_i \exp(t\psi(g_j^{-1}g_i)) \geqq 0$$

for each natural number n, each choice of complex numbers $\lambda_1, \ldots, \lambda_n$ and each choice of $g_1, \ldots, g_n$ in G. One way to verify this last inequality is by expressing the $n \times n$ matrix $(\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i))$ in the form $(\sum_{k=1}^{n} \mu_{ik}\mu_{kj})$, i.e., $(\mu_{\ell m})$ is the positive hermitian matrix square root of the "ψ-martix." If one inserts $\sum_{k=1}^{n} \mu_{ik}\mu_{kj} + \psi(g_j^{-1}) + \psi(g_i)$ for $\psi(g_j^{-1}g_i)$, then uses the exponential series expansion followed by some tedious manipulations, one gets the desired result. This was our first proof.

A much simpler proof uses the easily verified fact that if $A = (a_{ij})$, $B = (b_{ij})$ are two positive hermitian matrices, then $A \circ B = (a_{ij}b_{ij})$, the "component-wise" product (sometimes called the Schur product) of A and B is also a positive hermitian matrix, cf. p. 9 **[10]**, p. 683, vol. II **[17]**. Thus

$$\begin{aligned}&\sum_{i,j=1}^{n} \lambda_i\bar{\lambda}_i \exp(t\psi(g_j^{-1}g_i))\\ &= \sum_{i,j=1}^{n} \exp(t\{\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i)\})[\lambda_i \exp(t\psi(g_i))][\bar{\lambda}_j \exp(\overline{t\psi(g_j)})]\\ &\geqq 0,\end{aligned}$$

since if $A = (\psi(g_j^{-1}g_i) - \psi(g_j^{-1}) - \psi(g_i))$ is a positive, hermitian, $n \times n$ matrix, the "Schur exponential" $e^{tA} \equiv I + tA + t^2A \circ A + t^3A \circ A \circ A + \cdots$ is also positive hermitian for $t \geqq 0$. Note that we used the fact that $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$.

COROLLARY. *Let ψ be a continuous function. Then ψ is a semitangent vector at* $\mathbf{1}$ *in $P(G)_1$, i.e., $\psi \in N_0(G)$, if and only if $\{e^{t\psi}\}_{t\geqq 0}$ is a one-parameter semigroup in $P(G)_1$.*

REMARK. It is fairly easy to see that a function ψ satisfies $\psi(g^{-1}) = \overline{\psi(g)}$ for all $g \in G$ and equation (1) if and only if ψ satisfies equation (1′), viz., $(\psi(g_j^{-1}g_i) - \overline{\psi(g_j)} - \psi(g_i))$ is a positive definite hermitian matrix for each natural number n and each choice of $g_1, \ldots, g_n$ in G, i.e.,

$$\sum_{i,j=1}^{n} \{\psi(g_j^{-1}g_i) - \overline{\psi(g_j)} - \psi(g_i)\}\lambda_i\bar{\lambda}_j \geqq 0 \tag{1′}$$

for each natural number n, each choice of complex numbers $\lambda_1, \ldots, \lambda_n$ and each choice of $g_1, \ldots, g_n$ in G. If ψ satisfies either equation (1) or (1′) then $\psi(e) \leqq 0$.

DEFINITION 4. A function ψ which satisfies (1′) is said to be of negative type, and we write $\psi \in N(G_d)$. The continuous functions of negative type on G are denoted $N(G)$.

REMARK. The term negative type is often used synonomously with the term negative definite. The former terminology is perhaps more accurate, but the latter seems to be more prevalent, at least in mathematics written in English.

It is immediate from the definition of $N(G_d)$, $N(G)$, $N_0(G_d)$, and $N_0(G)$ that they are convex cones, i.e., $\lambda\psi_1 + \mu\psi_2$ is in the set if ψ_1, ψ_2 are and $\lambda, \mu \geqq 0$. If ψ belongs to any of the four cones, then $\bar{\psi}$, the complex conjugate, as well as Re ψ also belongs. We have that $N(G_d)$ and $N_0(G_d)$ are closed under pointwise convergence while $N(G)$ and $N_0(G)$ are closed in the compact-open topology, i.e., topology of uniform convergence on compact sets. Finally the range of $\psi \in N(G_d)$ lies in the closed left-half plane of the complex numbers, viz., 2Re $\psi(g) \leqq \psi(e) \leqq 0$ for all $g \in G$. It is not hard to see that $\psi \in N(G_d)$ actually implies Re $\psi(g) \leqq \psi(e) \leqq 0$ for all $g \in G$.

The unnormalized version of the corollary immediately above is as follows.

COROLLARY. *Let ψ be a continuous function on G. Then ψ is of negative type, i.e., $\psi \in N(G)$, if and only if* (i) $\psi(e) \leqq 0$, *and* (ii) $e^{t\psi}$ *is of positive type for all* $t \geqq 0$.

This result was first discovered by Schoenberg for $G = \mathbf{R}^1$ in [33]; and his proof in a real number context generalizes, modulo some technicalities with complex numbers.

The corollary immediately above can be strengthened to the following.

COROLLARY. *Let ψ be a complex function on G. Then ψ is continuous and of negative type, i.e., $\psi \in N(G)$, if and only if $\psi(e) \leqq 0$ and $e^{t\psi(\cdot)}$ is continuous and of positive type, i.e., $e^{t\psi(\cdot)} \in P(G)$, for each $t > 0$.*

PROOF. The only if part of this corollary is contained in the above corollary. Conversely, suppose $\psi(e) \leqq 0$ and that $e^{t\psi(\cdot)}$ is in $P(G)$ for each $t > 0$. Consider that

$$\frac{1}{1-\psi(g)} = \int_0^\infty e^{-t}e^{t\psi(g)}\,dt$$

holds for each $g \in G$, because $t \mapsto e^{-t(1-\psi(g))}$ is continuous and integrable (since $|e^{t\psi(g)}| \leqq e^{t\psi(e)} \leqq 1$) on $[0, \infty)$ for each $g \in G$. However, we claim that $\int_0^\infty e^{-t}e^{t\psi(g)}\,dt$ is continuous for g in G. To see this let g_α converge to g_0 in G. Then

$$\left|\int_0^\infty e^{-t}e^{t\psi(g_\alpha)}dt - \int_0^\infty e^{-t}e^{t\psi(g_0)}dt\right| \leqq 2^{1/2}\int_0^\infty e^{-t}e^{(t/2)\psi(e)}\,|e^{t\psi(e)} - e^{t\psi(g_\alpha^{-1}g_0)}|^{1/2}\,dt$$

since $e^{t\psi(\cdot)}$ is of positive type, cf. 13.4.7 [8]. Given $\varepsilon > 0$, choose $N \in [0, \infty)$ so large that $e^{-N} < \varepsilon/4$. Then the above integral equals

$$2^{1/2}\int_0^N e^{-t}f_\alpha(t)dt + 2^{1/2}\int_N^\infty e^{-t}f_\alpha(t)dt$$

where

$$f_\alpha(t) = e^{(t/2)\psi(e)}\,|e^{t\psi(e)} - e^{t\psi(g_\alpha^{-1}g^0)}|^{1/2}.$$

The last integral is no more than $2e^{-N} < \varepsilon/2$. The first integral of the sum can be made less than $\varepsilon/2$ for sufficiently large α. This follows since $f_\alpha(t)$ is continuous in t for each α, and by hypothesis $f_\alpha(t) \to 0$ as $\alpha \to \infty$, for each $t \in [0, N]$. Thus by the usual compactness argument for sufficiently large α, $f_\alpha(t) < \varepsilon/4$, for $t \in [0, N]$, and we are done.

If ψ belongs to $N(G)$ then $\psi - \psi(e)$ belongs to $N_0(G)$; in other words, any $\psi \in N(G)$ is easily normalized so as to belong to $N_0(G)$. For this reason we will most often restrict our attention to $N_0(G)$. The fact that $\psi - \psi(e) \in N_0(G)$ for all $\psi \in N(G)$ follows from an alternate characterization of negative type functions given by Schoenberg in [33].

PROPOSITION 4. *A function ψ belongs to $N(G_d)$ if and only if $\psi(e) \leqq 0$; $\psi = \psi^\flat$ and*

$$(1'') \qquad \sum_{i,j=1}^{n} \psi(g_j^{-1}g_i)\bar{\lambda}_j\lambda_i \geqq 0$$

for any choice of natural number n, any choice $g_1, \ldots, g_n \in G$, and any choice of complex numbers $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 0$.

PROOF. We leave the reader to construct the proof or to look it up in [33], [5]. We warn the reader that the set of negative definite functions defined in [5] is $\{-\psi: \psi \in N(G_d)$ as herein defined$\}$. Recall also that by definition $\psi^\flat(g) = \overline{\psi(g^{-1})}$ for all $g \in G$ for any complex function ψ on G.

REMARK. The functions defined by equation (1″) in Proposition 4 are often called conditionally positive definite in the literature.

COROLLARY. *If $\psi \in N(G_d)$, then $\psi - \psi(e) \in N_0(G_d)$.*

PROOF. Let $\psi_0 = \psi - \psi(e)$, then $\psi_0(e) = 0$ and $\psi_0^\flat = \psi_0$, since $\psi(e) \leqq 0$ is real. Finally if $\sum_{i=1}^n \lambda_i = 0$, then

$$\sum_{i,j=1}^n \psi_0(g_j^{-1}g_i)\bar{\lambda}_j\lambda_i = \sum_{i,j=1}^n \psi(g_j^{-1}g_i)\, \bar{\lambda}_j\lambda_i \geqq 0$$

by Proposition 4, since $\psi \in N(G_d)$.

REMARK. We have already observed that $p \in P(G)_1$ (respectively $p \in P(G)$) implies that $p - 1$ (respectively, $p - p(e)$) is in $N_0(G)$ (respectively $N_0(G)$). This also follows from Proposition 4.

We now have the following corollary of the various equivalent formulations of negative type functions.

COROLLARY. Let ψ be a continuous complex-valued function on G. The following are equivalent:

(1) *ψ is of negative type*;

(2) $\int_G\int_G (\psi(y^{-1}x) - \overline{\psi(y)} - \psi(x))\overline{f(y)}f(x)\, dy\, dx \geqq 0$ *for each* $f \in C_c(G)$;

(2′) $\int_G\int_G (\psi(y^{-1}x) - \overline{\psi(y)} - \psi(x))d\overline{\mu(y)}\ \ d\mu(x) \geqq 0$ *for each (bounded) complex Borel measure μ with compact support*;

(3) $\psi(e) \leqq 0$, $\psi^\flat = \psi$, *and* $\int_G\int_G \psi(y^{-1}x)\overline{f(y)}f(x)\ dy\ dx \geqq 0$ *for each* $f \in C_c(G)$ *such that* $\int_G f(x)dx = 0$;

(3′) $\psi(e) \leqq 0$, $\psi^\flat = \psi$, *and* $\int_G\int_G \psi(y^{-1}x)d\overline{\mu(y)}\ d\mu(x) \geqq 0$ *for each (bounded) complex Borel measure μ with compact support such that* $\mu(G) = 0$.

PROOF. Note that $\psi^\flat(g) = \overline{\psi(g^{-1})}$ for all $g \in G$, and $f^\# = \Delta^{-1}f^\flat$ where

Δ is the modular function for G. We leave the proofs to the reader; however, note that (1) $\Rightarrow$ (2) $\Rightarrow$ (2′) $\Rightarrow$ (1) follows from the proof of 13.4.4 [8] with only slight modifications. Also (2) $\Rightarrow$ (3) and (2′) $\Rightarrow$ (3′) are trivial. The proof of implication (3) $\Rightarrow$ (3′) follows much the same lines as (2) $\Rightarrow$ (2′) with the observation that if μ has compact support and $\mu(G) = 0$ then for $f \in C_c(G)$ $\mu * f \in C_c(G)$ and $\int_G \mu * f = \mu(G) \int_G f(x)dx = 0$. The implication (3′) $\Rightarrow$ (2′) follows much as in the discrete case, cf. §7.5 [5].

REMARK. The condition $\int_G f(x)dx = 0$ or $0 = \mu(G) = \int_G d\mu(x)$ is equivalent to saying f or μ is in the kernel of the trivial 1-dimensional unitary representation, **1**.

REMARK. If for $f \in C_c(G)$ a continuous function ψ satisfies $\psi(f^\# * f) \geqq (\psi f)^\# * f + f^\# * (\psi f)$, $\geqq$ being operator order, i.e., $\omega(\psi(f^\# * f)) \geqq \omega(\psi f)^* \omega(f) + \omega(f)^* \omega(\psi f)$, then $\int \{\psi(x)(f^\# * f)(x) - (\psi f^\#) * f(x) - (f^\# * (\psi f))(x)\} 1 \; dx \geqq 0$. This reduces to (2), hence ψ is of negative type. A similar result holds if ψ is not necessarily continuous but ψ satisfies the above relationships with f replaced by $\sum_{i=1}^n \lambda_i \varepsilon_{g_i}$, a complex-linear combination of point masses.

There are two more properties of semitangents at **1**, or more generally, negative definite functions, which follow immediately from the definition. We give first the following.

DEFINITION 5. A non-negative, subadditive function ρ on G is called a seminorm on G, i.e., $\rho\colon G \to [0, \infty)$ and $\rho(g_1 g_2) \leqq \rho(g_1) + \rho(g_2)$.

PROPOSITION 5. *If $\psi \in N(G_d)$, then $\rho_\psi = |\psi|^{1/2}$ is a seminorm on G. If $\psi \in N_0(G_d)$ then we also have that $\rho_\psi(e) = 0$.*

PROOF. Equation (1′) for $n = 2$, which we call the *2-positivity condition* for negative type functions, is equivalent to saying that the matrix

$$\begin{pmatrix} \psi(e) - \overline{\psi(g_1)} - \psi(g_1) & \psi(g_2^{-1}g_1) - \overline{\psi(g_2)} - \psi(g_1) \\ \psi(g_1^{-1}g_2) - \overline{\psi(g_1)} - \psi(g_2) & \psi(e) - \overline{\psi(g_2)} - \psi(g_2) \end{pmatrix}$$

is non-negative hermitian for any $g_1, g_2 \in G$. Using the fact that $\psi^\flat = \psi$ and taking the necessarily non-negative determinant of the 2×2 matrix above we are led to

$$(2) \quad |\psi(g_1^{-1}g_2) - \overline{\psi(g_1)} - \psi(g_2)|^2 \leqq (\psi(e) - 2\mathrm{Re}\, \psi(g_1))(\psi(e) - 2\mathrm{Re}\, \psi(g_2)).$$

The right hand side of this expression is less than or equal to $4|\psi(g_1)| \, |\psi(g_2)|$; it then follows that $|\psi(g_1 g_2)| \leqq (|\psi(g_1)|^{1/2} + |\psi(g_2)|^{1/2})^2$.

COROLLARY. *Let $\psi \in N(G)$, then $H_\psi = \{g \in G\colon \psi(g) = 0\}$ is a closed subgroup.*

COROLLARY. *Let f be a function on a group G such that $f(g^{-1}) = f(g) =$*

Re $f(g) \leqq f(e) = 0$ *for all* g *in* G. *Then* f *satisfies the* 2*-positivity condition if and only if* $|f|^{1/2}$ *is a seminorm on* G.

REMARK. Proposition 5 was noticed in [33], footnote page 525. See [5] also. The seminorm property, being a consequence of the 2-positivity condition alone for negative type functions is indeed a much weaker condition than complete positivity. For example, every non-compact, compactly generated group has unbounded seminorms, cf. [20], but many groups, cf. Theorem 2 below, have only bounded negative type functions.

One very important consequence of Proposition 5 is that the "growth rate" of $|\psi(g)|$ as $g \to \infty$ of any $\psi \in N(G_d)$ is limited. We will use this idea later on. For now let us briefly observe that $|\psi(g^n)|^{1/2} \leqq n|\psi(g)|^{1/2}$, i.e., $|\psi(g^n)| \leqq n^2|\psi(g)|$ for every $g \in G$.

The last elementary property of negative type functions that we will discuss now is as follows.

PROPOSITION 6. *Let function* ψ *be in* $N(G_d)$. *Then* ψ *is an algebraic homomorphism into the additive complex numbers if and only if* Re $\psi(g) = \psi(e) = 0$ *for all* $g \in G$. *In this case* $\psi = i\gamma$ *where* γ *is an algebraic homomorphism of* G *into the additive reals, and all functions* ψ *of this form are in* $N(G_d)$.

PROOF. If Re $\psi(g) = \psi(e) = 0$ for all g in G, then since $\psi \in N(G_d)$ ψ satisfies equation (2). Thus $\psi(g_1^{-1}g_2) = \overline{\psi(g_1)} + \psi(g_2)$ for all g_1, g_2 in G. Replacing g_1 by g_1^{-1} we see that $\psi(g_1g_2) = \psi(g_1) + \psi(g_2)$ for all g_1, g_2 in G, i.e., ψ is a homomorphism. Conversely if ψ is a homomorphism then $\psi(g^{-1}) = -\psi(g)$ for all $g \in G$. However, Re $\psi \in N(G_d)$ if $\psi \in N(G_d)$; and for all $g \in G$, Re $\psi(g) \leqq \psi(e) = -\psi(e) = 0$. Thus Re $\psi(g^{-1}) = -$Re $\psi(g) \leqq 0$ implies that Re $\psi(g) = 0$ for all $g \in G$. Finally, if $\psi = i\gamma$, γ a real, additive homomorphism, then ψ easily satisfies equation (1′), i.e., $\psi \in N(G_d)$.

REMARK. See [5] also.

REMARK. If γ is a homomorphism of G into the additive reals than $-\gamma^2 \in N(G_d)$. This can be seen directly from the definition of $N(G_d)$.

Up through the proof of Theorem 1 we have developed our ideas essentially as we first discovered them. After having discovered equations (1) and (1′) we decided to search the literature to see if this type of function had been discussed before. Our search first brought us to [6]; and then we quickly discovered that "our" semitangents had been considered as early as 1938 by Schoenberg in [33], at least for real valued negative definite functions on $G = \mathbf{R}$. Sometime later we became aware of [5] which contains a fairly complete and concise account of negative type functions on abelian groups. Since much of the treatment in [5] works equally well for non-abelian groups we will refer to [5] (as we have already done) for needed properties of negative type functions even when the group under consideration is not abelian.

We have searched the literature relevant to this paper fairly thoroughly and have two remarks to make. The first is that the notion of semitangent (though never called such) occurs in a variety of contexts, as mentioned in the introduction. We will proceed to discuss the various "applications" of the notion of semitangent shortly. The fact that such diverse topics can be seen to be different manifestations of such a fundamental notion as differentiation lends interest and depth to the subject. Secondly, nowhere in the literature have we seen equations (1) or (1′) derived as a natural consequence of the combined notions of complete positivity and differentiation. This, of course, cannot be expected in treatments of the subject from a commutative point of view. It appears that our approach is new, and hopefully will have interesting consequences. For example, from any discrete sequence $\{p_{n_j}\}$ converging to $\mathbf{1}$ in $P(G)_1$ such that

$$\lim_{j\to\infty} n_j(p_{n_j}(\cdot) - \mathbf{1})$$

exists at each point in G we get a $\psi \in N_0(G)$ and hence a one-parameter semigroup in $P(G)_1$. This leads us to a method for actually constructing functions in $N_0(G)$, as we shall see later. We view this latter occurrence as "proof" that we have actually "differentiated" at $\mathbf{1}$ in $P(G)_1$.

Before leaving Theorem 1 too far behind we would like to observe another fact implicit in the first half of its proof. Let us begin with the following definition.

DEFINITION 6. A linear operator ∂ defined on a norm dense subspace of $C^*(G)$, denoted $\mathrm{Dom}(\partial)$, with values in $C^*(G)$ is called a semiderivation on $C^*(G)$ if $x \in \mathrm{Dom}(\partial)$ implies $x^* \in \mathrm{Dom}(\partial)$ and

$$\partial(x^*x) \geqq (\partial x^*)x + x^*\partial x \tag{3}$$

whenever $x \in \mathrm{Dom}(\partial)$ and $x^*x \in \mathrm{Dom}(\partial)$. We define a semiderivation on $W^*(G)$ to be a linear operator defined on a σ-weakly dense self-adjoint subspace of $W^*(G)$, denoted $\mathrm{Dom}_\sigma(\partial)$, which satisfies (3) whenever x and x^*x are in $\mathrm{Dom}_\sigma(\partial)$. Note that $\geqq$ is the operator order.

REMARK. If ψ is a continuous complex function on G and ∂ is a semiderivation such that $\partial\omega(f) = \omega(\psi f)$ or $f \in C_c(G)$, then ψ is of negative type. To see this consult the second corollary of Proposition 4. In a similar fashion if ∂ is a semiderivation such that $\partial\omega(g) = \psi(g)\omega(g)$ for $g \in G$, ψ a function on G, then ψ is of negative type (but not necessarily continuous). From this remark and the next proposition we see that the notion of a semiderivation (that "comes from" a function ψ) and the notion of ψ

being a semitangent are equivalent. We note in passing that there are semiderivations, in fact derivations of $W^*(G)$ or $C^*(G)$ which do not come from a function on G via pointwise multiplication on G. For example, an inner derivation determined by an element from $L^1(G)$ or $C^*(G)$ "comes from pointwise multiplication on the dual of G".

PROPOSITION 7. *Each semitangent ψ in $N_0G)$ defines a semiderivation, denoted ∂_ψ, which is a closed operator on $C^*(G)$. A semiderivation, again denoted ∂_ψ, is also defined on $W^*(G)$. If ψ is a tangent (at* **1** *in $P(G_1)$, then ∂_ψ is a derivation.*

PROOF. In either the $C^*(G)$ or $W^*(G)$ case we take the domain of ∂_ψ to be the "natural" one, i.e., those x in $C^*(G)$ (respectively $W^*(G)$) for which the norm limit as $t \to 0$ of $(1/t)[T_{e^{t\psi}}x - x]$ exists in $C^*(G)$ (respectively $W^*(G)$). We define for such x:

$$\partial_\psi x = \lim_{t \downarrow 0} (1/t)[T_{e^{t\psi}}x - x].$$

Now $\{T_{e^{t\psi}}\}_{t \geqq 0}$ is a one-parameter semigroup of norm-decreasing operators, i.e., contractions, on $C^*(G)$ (respectively $W^*(G)$) and $T_{e^{0\psi}} = T_1$ is the identity operator on $C^*(G)$ (respectively $W^*(G)$). On $C^*(G)$ the one-parameter semigroup is very nicely behaved in that we have the following continuity condition:

$$\lim_{t \downarrow 0} \|T_{e^{t\psi}}x - x\|_{C^*(G)} = 0 \text{ for every } x \in C^*(G).$$

This last condition is clear when $x = \omega(f)$, $f \in L^1(G)$; and a "3-epsilon" argument coupled with the norm density of $\omega(L^1(G))$ in $C^*(G)$ finishes the proof of the continuity condition. We thus can appeal to standard results for semigroups, cf. 13.35 **[32]**, chap. X **[19]**, to get that ∂_ψ on $C^*(G)$ is a closed densely defined linear operator.

Since the involution * is an isometry on $C^*(G)$ (respectively $W^*(G)$) we have

$$\partial_\psi(x^*) = \lim_{t \downarrow 0} (1/t)[T_{e^{t\psi}}x^* - x^*] = \{\lim_{t \downarrow 0} (1/t)[T_{e^{t\psi}}x - x]\}^* = (\partial_\psi x)^*$$

for every $x \in \mathrm{Dom}(\partial_\psi)$, (respectively $\mathrm{Dom}_\sigma(\partial_\psi)$). A trivial modification of the first half of the proof of Theorem 1 then shows that if x and x^*x are in $\mathrm{Dom}(\partial_\psi)$ (respectively $\mathrm{Dom}_\sigma(\partial_\psi)$) then equation (3) holds. Finally, since $\mathrm{Dom}(\partial_\psi) \subset \mathrm{Dom}_\sigma(\partial_\psi)$ and the former is σ-weakly dense in $W^*(G)$, the latter is also, cf., the following remarks also.

Now suppose that ψ is a tangent at **1**, i.e., ψ and $-\psi \in N(G)$; thus $\mathrm{Re}\ \psi \equiv 0$ and, by Proposition 5, $\psi = i\gamma$ where γ is an additive homomorphism of G into the reals. Thus $\{e^{it\gamma}\}$ is a one-parameter group of characters, and $\{T_{e^{it\gamma}}\}$ is a one-parameter group of isometries of $C^*(G)$ (or $W^*(G)$). Since $e^{it\gamma}$ is a character,

$$T_{e^{it\tau}}\,\omega(f_1 * f_2) = (T_{e^{it\tau}}\,\omega(f_1))(T_{e^{it\tau}}\,\omega(f_2))$$

for $f_1, f_2 \in L^1(G)$; it follows that

$$T_{e^{it\tau}}(x_1x_2) = (T_{e^{it\tau}}x_1)(T_{e^{it\tau}}x_2)$$

for all x_1, $x_2 \in W^*(G)$. (A norm density argument gets this result for x_1, $x_2 \in C^*(G)$. A σ-weak density argument, using the separate σ-weak continuity of multiplication, will do for x_1, $x_2 \in W^*(G)$.) Thus if $x, y \in \mathrm{Dom}(\partial_\psi)$ (resp. $\mathrm{Dom}_\sigma(\partial_\psi)$) then $xy \in \mathrm{Dom}(\partial_\psi)$ (resp, $\mathrm{Dom}_\sigma(\partial_\psi)$) and $\partial_\psi(xy) = (\partial_\psi x)y + x\partial_\psi y$ by the usual calculus argument. Note that $\mathrm{Dom}(\partial_\psi)$ and $\mathrm{Dom}_\sigma(\partial_\psi)$ are *-subalgebras in this case.

COROLLARY. *Each semitangent ψ in $N_0(G)$ defines a semiderivation, denoted $\partial_{\lambda(\psi)}$, which is a closed operator on $C^*_\lambda(G)$, the reduced C^*-algebra. A semiderivation, again denoted $\partial_{\lambda(\psi)}$, is also defined on $W^*_\lambda(G)$. If ψ is a tangent (at* **1** *in $P(G)_1$) then $\partial_{\lambda(\psi)}$ is a derivation.*

PROOF. The semigroup $\{e^{t\psi}\}_{t\geqq 0}$ in $P(G)_1$ induces a semigroup of completely positive maps of $C^*_\lambda(G)$ and $W^*_\lambda(G)$,

$$\{T_{e^{t\psi}}\}_{t\geqq 0},$$

(see Proposition 2). The proof of Proposition 7 now applies.

COROLLARY. *Each semitangent ψ in $N_0(G)$ defines a closed, normal operator on $L^2(G)$, denoted M_ψ, and*

$$M_{e^{t\psi}} = e^{tM_\psi},$$

$t \geqq 0$; where $M_{e^{t\psi}}$ is the bounded operator on $L^2(G)$ given by multiplication by $e^{t\psi}$ for each $t \geqq 0$. If ψ is a tangent (at **1** *in $P(G)_1$) then each $M_{e^{t\psi}}$ is unitary and M_ψ is skew-adjoint.*

PROOF. See 13.37 [**32**]. The condition

$$\|M_{e^{t\psi}}\xi - \xi\|_{L^2(G)} \to 0 \text{ as } t \to 0,$$

needed to apply 13.37 [**32**], follows since $e^{t\psi} \to$ **1** uniformly on compact sets of G.

REMARK. There is a dual Haar weight φ_λ on $W^*_\lambda(G)$ and a natural dual Hilbert space $L^2(W^*_\lambda, \varphi_\lambda)$ which is isometrically isomorphic with $L^2(G)$, cf. p. 158 [**36**]. There is thus a version of the last corollary on $L^2(W^*_\lambda, \varphi_\lambda)$. We will address this in more detail in another paper.

REMARK. The operators ∂_ψ, $\psi \in N_0(G)$, are examples of completely dissipative operators. Such operators are of some importance, for example in physics, [**24**], [**3**], [**15**]. Since (at $t = 0$) we can differentiate $T_{e^{t\psi}} \otimes I_n$, which acts on $W^*(G) \otimes M_n$ (or $C^*(G) \otimes M_n$), we get that $\partial_\psi \otimes I_n$ is a semiderivation for each natural number n.

REMARK. From the general semigroup theory we get

$$\frac{d}{dt}(T_{et\psi}x) = \partial_\psi T_{et\psi}x = T_{et\psi}\partial_\psi x \tag{†}$$

for $x \in \mathrm{Dom}(\partial_\psi)$ and

$$T_{et\psi}x = \lim_{\varepsilon \downarrow 0} \exp\Big(t(1/\varepsilon)[T_{e\varepsilon\psi} - T_1]x\Big) \tag{††}$$

for all $x \in C^*(G)$, with convergence being uniform for t varying over compact subsets of $[0, \infty)$, cf., [32], [19].

REMARK. In this group algebra case we can say a good deal more about $\mathrm{Dom}(\partial_\psi)$ and $\mathrm{Dom}_\sigma(\partial_\psi)$ than can be deduced from general semigroup theory on Banach spaces. In particular $\omega(C_c(G)) \subset \mathrm{Dom}(\partial_\psi)$ for each $\psi \in N_0(G)$. Recall that $C_c(G)$ is the space of continuous, compactly supported complex-valued functions on G. The linear space $C_c(G)$ is also an algebra (with convolution for product) with involution $f^\# = \Delta^{-1}f^\flat$, where Δ is the modular function on G and $f^\flat(g) = \overline{f(g^{-1})}$ for $g \in G$. This is important since it implies that equation (3) is not vacuous; in fact, $\partial_\psi\omega(f) = \omega(\psi f)$ and $\partial_\psi\omega(f^\# * f) \geqq (\partial_\psi\omega(f)^*)\omega(f) + \omega(f)^*\partial_\psi\omega(f)$ for all $f \in C_c(G)$. It is easy to see that $\omega(C_c(g))$ is in the domain of polynomial operators in several variables formed from the ∂_ψ, $\psi \in N_0(G)$. Thus "semidifferential equations" and inequalities can be formed and their solutions contemplated. We can also easily see, cf., proof of Theorem 1, that the linear span of $\{\omega(g) : g \in G\}$ is an algebra with involution in $\mathrm{Dom}_\sigma(\partial_\psi)$ which is not in $\mathrm{Dom}(\partial_\psi)$ if G is not discrete, cf., [1]. Thus the extension of ∂_ψ from $C^*(G)$ to $W^*(G)$ is often non-trivial. We thus have an abundance of "smooth elements" in $C^*(G)$ (and $W^*(G)$) in the sense that they can be semidifferentiated by all (several variable polynomials in) ∂_ψ, $\psi \in N_0(G)$. Questions about $\mathrm{Dom}(\partial_\psi)$ and $\mathrm{Dom}_\sigma(\partial_\psi)$ obviously remain and are begging to be discussed; and, indeed, we shall shortly return to discussing $\mathrm{Dom}(\partial_\psi)$, if only briefly. However, to investigate $\mathrm{Dom}(\partial_\psi)$ and $\mathrm{Dom}_\sigma(\partial_\psi)$ thoroughly here would take us too far from the "introductory" nature of this paper and increase its length beyond reasonable bounds. To get a glimpse of some of the intricacies that abound, even when $G = \mathbf{R}^n$ see p. 142ff [9].

To partially illustrate what we have done so far let us consider what are perhaps the two simplest examples. These examples occur in the context of the corollaries to Proposition 7.

EXAMPLE 1. Let $G = \mathbf{R}^1$; the arguments that follow are quite similar, except for notation, on $G = \mathbf{R}^k$, $k = 2, 3, \ldots$. If $p_n(\cdot) = e^{i(1/n(\cdot)}$, $n = 1, 2, \ldots$, then $\psi_1(x) = \lim_{n\to\infty} n(e^{i(x/n)} - 1) = ix$ for each $x \in \mathbf{R}^1$. Thus, since ψ_1 and $-\psi_1$ (where in a similar fashion $-\psi_1 = \lim_{n\to\infty} n(e^{-i(\cdot)/n} - 1)$

are semitangents at $\mathbf{1}$ to $P(\mathbf{R}^1)_1$; ψ_1 (or $-\psi_1$) is a tangent. As is well known the operator $M_{-\psi_1}$, multiplication by $-\psi_1$ on $L^2(\mathbf{R}^1)$, say, is unitarily equivalent via the Fourier-Plancherel transform to D, i.e., d/dy, the differentiation operator on $L^2(\hat{G}) = L^2(\mathbf{R}^1)$. Thus the identity $D\bar{f}f = (D\bar{f})f + \bar{f}Df$ becomes, if, say, f is a rapidly decreasing Schwartz function, $\psi_1(\hat{f}^\# * \hat{f}) = (\psi_1\hat{f}^\#) * \hat{f} + \hat{f}^\# * \psi_1\hat{f}$. Hence the non-negative "error term" in equation (3), viz., $\partial_\psi(x^*x) - (\partial_\psi x^*)x - x^*\partial_\psi x \geqq 0$, is 0 in this case, i.e., we are looking at a true derivation (obviously).

EXAMPLE 2. Let $p_{n^2}(\cdot) = \cos((\cdot)/n)$, $n = 1, 2, \ldots$; then

$$\psi_2(x) = \lim_{n\to\infty} n^2(p_{n^2}(x) - 1) = \lim_{n\to\infty} n^2(\cos(x/n) - 1) = -x^2$$

for each $x \in \mathbf{R}^1$. Thus ψ_2 is a semitangent (which is easily seen not to be a tangent). Clearly ψ_2 corresponds to D^2, or d^2/dy^2, on $\hat{G} = \mathbf{R}^1$. In particular, for f a Schwartz function, say, we have $D^2(\bar{f}f) = (D^2\bar{f})f + \bar{f}D^2f + 2D\bar{f}Df$, which transforms to

$$\psi_2(\hat{f}^\# * \hat{f}) = (\psi_2\hat{f}^\#) * \hat{f} + \hat{f}^\# * \psi_2 f + 2(-\psi_1\hat{f})^\# * (-\psi_1\hat{f})$$

and thus the non-negative "error term" in (3) is $2(-\psi_1\hat{f})^\# * (-\psi_1\hat{f})$.

The reader might ask: What about D^3? This transforms to $x \mapsto ix^3$ which is easily seen not to be of negative type. (Check equation (1′); or see that ix^3 grows faster than Proposition 5 allows, i.e., $|i(nx)^3| = n^3|ix^3|$ which is greater than $n^2|ix^3|$ for $x \neq 0$ and $n > 1$.)

This is perhaps an opportune moment to make one last comment on $\mathrm{Dom}(\partial_\psi)$. Integrability of a function f on G is not generally sufficient to guarantee that $\omega(f) \in \mathrm{Dom}(\partial_\psi)$. For example if $X_{[-1,1]}$ is the characteristic function of $[-1, 1] \subset \mathbf{R}^1$, then $X_{[-1,1]} * X_{[-1,1]}$ is not everywhere differentiable, yet it is the Fourier transform of an integrable function on $\mathbf{R}^1$. However, if $\psi f \in L^1(G)$, then $\partial_\psi\omega(f) = \omega(\psi f)$ and $\omega(f) \in \mathrm{Dom}\,\partial_\psi$ for $\psi \in N_0(G)$. Since Proposition 5 limits the growth of any ψ in $N_0(G_d)$ we are able to find a "rate of decay" for a function f which will imply a certain "degree of smoothness" of $\omega(f)$.

DEFINITION 7. Let $f \in L^1(G)$, then f is said to have a mean rate of decay of order d or more if there exists a compact neighborhood K of $e \in G$ such that

$$\text{(i)} \int_{\left(\bigcup_{n=1}^{\infty} K^n\right)^c} f(g)\, dg = 0; \text{ and}$$

$$\text{(ii)} \sum_{n=1}^{\infty} n^{d-2} \int_{K^n \setminus K^{n-1}} |f(g)|\, dg < \infty.$$

REMARK. We apply this notion here only for $d \geqq 2$. The symbol K^n denotes the product of K, n times with itself in G. Also $(\bigcup_{n=1}^{\infty} K^n)^c$ is the complement of $\bigcup_{n=1}^{\infty} K^n$ in G and $K^n \backslash K^{n-1}$ is the complement of K^{n-1} in K^n. Clearly the notion of mean rate of decay is related to the growth rate of the Haar measure of $K^n \backslash K^{n-1}$ for the choices of K which may be possible in G. For example, some groups have some order of polynomial growth, others exponential growth, and so forth. We now have the following result.

PROPOSITION 8. *If the complex-valued function* $f \in L^1(G)$ *on locally compact group* G *has mean rate of decay of order* 4 *or more, then* $\omega(f)$ *is in* $\mathrm{Dom}(\partial_\psi)$ *for all* $\psi \in N_0(G)$. *In this case we have* $\partial_\psi \omega(f) = \omega(\psi f)$ *for all* $\psi \in N_0(G)$, *i.e.,* $\omega(f)$ *is "completely semidifferentiable".*

PROOF. Given $\psi \in N_0(G)$, let $M = \sup\{|\psi(x)| : x \in K\}$. We have for some K

$$\int_G |\psi(x)||f(x)|\, dx = \sum_{n=1}^{\infty} \int_{K^n \backslash K^{n-1}} |\psi(x)||f(x)|\, dx$$

$$\leqq \sum_{n=1}^{\infty} Mn^2 \int_{K^n \backslash K^{n-1}} |f(x)|\, dx < \infty.$$

Note that $\sup\{|\psi(x)| : x \in K^n\} \leqq n^2 M$.

REMARK. If f has mean rate of decay greater than d for all $d \geqq 2$ then $\omega(f)$ is in the domain of any polynomial, semidifferential operator (with constant coefficients, say).

REMARK. Definition 7 and Proposition 8 make sense for measures. Using this generalization one can further generalize the second corollary following Proposition 4.

REMARK. There are groups for which each $\psi \in N(G)$ is bounded, cf. Theorem 2 below. For these groups semidifferentiability as thus far developed is not a very restrictive notion, e.g., $\mathrm{Dom}\, \partial_\psi = C^*(G)$ for ψ bounded.

We have thus far seen that the situation regarding the semigroup $P(G)_1$, the semitangent space at $\mathbf{1}$, namely the cone $N_0(G)$, and the exponential map of $N_0(G)$ into $P(G)_1$ bears a strong resemblance to the Lie group situation, with, of course, major differences. Nevertheless, it is worthwhile to keep the analogy in mind for it suggests many directions of further study which we will pursue here and elsewhere. For example, by analogy with the classical situation, two computational techniques are suggested for finding semitangent vectors.

The first technique, though quite simple, has applications, cf., Theorem 2 below. It is an "additive technique" and is simply this. For each $p \in P(G)_1$, $p - \mathbf{1} \in N_0(G)$, as we have noted before. Linear combinations of the form $\sum_{k=1}^{n} \lambda_k(p_k - \mathbf{1})$, n a natural number, $\lambda_k \geqq 0$ for $k = 1, 2, \ldots, n$, are again in $N_0(G)$. Any series $\sum_{k=1}^{\infty} \lambda_k(p_k - \mathbf{1})$, with $\lambda_k \geqq 0$, which converges pointwise yields an element in $N_0(G_d)$. If, say, uniform convergence on compact subsets of G can be arranged, then the series converges to an element of $N_0(G)$. Since this computation occurs in the cone of semitangents it might be called a "Lie cone" technique. For some applications of this technique see **[2]**.

The second computational technique is of a "multiplicative" nature, and though basically simple is rather delicate. The essential idea (which we adapt below) goes as far back as von Neumann's studies of matrix groups, cf. Chap. 24 **[19]**; and it is this. Select a sequence $\{p_n\} \subset P(G)_1$ which converges to $\mathbf{1} \in P(G)_1$ (in the compact-open topology) "at such a rate" that (at least a subsequence of) $\{p_n^n\}$ converges (in the compact-open topology) to some $p_\infty \in P(G)_1$ with $p_\infty \neq \mathbf{1}$. Hopefully one then has that $p_\infty = e^\psi$ for some $\psi \in N_0(G)$, e.g., $\log p_\infty = \psi$ makes sense.

Let us go through an example of this computation in a simple but nontrivial situation. Let G be a discrete group, and let $\{x_1, x_2, x_3, \ldots, x_n\} = K$ be a finite subset. Suppose also that $G = \bigcup_{n=1}^{\infty} K^n$, i.e., G is finitely generated. Now let $\|f\| = \sup\{|f(x)|: x \in K\}$ for f any complex-valued function on K. If $\{p_n\}$ is a sequence in $P(G)_1$ which converges to $\mathbf{1}$ pointwise with $p_n \neq \mathbf{1}$ for all n, then $\|p_n - \mathbf{1}\| > 0$ for all n. (If $p_n(x) = 1$ for all $x \in K$, then $p_n(x) = 1$ for all $x \in G$. This is the only place where we use the fact that K generates G.) Now select ε, $0 < \varepsilon < 1$. Let p_{n_0} be the first element of $\{p_n\}$ such that $\|p_{n_0} - \mathbf{1}\| < \varepsilon/2$. Now let n_1 be the largest natural number such that $\|p_{n_0} - \mathbf{1}\| < \varepsilon/2n_1$. Relabel p_{n_0} as p_{n_1}. Then we have $\varepsilon/2(n_1 + 1) \leqq \|p_{n_1} - \mathbf{1}\| < \varepsilon/2n_1$. Now let $p_{n_{00}}$ be the first element of $\{p_n\}$ with $n_{00} > n_0$ such that $\|p_{n_{00}} - \mathbf{1}\| < \varepsilon/2(n_1 + 1)$. Let n_2 be the largest natural number so that $\|p_{n_{00}} - \mathbf{1}\| < \varepsilon/2n_2$. Relabel $p_{n_{00}}$ as p_{n_2}, and we have $\varepsilon/2(n_2 + 1) \leqq \|p_{n_2} - \mathbf{1}\| < \varepsilon/2n_2$ with $n_2 > n_1$. Continue by induction, obtaining a sequence $\{p_{n_j}\}_{j=1}^{\infty}$ such that

$$(*) \qquad \varepsilon/2(n_j + 1) \leqq \|p_{n_j} - \mathbf{1}\| < \varepsilon/2n_j, \; j = 1, 2, \ldots,$$

while p_{n_j} converges to $\mathbf{1}$ as $j \to \infty$.

A sequence of elementary estimates yields

$$(**) \qquad \varepsilon/4 < \|p_{n_j}^{n_j} - \mathbf{1}\| < 3\varepsilon/4$$

for sufficiently large j.

Now $\{p_{n_j}^{n_j}\}$ has a convergent subsequence (which we will again denote $\{p_{n_j}^{n_j}\}$ to simplify notation) in $P(G)_{\leqq 1}$, the collection of continuous functions of positive type of norm less than or equal to one, since $P(G)_{\leqq 1}$ is $\sigma(B(G), C^*(G))$ compact. So there is a $p_\infty \in P(G)_{\leqq 1}$ such that

$$\sigma(B(G), C^*(G)) - \lim_{j \to \infty} p_{n_j}^{n_j} = p_\infty,$$

in particular $\omega(\varepsilon_e) \in C^*(G)$ and

$$\lim_{j\to\infty} \langle p_{n_j}^{n_j}, \omega(\varepsilon_e)\rangle = \lim_{j\to\infty} (p_{n_j}(e))^{n_j} = 1 = p_\infty(e).$$

Thus $p_\infty \in P(G)_1$ and $p_{n_j}^{n_j}$ converges to p_∞ pointwise on G. We also have that $p_\infty \neq \mathbf{1}$ since $\varepsilon/4 \leqq \|p_\infty - \mathbf{1}\| \leqq 3\varepsilon/4$.

Now define

$$\psi(g) = \lim_{j\to\infty} n_j(p_{n_j}(g) - 1)$$

for each $g \in K$. We can see that this limit exists as follows:

$$|(p_{n_j}(g)^{n_j} - 1| < \varepsilon < 1$$

for all n_j and all $g \in K$. Thus the binomial series gives:

$$\begin{aligned} p_{n_j}(g) &= (1 + (p_{n_j}(g))^{n_j} - 1)^{1/n_j} \\ &= 1 + \sum_{k=1}^{\infty} \binom{1/n_j}{k} ((p_{n_j}(g))^{n_j} - 1)^k, \end{aligned}$$

hence

$$n_j(p_{n_j}(g) - 1) = \sum_{k=1}^{\infty} n_j \binom{1/n_j}{k} ((p_{n_j}(g))^{n_j} - 1)^k.$$

The series

$$\sum_{k=1}^{\infty} n \binom{1/n}{k} y^k$$

converges uniformly with respect to n and y for $\|y\| \leqq \rho < 1$. Thus for

$$\begin{aligned} \psi(g) &= \lim_{j\to\infty} n_j(p_{n_j}(g) - 1) \\ &= \sum_{k=1}^{\infty} ((-1)^{k-1}/k)(p_\infty(g) - 1)^k = \log p_\infty(g) \end{aligned}$$

and $\psi(g) \neq 0$ for at least one $g \in K$.

We now show that

$$\psi(g) = \lim_{j\to\infty} n_j(p_{n_j}(g) - 1) = \log p_\infty(g)$$

holds for all $g \in G$. Of course, if $|p_\infty(g) - 1| < 1$, then for sufficiently large j,

$$|(p_{n_j}(g))^{n_j} - 1| \leqq \rho < 1 \text{ for some } \rho > 0.$$

Thus the same argument as above shows that $\psi(g) = \lim_{j\to\infty} n_j(p_{n_j}(g) - 1)$ exists and $\psi(g) = \log p_\infty(g)$. If $|p_\infty(g) - 1| \geqq 1$, then there exists a complex number $e^{i\theta(g)}$ such that $|e^{i\theta(g)}p_\infty(g) - 1| < 1$. Thus

$$|e^{i\theta(g)}(p_{n_j}(g)^{n_j} - 1| = |(\exp(i\theta(g)/n_j)p_{n_j}(g))^{n_j} - 1| \leqq \rho < 1$$

for some $\rho > 0$ and sufficiently large j. By the binomial series argument above we then have that $\lim_{j\to\infty} n_j(\exp(i\theta(g)/n_j)p_{n_j}(g) - 1) = \log e^{i\theta(g)}p_\infty(g)$. But

$$\begin{aligned} &n_j(\exp(i\theta(g)/n_j)p_{n_j}(g) - 1) \\ &\quad = n_j(p_{n_j}(g) + (i\theta(g)/n_j)p_{n_j}(g) + (1/n_j^2)R_j(g) - 1) \end{aligned}$$

where $\{R_j(g)\}$ is a bounded complex-valued function of j. Thus

$$\begin{aligned} &\lim_{j\to\infty} n_j(\exp(i\theta(g)/n_j)p_{n_j}(g) - 1) \\ &\quad = \lim_{j\to\infty} \{n_j(p_{n_j}(g) - 1) + i\theta(g)p_{n_j}(g) + (1/n_j)R_j(g)\} \\ &\quad = \lim_{j\to\infty} n_j(p_{n_j}(g) - 1) + i\theta(g) \\ &\quad = \log e^{i\theta(g)}p_\infty(g) \end{aligned}$$

exists and we have

$$\begin{aligned} \psi(g) &= \lim_{j\to\infty} n_j(p_{n_j}(g) - 1) \\ &= \log e^{i\theta(g)}p_\infty(g) - i\theta(g) \\ &= \log p_\infty(g) \end{aligned}$$

for suitably defined continuation of the log function.

REMARK. It is easy to see that $|p_\infty(g)| \geqq \exp(-n^2\varepsilon/2)$ for all $g \in K^n$. (Use Proposition 5 to get $1 - n_j^2\varepsilon/2 \leqq |p_{n_j}(g)|$ for $g \in K^n$.)

In order to be certain that there are sequences $\{p_n\} \subset P(G)_1$ converging to **1** it is best to assume G is σ-compact, i.e., countable if G is discrete. If G is separable then $P(G)_1$ with the compact-open topology is a metrizable topological space. The assumption we made in the argument above, viz., G is finitely generated, is really not necessary. All we need in order that a ψ, defined as in the above discussion by a sequence $\{p_n\}$ converging to **1**, be not identically zero is that there be a point $g \in G$ such that $|p_n(g) - 1| > 0$ for all n. We have thus proved the following proposition.

PROPOSITION 9. *Let G be a countable discrete group. Let $\{p_n\}$ be a sequence in $P(G)_1$ converging pointwise to $\mathbf{1} \in P(G)_1$. If there is a $g \in G$ such that $|p_n(g) - 1| < 0$ for all n, then there exists a (possibly relabeled) subsequence $\{p_{n'}\}$* of *$\{p_n\}$ so that*

$$\psi(g) = \lim_{n'\to\infty} n'(p_{n'}(g) - 1)$$

exists for all $g \in G$ and $\psi \in N_0(G)$, $\psi \not\equiv 0$.

COROLLARY. *If G is a finitely generated discrete group and $\{p_n\}$ is a sequence in $P(G)_1$ converging to $\mathbf{1}$ in $P(G)_1$ with the compact-open topology, such that $p_n \neq \mathbf{1}$ for all n, then there exists a (possibly relabeled) subsequence $\{p_{n'}\}$ of $\{p_n\}$ such that*

$$\psi(g) = \lim_{n' \to \infty} n'(p_{n'}(g) - 1)$$

exists for all $g \in G$, $\psi \in N_0(G)$ and $\psi \not\equiv 0$.

REMARK. Such sequences as hypothesised in Proposition 9 always exist, provided G has more than one point.

REMARK. We have a certain uniqueness result here, in that up to a positive multiple the ψ of Proposition 9 depends only on its defining sequence. If

$$\psi = \lim_{n' \to \infty} n'(p_{n'} - 1)$$

and n'' is another subsequence of the natural numbers which converges to ∞ as $n' \to \infty$, then

$$\begin{aligned} \lim_{n' \to \infty} n''(p_{n'} - \mathbf{1}) &= \lim_{n' \to \infty} (n''/n')n'(p_{n'} - \mathbf{1}) \\ &= \lambda\psi, \\ \text{if } \lim_{n' \to \infty} (n''/n') &= \lambda \in [0, \infty]. \end{aligned}$$

We now briefly turn our attention to generalizing the foregoing results from the discrete case to the (continuous) locally compact case. Our first observation is of a negative nature. We exhibit a sequence $\{p_n\} \subset P(G)_1$ converging to $\mathbf{1}$, uniformly on compact sets, such that no matter what subsequence is chosen or how the "rate of convergence" is adjusted the (continuous) $\psi \in N_0(G)$ obtained is identically zero. The reader may note that $G = \mathbf{R}^1$ exhibits the following example.

Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of compact neighborhoods of $e \in G$ with the interior of U_n containing U_{n+1} for all n and $\bigcap_{n=1}^{\infty} U_n = \{e\}$. Let $q_n \in P(G)_1$ have support contained in U_n, cf. [12]. Now let

$$p_n = 2^{-1/n}\mathbf{1} + (1 - 2^{-1/n})q_n.$$

Clearly $p_n \in P(G)_1$. Thus given any $g \neq e$ in G, for sufficiently large n, $p_n(g) = 2^{-1/n}$; and hence $\lim_n p_n = \mathbf{1}$ in $P(G)_1$ with the compact-open topology. Thus no matter what subsequence we choose, or the rate of convergence chosen we must obtain that ψ is constant for all $g \in G$, $g \neq e$. If $\psi \in N_0(G)$, then continuity requires that $\psi(g) = 0$ for all $g \in G$.

This example is not disappointing. It merely makes evident the fact that we need to impose some condition (such as we imposed in Proposition 9) to guarantee that the semiderivatives ψ obtained by the "differentiation process" (of Proposition 9) are not identically zero. The careful reader will find that the proof of Proposition 9 generalizes to arbitrary locally compact G, except at one crucial point. When a $\sigma(P(G), C^*(G))$- convergent subsequence of $\{p_{n_j}^{n_j}\}_{n_j}$ is obtained we have no a priori guarantee that the limit, p_∞, is in $P(G)_1$. In order to generalize the proof we must assume some condition which will guarantee that a limit point (not equal to **1**) of $\{p_{n_j}^{n_j}\}$ will be in $P(G)_1$. Suppose, in addition to having our sequence $\{p_{n_j}\}$ satisfy (*) in the proof of Proposition 9, we have that $\{p_{n_j}\}$ also satisfies condition (C): Given any $\eta > 0$ there exists a neighborhood of e, U_η, such that for all n_j, $\sup\{|p_{n_j}(g) - 1|: g \in U_\eta\} < \eta/2n_j$. It then follows that

$$\sup\{|p_{n_j}(g)^{n_j} - 1|: g \in U_\eta\} \leqq (3/4)\eta$$

for all n_j and hence if p_∞ is a $\sigma(P(G), C^*(G))$ limit point of $\{p_{n_j}^{n_j}\}$ that

$$\|p_\infty\| \geqq |(1/|U_\eta|) \int_G p_\infty(y)\chi_{U_\eta}(g)\, dg| \geqq 1 - 3\eta/4,$$

where dg is left-Haar measure on G, χ_{U_η} is the characteristic function of U_η and $|U_\eta| = \int_G \chi_{U_\eta}(g)\, dg$. Since $\eta > 0$ was arbitrary, $\|p_\infty\| = 1$, thus $p_\infty \in P(G)_1$. This said, we leave to the reader the routine task of stating and proving an appropriate generalization of Proposition 9 to "continuous" σ-compact, locally compact groups.

Although our condition (C) as well as other conditions on $P(G)_1$ can be verified in certain circumstances, we have found them to be unwieldy in general. We have found it usually simpler to work in $N_0(G)$ (using the additive "Lie cone" technique mentioned at the beginning of this section) in order to obtain results about $N_0(G)$ and one-parameter semigroups in $P(G)_1$. As an example of this we offer Theorem 2 below, cf. [**2**].

3. Cohomology. The central role that functions of positive type play in the representation theory of locally compact groups and their associated algebras is in large part due to a well-known device called the Gelfand-Naimark-Segal (or G.N.S.) construction, cf. 2.4.4. [**8**]. Very briefly if $A = M_1(G)$ is the Banach (convolution) algebra with involution of the (bounded) complex Borel measures on G, then a continuous positive definite function p on G defines a pre-Hilbert space structure on A (modulo N_p), i.e.,

$$\begin{aligned}(\xi + N_p|\xi + N_p)_p &= \int_G p(g)\, d\xi^\# * \xi(g) \\ &= \int_G\int_G p(h^{-1}g)\, d\xi(g)\, d\underline{\xi}(h)\end{aligned}$$

for $\xi \in A$ where

$$N_p = \left\{\eta \in A: \int_G p(g)\, d\eta^\# * \eta(g) = 0\right\}$$

is a left-ideal in A. A continuous representation π_p of A on Hilbert space H_p, the completion of A/N_p with respect to $(\cdot|\cdot)_p$, is determined by $\pi_p(\mu)(\xi + N_p) = \mu * \xi + N_p$ for $\xi \in A$. The restriction of π_p to $G \subset A$ is a continuous unitary representation of G. In the above construction we could have taken A to be $L^1(G)$, the Haar-integrable functions on G, or $C_c(G)$.

In analogy with the G.N.S. construction above we have the following for $\psi \in N_0(G)$. This time let $A = M_c^1(G)$, the convolution algebra with involution of (bounded) complex Borel measures with compact support. (Note that the role of compact support is to assure that $(\xi|\eta)_\psi$, defined below, is finite for ξ, $\eta \in A$. Using Definition 7 we could choose A to be a larger class of measures.)

We define a sesquilinear form on A as follows:

$$(\xi|\eta)_\psi = \int [\psi(h^{-1}g) - \overline{\psi(h)} - \psi(g)]\, d\xi(g) d\bar{\eta}(h)$$

where $\bar{\eta}$ is the complex conjugate of measure $\eta \in A$. Let $\|\xi\|_\psi = (\xi|\xi)_\psi^{1/2}$. By the second corollary following Proposition 4 $(\cdot|\cdot)_\psi$ is indeed a pre-inner product and it follows by a routine calculation that $(\Pi_\psi(g)\xi|\Pi_\psi(g)\eta)_\psi = (\xi|\eta)_\psi$ for all ξ, $\eta \in A$, where $\Pi_\psi(g)\xi = \varepsilon_g * \xi - \xi(G)\varepsilon_g$, with ε_g the unit point mass at $g \in G$ and $\xi(G) = \int_G d\xi(h)$ for $\xi \in A$. [Note that $\Pi_\psi(g)A \subset A_0 = \{\xi \in A: \xi(G) = 0\}$.] The subspace $N_\psi = \{\xi \in A: \|\xi\|_\psi = 0\}$ is invariant under $\Pi_\psi(g)$ for all $g \in G$; and hence a continuous unitary representation Π_ψ of G on the Hilbert space H_ψ, the completion of A/N_ψ with respect to $(\cdot|\cdot)_\psi$, is determined by $\pi_\psi(g)(\xi + N_\psi) = \varepsilon_g * \xi - \xi(G)\varepsilon_g + N_\psi$. [Note that continuity of π_ψ at e, hence at g, in G follows by a straightforward calculation.]

We thus have the following result.

PROPOSITION 10. *Given $\psi \in N_0(G)$ there exists a pair (π_ψ, c_ψ) consisting of a continuous unitary representation π_ψ of G on a Hilbert space H_ψ, together with a continuous cocycle $c_\psi: G \to H_\psi$ for π_ψ, i.e.,*

$$c_\psi(gh) = c_\psi(g) + \pi_\psi(g)c_\psi(h).$$

PROOF. All that remains is to define c_ψ. Let $c_\psi(g) = \varepsilon_g + N_\psi \in H_\psi$ for $g \in G$. Then for g, $h \in G$, $c_\psi(gh) = \varepsilon_{gh} + N_\psi = (\varepsilon_g + N_\psi) + (\varepsilon_{gh} - \varepsilon_g + N_\psi) = c_\psi(g) + \pi_\psi(g)c_\psi(h)$. Since $\|c_\psi(g) - c_\psi(h)\|_\psi^2 = -2\mathrm{Re}\, \psi(h^{-1}g)$, c_ψ is continuous.

REMARK. If ψ is replaced by $\psi + i\chi$ where χ is any continuous homomorphism of G into the additive reals, then $H_{\psi+i\chi} = H_\psi$, $c_{\psi+i\chi} = c_\psi$ and $\pi_{\psi+i\chi} = \pi_\psi$.

That functions of negative type are related to cohomology of group representations (or group actions) is at least hinted at in the form that is taken by the kernel $[\psi(h^{-1}g) - \overline{\psi(h)} - \psi(g)]$. Let us recall the basic definitions of group cohomology.

General references for group cohomology are **[26]**, **[11]**. Constant maps of G into a fixed abelian group Γ, upon which G acts, are called 0-cochains. The set of functions from G to Γ is called the set of 1-cochains, and in general the set of functions of n variables in G to Γ is called the set of n-cochains. The set of n-cochains is an abelian group if addition of n-cochains is defined pointwise. If f is a 1-cochain, then by definition of δ, $\delta f(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$. The group homomorphism δ is called the coboundary operator (applied to 1-cochains) and its range is by definition the set of 2-coboundaries. The 2-coboundary δf is said to be cobounded by f. This basic cohomological structure has already occurred both explicitly and implicitly.

First if $\Gamma = H_\psi$ and G acts on H_ψ via π_ψ the map c_ψ defined in Proposition 10 satisfies $\pi_\psi(g)c_\psi(h) - c_\psi(gh) + c_\psi(g) = 0 = \delta c_\psi(g, h)$. Thus c_ψ is in the kernel of (group homomorphism) δ. By definition 1-cochains which are mapped by δ to the identically zero 2-cochain are called 1-cocycles. Thus c_ψ is a 1-cocycle (relative to the action π_ψ); sometimes such maps are called "crossed homomorphisms." Here the symbol $Z^1(G, H(\pi_\psi))$ denotes all such continuous crossed homorphisms for π_ψ. We note in passing that the operator δ applied to a 0-cochain f is defined by $\delta f(g) = \pi_\psi(g)v - v$, for $g \in G$, where $f(g) = v \in H_\psi$ for $g \in G$. The range of δ applied to the 0-cochains is denoted $B^1(G, H(\pi_\psi))$, the 1-coboundaries for π_ψ; this being a subgroup of $Z^1(G, H(\pi_\psi))$, we get the quotient group $H^1(G, H(\pi_\psi)) = Z^1(G, H(\pi_\psi))/B^1(G, H(\pi_\psi))$, the first ("continuous") cohomology group of π_ψ.

Second, if $\Gamma =$ the additive real line and G acts trivially on Γ, i.e., leaves all points fixed, then $\delta\psi(h^{-1}, g) = \psi(g) - \psi(h^{-1}g) + \overline{\psi(h)}$, so our kernel is a coboundary with a "twist." Since $(c_\psi(g)|c_\psi(h))_\psi = \psi(h^{-1}g) - \overline{\psi(h)} - \psi(g)$, we have that $-\delta\psi(h, g) = (c_\psi(g)|c_\psi(h^{-1}))_\psi$ is cobounded by $-\psi$. In particular $\mathrm{Re}(c_\psi(g)|c_\psi(h^{-1}))_\psi$ is cobounded by $-\mathrm{Re}\ \psi(\cdot) = (1/2)\|c_\psi(\cdot)\|_\psi^2$ and $\mathrm{Im}(c_\psi(g)|c_\psi(h^{-1}))_\psi = -\mathrm{Im}(c_\psi(g^{-1})|\pi_\psi(g^{-1})c_\psi(h^{-1}))_\psi$ is cobounded by $-\mathrm{Im}\ \psi(\cdot)$.

The relevance of these last observations will be more evident after the next Proposition. Let us first recall how δ, the coboundary operator, is applied to 2-cochains. Namely $\delta f(g_1, g_2, g_3) = g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2)$. When δ is applied to 2-cochains, its kernel is called the set of 2-cocyles. (Sometimes 2-cocycles are called "factor sets" because of their role, mentioned below, in group extensions.) If f is any 1-cochain, $\delta(\delta f) = 0$; thus any 2-coboundary is a 2-cocycle. The symbol $Z^2(G, H(\pi_\psi))$ denotes the set of all continuous 2 cocycles for π_ψ and $B^2(G, H(\pi_\psi))$ the continuous 2-coboundaries for π_ψ. The quotient group $H^2(G, H(\pi_\psi)) =$

$Z^2(G, H(\pi_\psi))/B^2(G, H(\pi_\psi))$ is the ("continuous") second cohomology group for π_ψ.

We are now ready to state and prove a "converse" of Proposition 10. We first saw this proposition proved by C.C. Moore in the Moore-Rieffel seminar on group representations, Berkeley.

PROPOSITION 11. *Given a continuous complex Hilbert space valued* 1-*cocycle c for continuous unitary representation π of G on H_π there exists a continuous function ψ of negative type* (*zero at the identity*) *such that $c = c_\psi$ and $\pi = \pi_\psi$ where π_ψ and c_ψ are as defined in the proof of Proposition* 10. *The function ψ is either defined on G or $\tilde{G}$, an extension of G. The group $\tilde{G}$ may be taken to be a central extension of G by* **R**, *the additive reals, with respect to the trivial action of G on* **R**. *In fact $\tilde{G}$ may be taken to be the "multiplier" extension defined by the "multiplier", i.e., factor set or* 2-*cocycle*, Im $(c(h), c(g^{-1}))$, *for $g, h \in G$.*

REMARK. The equalities $c = c_\psi$ and $\pi = \pi_\psi$ are in the sense of unitary equivalence. Thus if $\psi \in N_0(G)$ there is an isometry u from H_c, the closure of the linear span of $\{c(g)\colon g \in G\}$, onto H_ψ, the closure of the linear span of $\{c_\psi(g)\colon g \in G\}$, such that $u(\sum_{i=1}^n \lambda_i c(g_i)) = \sum_{i=1}^n \lambda_i c_\psi(g_i)$ for any $g_1, \ldots, g_n \in G$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, the complex numbers. We also have $u\pi(g)u^{-1}\xi = \pi_\psi(g)\xi$ for all $g \in G$, $\xi \in H_\psi$. In the case where $\psi \in N_0(\tilde{G})$, $\tilde{G}$ defined below; we have that $c_\psi(s, g) = c_\psi(0, g)$ and $\pi_\psi(s, g) = \pi_\psi(0, g)$ for all $s \in \mathbf{R}$, $g \in G$. In this case there is an isometry $u\colon H_c \to H_\psi$ such that $u(\sum_{i=1}^n \lambda_i c(g_i)) = \sum_{i=1}^n \lambda_i c_\psi(0, g_i)$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, $g_1, \ldots, g_n \in G$; and $u\pi(g)u^{-1}\xi = \pi_\psi(0, g)\xi$ for $\xi \in H_\psi$, $g \in G$.

PROOF. The main idea is this. If a complex-valued function ψ_0 on G satisfies $\psi_0(e) = 0$, $\psi_0(g^{-1}) = \overline{\psi_0(g)}$ for all g in G, and ψ_0 cobounds $(c(g)\,|\,c(h))$ with a "twist", i.e.,

$$\delta\psi_0(h^{-1}, g) = \psi_0(g) - \psi_0(h^{-1}g) + \overline{\psi_0(h)} = (c(g)\,|\,c(h)),$$

then $-\psi_0 \in N_0(G)$. This follows from Theorem 1, equation (1). Now although $(c(\cdot), c(\cdot))$ may not be cobounded on G it is always cobounded on a suitable extension of G. This is what we will now show.

If c is a 1-cocycle for π and we define $\psi_1(g) = (1/2)(c(g)\,|\,c(g))$ for all $g \in G$, then it is easy to show that $\delta\psi_1(h^{-1}, g) = \mathrm{Re}(c(g)\,|\,c(h))$. Thus if 1-cocycle c has range in a real Hilbert space, i.e., $(c(g)\,|\,c(h))$ is real for all $h, g \in G$, then $-\psi_1 \in N_0(G)$. In general it is easy to show (taking the action of G on $\mathbf{R}$ to be trivial) that $m(g, h) = \mathrm{Im}(c(h)\,|\,c(g^{-1}))$ for $g, h \in G$ is a 2-cocycle, sometimes called a factor set; or were we to look at $e^{im(g,h)}$, the latter would often be called a multiplier. By p. 112 [**26**] the group $\tilde{G} = \mathbf{R} \times_m G$, where $(s, g)(t, h) = (s + t + m(g, h), gh)$ for $s, t \in \mathbf{R}$, $g, h \in G$, is indeed a locally compact group. If ψ_2 is the map $(s, g) \in \tilde{G} \mapsto$

$-s \in \mathbf{R}$, then $\delta(\psi_2)((-t, h^{-1}), (s, g)) = \operatorname{Im}(c(g) | c(h))$. Extending ψ_1 trivially to $\tilde{G}$, i.e., $\psi_1(s, g) = \psi_1(g)$ for all $s \in \mathbf{R}$, let $\psi_0 = \psi_1 + i\psi_2$ on $\tilde{G}$. We have that $\psi_0(0, e) = 0$, and since $m(g^{-1}, g) = 0$ for all $g \in G$ $\psi_0(s, g)^{-1}) = \overline{\psi_0(s, g)}$. Also $\delta\psi_0((t, h)^{-1}, (s, g)) = (c(g), c(h))$. Thus $\psi = -\psi_0 \in N_0(\tilde{G})$.

COROLLARY. *If the cocycle of Proposition* 11 *is real Hilbert space valued, then the* ψ *of Proposition* 11 *can always be taken in* $N_0(G)$.

REMARK. The function ψ of Proposition 11 is unique up to an $i\chi$ where χ is a continuous homorphism of $\tilde{G}$ (or G) into $\mathbf{R}$. Thus Propositions 10 and 11 say roughly that modulo the tangents in $N_0(G)$, the semitangents are in a fairly precise manner a description of the Hilbert space valued continuous crossed-homomorphisms of G, i.e., continuous 1-cocycles for unitary representations. In fact the map $\psi \mapsto (\pi_\psi, c_\psi)$ of Proposition 10 and its "inverse" defined by Proposition 11 indeed define a one-to-one correspondence if one restricts attention to real negative definite functions ψ and real Hilbert space valued cocycles c.

REMARK. The conclusion of Proposition 11 can be simplified to say that $\psi \in N_0(G)$, if every real valued 2-cocycle is known to be a coboundary, e.g., for separable, simply-connected semisimple Lie groups. Passing from Borel measurability to continuity can be done by using the second corollary to Definition 4 and Theorem 3.1 **[29]**. See also V.S. Varadarajan, Geometry of Quantum Theory, Vol. II, Chapter X.

As a simple application of these ideas we can generalize Proposition 7.13 of **[5]** to all locally compact groups and at the same time simplify the proof.

PROPOSITION 12. *Let* $\psi \in N(G)$ *and suppose* ψ *is bounded on* G, *then there exists a constant* $m \geqq 0$ *such that* $m + \psi \in P(G)$; *i.e.*, $m + \psi$ *is a continuous function of positive type.*

PROOF. Construct π_ψ and c_ψ as in Proposition 10. Since ψ is bounded, Re ψ is bounded, hence c_ψ is bounded and is hence a 1-coboundary, by 3.4 or 3.7 **[21]**, i.e., there is a $v \in H_\psi$ (without loss of generality $v \neq 0$) such that $c_\psi(g) = \pi_\psi(g)v - v$. (Note that we can find this v directly by applying the Ryll-Nardzewski fixed point theorem, cf. **[14]**, to the action $g \cdot w = \pi_\psi(g)w - c_\psi(g)$ of g on H_ψ.) Thus

$$\begin{aligned} \psi(h^{-1}g) - \overline{\psi(h)} - \psi(g) &= (c_\psi(g) | c_\psi(h))_\psi \\ &= (\pi_\psi(g)v - v | \pi_\psi(h)v - v)_\psi \\ &= (\pi_\psi(h^{-1}g)v | v)_\psi - (\pi_\psi(g)v | v)_\psi \\ &\quad - \overline{(\pi_\psi(h)v | v)_\psi} + \|v\|_\psi^2. \end{aligned}$$

Let $p(k) = (\pi_\psi(k)v | v)_\psi$ for all $k \in G$, where $p(\cdot)$ is continuous and of positive type. Thus for g, $h \in G$ $\psi(h^{-1}g) - \overline{\psi(h)} - \psi(g) = (p(h^{-1}g) - m) - \overline{(p(h) - m)} - (p(g) - m)$ where $m = \|v\|_\psi^2 > 0$. Hence $\chi = \psi + m - p$ satisfies $\chi(h^{-1}g) = \overline{\chi(h)} + \chi(g) = \chi(h^{-1}) + \chi(g)$, for g, $h \in G$, i.e., χ is a continuous homomorphism of G into the complex numbers, and χ is of negative type. By Proposition 6, $\chi = i\gamma$ where γ is a continuous homomorphism of G into the additive reals. Finally $\psi + m = p + i\gamma$ is bounded on G, hence $\gamma(g) = 0$ for all $g \in G$. In the first sentence of this proof we tacitly assumed $\psi \in N_0(G)$; but if $\psi \in N(G)$ we can without loss replace ψ by $\psi - \psi(e)$.

COROLLARY. *If $\psi \in N(G)$ and* Re ψ *is bounded on G, then there exists a homomorphism γ of G into the additive reals and a constant $m > 0$ such that $\psi + m - i\gamma \in P(G)$, i.e., is continuous and of positive type.*

4. Isometric Imbeddings of a Group in Hilbert Space. It is perhaps fitting that as we near the end of this paper we return to the beginning of the subject as it appears in the early papers of von Neumann and Scheonberg, **[28]**, **[33]**. One of the problems of interest to them was the construction of all (semi) metrics ρ on $\mathbf{R}$, the real line, which allowed $(\mathbf{R}, \rho)$ as a metric space to be imbedded isometrically in a (real) Hilbert space. In particular they were interested in finding *screw functions* (their terminology) F on $\mathbf{R}$ so that $\rho(x, y) = F(x - y)$ for x, $y \in \mathbf{R}$. One of the elementary screw functions found was $F(x) = |\sin(\omega x)|$, ω fixed in $\mathbf{R}$. We can now give a rather novel proof that this is indeed a screw function. First of all $\sin^2(\omega x) = (1 - \cos(2\omega x))/2$ for $x \in \mathbf{R}$ is easily seen to be in $-N_0(\mathbf{R})$ since $\cos(2\omega x) = \operatorname{Re} e^{i2\omega x}$, for $x \in \mathbf{R}$, is in $P(\mathbf{R})_1$. By Proposition 5, $|\sin(\omega x)| = (\sin^2(\omega x))^{1/2}$ for $x \in \mathbf{R}$ is a seminorm on $\mathbf{R}$. From our considerations of cohomology we have a cocycle c which imbeds $\mathbf{R}$ into a (real) Hilbert space H so that $\|c(x) - c(y)\|_H = (2|\sin \omega(x - y)|)^{1/2}$. We might note in passing that if $\psi \in N(G)$, then $-(-\psi)^\alpha$ (suitably defined when ψ is complex) for $\alpha \in (0, 1)$ is also of negative type. A useful formula in this regard, used also by von Neumann and Scheonberg, is

$$x^\alpha = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-xs}) s^{-\alpha - 1}\, ds \quad \text{for } x > 0,$$

Γ the gamma-function, $\alpha \in (0, 1)$. This last formula is a special case of a Bernstein function; a subject discussed fully in **[5]** Chapter II §9, wherein it is shown that if $-\psi \in N(G)$ then the composition $f \circ (-\psi) \in -N(G)$ whenever f is a Bernstein function continuous on the closed right-half plane and holomorphic in the open right-half plane. It is interesting to note that $-|\cdot|^\alpha \in N_0(\mathbf{R})$ for $\alpha \in (0, 2]$, but not for $\alpha > 2$.

We can now relatively painlessly solve the von Neumann-Schoenberg "screw function problem" for any locally compact group G, and simultaneously establish a direct link with the Lévy-Khinchin formula from probability theory.

We start with a theorem which in a slightly simplified form appeared as early as 1928 in a paper by K. Menger, [27]. We take the following from [40] Lemma 2.3, Theorem 2.1 and Remark 3.2.

PROPOSITION 13. *A (semi) metric space* (X, ρ) *is embeddable isometrically in some (real) Hilbert space H if and only if fixing some point* $x_0 \in X$ *the quadratic form*

$$\sum_{i,j=1}^{n} (1/2)\{\rho(x_0, x_j)^2 + \rho(x_0, x_i)^2 - \rho(x_j, x_i)^2\}\lambda_j\lambda_j \geqq 0$$

for all choices of natural number n, all choices $x_1, \ldots, x_n \in X$, *and all choices* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}$.

REMARK. As a corollary of the ideas in Proposition 13 we find that ℓ^p $(p \neq 2)$ cannot be imbedded isometrically in any Hilbert space, cf. [40]. By (X, ρ) being isometrically embedded in H we mean, of course, that there is a function $\varphi: X \to H$ such that $\rho(x, y) = \|\varphi(x) - \varphi(y)\|_H$ for all $x, y \in X$. The 1/2 factor in the above quadratic form is irrelevant to us at this time and can be dropped.

If a continuous function F on G is to be a screw function for ρ, a continuous semi-metric on G, we must have by definition that $F(x^{-1}y) = \rho(x, y)$ for any $x, y \in G$. It is, of course, convenient to pick the x_0 of Proposition 13 to be the identity, e, of G; and we shall do so. We then easily find that a necessary and sufficient condition for a non-negative, real valued function F to be a screw function is that $-F^2 \in N_0(G)$.

We can not yet say we have "solved" the screw function problem unless we have a "concrete" description of the (continuous) functions of negative type on G. It turns out that a convenient description is possible; namely, one can express an arbitrary $\psi \in N_0(G)$ as an integral over extreme rays in $N_0(G)$ following Choquet (at least for separable G), [30]. E. Larsen, in his Ph.D. thesis [23], is giving a description of this process among other things. Of course, if $G = \mathbf{R}$ the results of this process of decomposing a $\psi \in N_0(G)$ as "barycenter of extreme rays" yields the classical Lévy-Khinchin formula characterizing infinitely divisible laws in probability. See [6] for a proof (not using Choquet's theorem) that $N(\mathbf{R}^n)$ is given exactly by the Lévy-Khinchin formula.

5. A Characterization of $\mathbf{N_0(G)}$ using Difference and Differential Operators on G. A somewhat different approach to characterizing $N_0(\mathbf{R}^n)$ is taken by I.M. Gelfand and N. Ya. Vilenkin in [13] Chapter II, Section 4, wherein they show that conditionally positive definite functions on $\mathbf{R}^n$ (of order 1) are characterized by a Lévy-Khinchin type formula. We will now show that a weak version of the definition they take for "conditionally positive definite generalized function on $\mathbf{R}^n$ of order 1" can be generalized to any Lie group G and that it coincides with our definition of $N(G)$. For lack of an immediate need and enough space we stop short of pushing all of this approach to the generality of locally compact G, using say **[22]**.

Let G be a Lie group and $\mathscr{G}_\ell$ its Lie algebra of left invariant (analytic) vector fields, $\mathscr{G}_r$ its Lie algebra of right invariant (analytic) vector fields. Let $\mathscr{D}(G)$ be the locally convex (nuclear) topological vector space of infinitely differentiable functions with compact support on G equipped with the usual inductive limit topology. Let $\mathscr{D}'(G)$ be the continuous, complex-valued linear functionals on $\mathscr{D}(G)$, i.e., the distributions on G. If L is a linear map of $\mathscr{D}(G)$ into $\mathscr{D}(G)$ let $L\mathscr{D}(G) \equiv \{L\varphi \colon \varphi \in \mathscr{D}(G)\}$. For $g \in G$ let $\lambda(g)$ (respectively, $\rho(g)$) denote convolution on the left (respectively, the right) by the measure ε_g, the unit point mass at g. Also if $\varphi \in \mathscr{D}(G)$, $\varphi^\#(g) = \Delta(g^{-1})\bar{\varphi}(g^{-1})$ for $g \in G$, where $\bar{\varphi}$ is the complex congugate of φ and Δ is the modular function of G. If S is a subset of $\mathscr{D}(G)$ we let $\langle S \rangle^-$ be the closed subspace of $\mathscr{D}(G)$ generated by the complex-linear span of S. We denote by $\mathscr{D}(G)_0 = \{\varphi \in \mathscr{D}(G) \colon \int_G \varphi(g)dg = 0\}$, i.e., the kernel of left Haar measure viewed as a distribution. We have the following Lemma.

LEMMA. *If $X_1, \ldots, X_n$ is a basis of $\mathscr{G}_r$, then,*

$$\langle \bigcup_{i=1}^{n} X_i\mathscr{D}(G) \rangle^- = \mathscr{D}(G)_0 = \langle \{(\lambda(g) - \lambda(e))\mathscr{D}(G) \colon g \in G\} \rangle^-.$$

PROOF. Let u be in $\mathscr{D}'(G)$, and let $\langle u, X_i\varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(G)$, $i = 1, \ldots, n$. Now $u * \varphi \in \mathbf{C}^\infty(G)$, the infinitely differentiable functions on G, p. 489 **[39]**. Also for all $Y \in \mathscr{G}_\ell$, $Y(u * \varphi) = u * (Y\varphi) = 0$ since $0 = \langle u, (Y^\vee)^g \lambda(g)\varphi^\vee \rangle = \langle u, \lambda(g)(Y^\vee \varphi^\vee) \rangle = \langle u, \lambda(g)(Y\varphi)^\vee \rangle = u * (Y\varphi)(g)$ for all $g \in G$. Note that $(Y^\vee)^g \in \mathscr{G}_r$ is the image of $Y \in \mathscr{G}_\ell$ under the map of $\mathscr{G}_\ell$ onto $\mathscr{G}_r$ determined by $x \in G \mapsto x^{-1} \in G$ followed by the inner automorphism $x \in G \mapsto gxg^{-1} \in G$. Also $\varphi^\vee(x) = \varphi(x^{-1})$ for all $x \in G$, by definition. Thus $u * \varphi$ is a constant function on G. Letting φ run through an approximate identity at $e \in G$ we find that u is a scalar multiple of Haar measure. Now for $i = 1, \ldots, n$, $X_i\mathscr{D}(G) \subset \mathscr{D}(G)_0$, since $X_i\varphi = \lim_{t \to 0} (1/t)[\lambda(\exp(-tX_i)) - \lambda(e)]\varphi$; and $[\lambda(g) - \lambda(e)]\varphi \in \mathscr{D}(G)_0$ for all $g \in G$. If $\langle \bigcup_{i=1}^n X_i\mathscr{D}(G) \rangle^- \subsetneqq \mathscr{D}(G)_0$ then by the Hahn-Banach separation theorem **[32]** Theorem 3.5 there is a $u_0 \in \mathscr{D}'(G)$ such that u_0 vanishes on $\langle \bigcup_{j=1}^n X_i\mathscr{D}(G) \rangle^-$ but not on $\mathscr{D}(G)_0$. But u_0 is a scalar multiple of Haar measure and we are done. A similar even easier argument shows that $\langle \{[\lambda(g) - \lambda(e)]\mathscr{D}(G) \colon g \in G\} \rangle^- = \mathscr{D}(G)_0$, cf. Proposition 5.2.1.2, **[39]**.

If $D = \sum_{i=1}^n a_i X_i$ is in $\mathscr{G}_\ell$ where $a_i \in \mathbf{C}$ $i = 1, 2, \ldots, n$, define $\bar{D} \in \mathscr{G}_r$ by $\bar{D}\varphi = (D\varphi^\#)^\#$ for all $\varphi \in \mathscr{D}(G)$. Then $D\bar{D}(\varphi * \varphi^\#) = D(\varphi * \bar{D}\varphi^\#) = (D\varphi) * (D\varphi)^\#$. Similarly we define $L_g\varphi = [\lambda(g) - \lambda(e)]\varphi$ and $\bar{L}_g\varphi = (L_g\varphi^\#)^\#$ for $\varphi \in \mathscr{D}(G)$ and $g \in G$. Then $L_g\bar{L}_g(\varphi * \varphi^\#) = (L_g\varphi) * (L_g\varphi)^\#$ for $\varphi \in \mathscr{D}(G)$. First we have the following result.

PROPOSITION 14. *A continuous function ψ (viewed as a distribution) is in $N(G)$ if and only if $\psi(e) \leqq 0$, $\psi^{\flat} = \psi$, and either*

(1) $\left\langle \psi, \left(\sum_{i=1}^{n} a_i X_i \varphi_i\right) * \left(\sum_{i=1}^{n} a_i X_i \varphi_i\right)^{\#} \right\rangle \geqq 0$ *for all* $\varphi_1, \ldots, \varphi_n \in \mathscr{D}(G)$ where $a_i \in C$, $i = 1, \ldots, n$; *and* $X_1, \ldots, X_n$ *is a fixed basis of* $\mathscr{G}_r$, *n a positive integer,*

or

(2) $\left\langle \psi, \left(\sum_{i=1}^{n} \lambda_i L_{g_i} \varphi_i\right) * \left(\sum_{i=1}^{n} \lambda_i L_{g_i} \varphi_i\right)^{\#} \right\rangle \geqq 0$ *for all* $\varphi_1, \ldots, \varphi_n \in \mathscr{D}(G)$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, $g_1, \ldots, g_n \in G$, *n a positive integer.*

PROOF. Inequality (1) says that $\int_G \psi(g)\eta * \eta^{\#}(g)dg \geqq 0$ for all $\eta \in \mathscr{D}_0(G)$ by the lemma. Now if $f \in C_c(G)$ has integral zero, then for all i, $f * \varphi_i \in \mathscr{D}(G)_0$ where $\{\varphi_i\}$ is an approximate unit in $\mathscr{D}(G)$ at $e \in G$. This implies that $\int_G \psi(g) f * f^{\#}(g)dg \geqq 0$ for all $f \in C_c(G)$ such that $\int_G f(g)dg = 0$. Thus by part (3) of the second corollary to Proposition 4, $\psi \in N(G)$. The converse is clear. The proof involving inequality (2) is similar.

PROBLEM. What class of distributions ψ satisfy $\langle \psi, D\bar{D}(\varphi * \varphi^{\#})\rangle \geqq 0$ or $\langle \psi, L_g \bar{L}_g(\varphi * \varphi^{\#})\rangle \geqq 0$ for all $\varphi \in \mathscr{D}(G)$, where D, $\bar{D}$, L_g, $\bar{L}_g$ are defined as above?

REMARK. In the sense of distributions $D\bar{D}\psi$ is of positive type, if $\psi \in N(G)$. Thus ψ is a derivative on the dual of G as well as a "global antiderivative" of sorts on G.

REMARK. The formulation of functions of negative type given by equation (2) generalizes easily to the general locally compact case. Also Proposition 14 can be stated using $\mathscr{G}_r$ and ρ.

6. Some Final Remarks. We will close this paper with a brief mention of some applications and examples. First of all, when looking for examples of functions of negative type one encounters basically two types of groups. First there are those groups G for which $\{\mathbf{1}\}$, viewed as the set containing the trivial one-dimensional continuous irreducible unitary representation of G, is open (with respect to the Fell topology) in $\hat{G}$, the collection of all continuous, irreducible unitary representations of G. Such groups are said to be Kazhdan groups or groups with property (T). Second, there are groups without property (T), i.e., $\{\mathbf{1}\}$ is not open in $\{\hat{G}\}$. If $\{\mathbf{1}\}$ is an isolated point in $\hat{G}$ it seems intuitively clear that one cannot "differentiate classically," even though $\{\mathbf{1}\} \subset P(G)_1$ is not strictly isolated in the compact-open topology of $P(G)_1$. We are led to the following theorem.

THEOREM 2. *Let G be a locally compact, σ-compact group. The following are equivalent*:

(1) *G has property (T)*;

(2) *Every $\psi \in N_0(G)$ is bounded as a function on G*;

(3) *Every semiderivation ∂_ψ, induced on $C^*(G)$ by a $\psi \in N_0(G)$, is bounded as an operator on $C^*(G)$*;

(4) *$H^1(G, H(\pi)) = 0$ for all continuous unitary representations π of G.*

PROOF. A rigorous proof for $(1) \Leftrightarrow (2)$ requires some attention to preliminary details (especially for non separable G), but the basic idea is that for a G with property (T) a net $\{p_\alpha\} \subset P(G)_1$ can only approach $\mathbf{1} \in P(G)_1$ by eventually converging uniformly on G. Then using a "Lie cone" technique, i.e., working in $N_0(G)$ as opposed to $P(G)_1$, the result can be established, c.f. **[2]**. To prove that (3) implies (2) we first note that for $f \in L^1(G)$ with $f(g) \geqq 0$ for all $g \in G$ that

$$\|f\|_{L^1(G)} = \|\omega(f)\|_{C^*(G)},$$

13.11.1 **[8]**. Thus for $\psi \in N_0(G)$, ψ real, if ∂_ψ is bounded, we must have that

$$\|\partial_\psi\| \geqq \|\partial_\psi \omega(f)\|_{C^*(G)} = \int_G -\psi(g)f(g)dg$$

at least for non-negative, compactly supported functions $f \in L^1(G)$ with $\|f\|_{L^1(G)} \leqq 1$. An argument by contradiction shows ψ is bounded as a function on G, hence (2) follows, even for complex ψ, cf. §7.14 **[5]**. Conversely, if $\psi \in N_0(G)$ is bounded, by Proposition 12, $\psi = p - m$ for $p \in P(G)$, m positive constant. Thus

$$\begin{aligned}\|\partial_\psi \omega(f)\|_{C^*(G)} &= \|\omega(pf - mf)\|_{C^*(G)} \\ &\leqq \|\omega(pf)\|_{C^*(G)} + m\|\omega(f)\|_{C^*(G)} \\ &\leqq (m + p(e))\|\omega(f)\|_{C^*(G)}.\end{aligned}$$

Thus (2) implies (3).

To see that (2) implies (4) we observe that if there is a continuous unitary representation π with non-trivial cocycle c, then c is not bounded in $H(\pi)$ by the proof of Proposition 12. Thus $\psi(g) = -\|c(g)\|^2$ is an unbounded function, and $\psi \in N_0(G)$. Conversely, (4) implies (2). For if each cocycle for every unitary representation is trivial, hence bounded, then Re ψ is bounded for all $\psi \in N_0(G)$. Thus by the corollary of Proposition 12, any $\psi \in N_0(G)$ is of the form $\psi = p - m + i\gamma$ where $p \in P(G)$, m is a constant, and γ is a homomorphism of G into the additive reals. But $-\gamma^2 \in N_0(G)$ is real, hence bounded; thus γ is bounded. Note that (1) implies (4) is known, but existing proofs are different from ours, cf. **[38]**.

In **[2]** we establish the equivalence between various ways in which $\psi \in N_0(G)$ can be unbounded and certain properties of G. As for concrete examples, §10 of **[5]** provides several examples on $\mathbf{R}^n$; indeed, in principle all examples on $\mathbf{R}^n$ are provided by the Lévy-Khinchin formula. In the case of non-commutative groups certain very interesting examples have been discovered. Among these is the fact that in $\mathbf{F}_2$, the free group on two generators, $s \in \mathbf{F}_2 \mapsto -|s| =$ (negative) word length of s, is a function of negative type. The word length function is also of negative type on $\mathbf{F}_n$ where n is a finite natural number [**16**].

In his thesis, [**23**], E. Larsen investigates several examples. Sometimes one is fortunate in that a "nicely" parametrized family of positive definite functions converging to $\mathbf{1}$ is known. Such is the case for $SL(2, \mathbf{R})$. Since $SL(2, \mathbf{R})$ is a group without property (T) it has at least one unbounded negative type function which in "suitable" coordinates one can compute to be of the form $-\ln(\cosh t)$, $-\infty < t < \infty$. Such unbounded elements of $N_0(G)$ do not exist for $G = SL(n, \mathbf{R})$, $n > 2$ [**7**].

We have said nothing about differentiation at points $p_0 \in P(G)_1$ other than $p = \mathbf{1}$. To develop this part of the theory here would take us many pages and beyond our aim of an "introduction." We close with an example of an interesting derivative on the dual of the "$ax + b$ group" $(a > 0)$ at the point p_0 where $p_0(a, b) = (2a/(a^2 + 1))^{1/2}(a > 0)$. It can be shown that $p_t(a, b) = (2a/(a^2 + 1))^{1/2} \exp((-t/2)\{b^2/(a^2 + 1) + (\ln\ a)^2/2\})$ is continuous and positive definite for $t \geqq 0$, yet the function of (a, b) in the exponent is not of negative type. Thus the space of semitangents at some points $p_0 \in P(G)_1$ is strictly larger than $N_0(G)$. ($N_0(G)$ is obviously always contained in the space of semitangents at any $p_0 \in P(G)_1$). The larger problem of which this last example is a part will be treated in a sequel to this paper.

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