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REARRANGEMENTS OF DIVERGENT SERIES

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ABSTRACT. Let $\sum a_k$ be a divergent series of positive numbers. The rate of divergence of $\sum a_k$ is related to the behavior of subseries and to rearrangements of the series. We show the rate of divergence of $\sum a_k$ is determined by the convergent subseries of $\sum a_k$ and also show that the rate of divergence can be changed, through rearrangement, to give some other predesignated rate of divergence.

1. Introduction. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $a_k \to 0$ and $\sum a_k = \infty$. In this paper we consider the rate of divergence of the partial sums $A_n = \sum_1^n a_k$ as $n \to \infty$. We prove some results concerning the rate of growth of these partial sums and how it may be altered through rearrangement of the series.

Our first results, making up the second section of the paper, show that the rate of growth of A_n as $n \to \infty$ is determined by the convergent subseries of $\sum a_k$. These results are, in a sense, "inverse" to previous results of Banerjee and Lahiri [1] and Salat [6]. (These papers consider the sums of convergent subseries of divergent series and how "often" a particular positive P can be the sum of a convergent subseries).

The third section of the paper concerns arrangements of $\sum a_k$. Let π: $\mathbf{Z}^+ \to \mathbf{Z}^+$ be a permutation of the positive integers. In [2], [3] and [4], Diananda found conditions under which $\sum_1^n a_k$ and $\sum_1^n a_{\pi(k)}$ are asytomtic as $n \to \infty$. In [7], Stenberg also considered rearrangements of divergent series and studied the divergent subseries of the rearrangements. Our main result in the third section considers another aspect of the rearrangement question. We show that given $f(x)$ positive, concave, increasing to ∞ on $(0, \infty)$ with $f(x + 1) - f(x) \to 0$ as $x \to \infty$ and

$$\limsup_{n\to\infty} \frac{f(n)}{A_n} \leqq 1,$$

there is a permutation π with $\sum_1^n a_{\pi(k)} \sim f(n)$. (Compare Riemann's classical theorem on rearrangement of conditionally convergent series [5]).

In this paper, all series are to be series of positive real numbers with

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terms tending to 0. For a series $\sum a_k$ and a positive integer n, $A_n = \sum_{k=1}^{n} a_k$ will denote the n-th partial sum of $\sum a_k$.

2. Subseries and asymptotic behavior. We first state a definition [5, p. 279] to clarify the manner in which we measure the asymptotic behavior of a series.

DEFINITION 1. Let $\sum a_k$ and $\sum b_k$ be divergent series. We will say that $\sum a_k$ and $\sum b_k$ diverge at the same rate if

$$0 < \alpha = \liminf_{n\to\infty} B_n/A_n \leqq \limsup_{n\to\infty} B_n/A_n = \beta < +\infty \tag{1}$$

If $\alpha = \beta = 1$ in (1), we shall say the two series are asymptotic and write $A_n \sim B_n$ or $\sum a_k \sim \sum b_k$.

The next theorem shows that for series with terms tending to zero, the growth of A_n as $n \to \infty$ is to some degree determined by the convergent subseries of $\sum a_k$.

THEOREM 2. *Let $\sum a_k$ and $\sum b_k$ be series with* $\lim a_k = \lim b_k = 0$. *Suppose that for each increasing sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers, $\sum a_{k_j}$ and $\sum b_{k_j}$ both converge or both diverge. Then $\sum a_k$ and $\sum b_k$ either both converge or both diverge at the same rate.*

PROOF. If every subseries of $\sum a_k$ (and hence of $\sum b_k$) converges, there is nothing to prove. So suppose $\sum a_k$ has some convergent and some divergent subseries. If the conclusion of Theorem 2 does not hold, then we may assume

$$\limsup_{n\to\infty} B_n/A_n = +\infty. \tag{2}$$

Consider the set S of positive integers defined by $S = \{k\colon b_k \geqq a_k\}$. Let $k_1, k_2, \ldots$ denote the elements of S listed in increasing order. By (2), it follows that $\sum b_{k_j}$ and $\sum a_{k_j}$ diverge and

$$\limsup_{n\to\infty} \tilde{B}_{k_n}/\tilde{A}_{k_n} = +\infty$$

where, $\tilde{A}_{k_n} = \sum_1^n a_{k_j}$ and similarly for $\tilde{B}_{k_n}$.

We now reassociate the terms of $\sum b_{k_j}$ as follows. Let

$$\mathscr{B}_1 = b_{k_1} + b_{k_2} + \cdots + b_{k_{n_1}}$$

where $n_1 = \min\{n\colon \sum_{j=1}^{n} b_{k_j} \geqq 1\}$. Having defined $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_m$ and associated positive integers $n_1 < \cdots < n_m$, let

$$\mathscr{B}_{m+1} = b_{k_{n_m+1}} + \cdots + b_{k_{n_{m+1}}}$$

where $n_{m+1} = \min\{n\colon \sum_{j=n_m+1}^{n} b_{k_j} \geqq 1\}$. We then define $\{\mathscr{A}_m\}_{m=1}^{\infty}$ by

$$\mathscr{A}_m = \sum_{j=n_{m-1}+1}^{n_m} a_{k_j}, \qquad m = 1, 2, 3, \ldots \text{ (taking } n_0 = 0).$$

It then follows that

$$\limsup_{n\to\infty} \sum_1^n \mathscr{B}_m / \sum_1^n \mathscr{A}_m = +\infty \tag{3}$$

and since $b_k \to 0$,

$$\lim_{m\to\infty} \mathscr{B}_m = 1. \tag{4}$$

Now from (3) we must have $\liminf_{m\to\infty} \mathscr{A}_m/\mathscr{B}_m = 0$. Thus we may select an increasing sequence $\{m_j\}$ of positive integers with

$$\mathscr{A}_{m_j}/\mathscr{B}_{m_j} \leqq 1/j^2, \qquad j = 1, 2, \ldots. \tag{5}$$

By (4) and (5) we see that $\sum \mathscr{B}_{m_j}$ diverges while $\sum \mathscr{A}_{m_j}$ converges. Breaking the $\mathscr{B}_{m_j}$'s and $\mathscr{A}_{m_j}$'s into their component b_{k_j}'s and a_{k_j}'s gives a subseries of $\sum b_k$ that diverges while the corresponding subseries of $\sum a_k$ converges. This contradiction shows (2) cannot hold, completing the proof of the theorem.

As immediate corollaries of Theorem 2 we have the following two results.

Corollary 3. *Let $\sum a_k$ and $\sum b_k$ be series satisfying the hypotheses of Theorem 2, and let $\{k_j\}_{j=1}^{\infty}$ be an increasing sequence of positive integers. Then $\sum a_{k_j}$ and $\sum b_{k_j}$ either both converge or both diverge at the same rate.*

Corollary 4. *Let $\sum a_k$ and $\sum b_k$ be series satisfying the hypotheses of Theorem 2, and let π be any permutation of the positive integers. Then $\sum a_{\pi(k)}$ and $\sum b_{\pi(k)}$ either both converge or both diverge at the same rate.*

In §4 we give an example showing the converse of Theorem 2 does not hold even if $\sum a_k$ and $\sum b_k$ are series of decreasing terms. However, the converse to Corollary 4 is valid as is shown by the next theorem. We first state a definition for notation.

Definition 5. Let $\{a_k\}$ and $\{b_k\}$ be sequences of positive numbers. For positive M, let $N_{A,B}^M = \{k : a_k \geqq Mb_k\}$.

Theorem 6. *Let $\sum a_k$ and $\sum b_k$ be infinite series with* $\lim a_k = \lim b_k = 0$. *Then the following are equivalent.*

(i) *There exists $M > 0$ such that*

$$\sum_{k\in N_{A,B}^M} a_k \text{ and } \sum_{k\in N_{B,A}^M} b_k$$

both converge.

(ii) *For each increasing sequence* $\{k_j\}$ *of positive integers* $\sum a_{k_j}$ *and* $\sum b_{k_j}$ *either both converge or both diverge.*

(iii) *For each permutation* π *of the positive integers,* $\sum a_{\pi(k)}$ *and* $\sum b_{\pi(k)}$ *either both converge or both diverge at the same rage.*

PROOF. It is easy to prove that (i) implies (ii), and (ii) implies (iii) by Corollary 4. To prove (iii) implies (i), we assume (i) does not hold and construct a permutation π that contradicts (iii). If (i) does not hold, we may assume

$$\sum_{k \in N_{A,B}^M} a_k$$

diverges for each positive integer M. Define π inductively as follows.

Let $\pi(1) = 1$. Take $\tilde{N}_{A,B}^4 = N_{A,B}^4 - \{1\}$ and let $k_1^{(2)}, k_2^{(2)}, \ldots, k_m^{(2)}, \ldots$ denote the elements of $\tilde{N}_{A,B}^4$ in increasing order. Let

$$n_2 = 1 + \min\left\{n \geqq 2 \colon a_1 + \sum_{j=1}^{n-1} a_{k_j^{(2)}} \geqq 2\left(b_1 + \sum_{j=1}^{n-1} b_{k_j^{(2)}}\right)\right\}.$$

We observe that n_1 must be finite since

$$a_{k_j^{(2)}} \geqq 4 b_{k_j^{(2)}}$$

for all j and

$$\sum a_{k_j^{(2)}}$$

diverges. Define $\pi(j+1) = k_j^{(2)}$ for $1 \leqq j \leqq n_2 - 1$; thus $\pi(j)$ is now defined for $1 \leqq j \leqq n_2$. Now suppose $n_m (m \geqq 2)$ has been produced and $\pi(j)$ has been defined for $1 \leqq j \leqq n_m$. We let

$$\pi(n_m + 1) = \min\{k \colon k \neq \pi(j),\ 1 \leqq j \leqq n_m\}. \tag{6}$$

Let

$$\tilde{N}_{A,B}^{(m+1)^2} = N_{A,B}^{(m+1)^2} - \{\pi(j) \colon 1 \leqq j \leqq n_m + 1\}$$

and let $k_1^{(m+1)}, k_2^{(m+1)}, \ldots$ denote the elements of $\tilde{N}_{A,B}^{(m+1)^2}$ in increasing order. We take

$$\begin{aligned} n_{m+1} = n_m + \min\Bigg\{n \geqq 2 \colon & \sum_{j=1}^{n_m+1} a_{\pi(j)} + \sum_{j=1}^{n-1} a_{k_j^{(m+1)}} \\ & > (m+1)\left(\sum_{j=1}^{n_m+1} b_{\pi(j)} + \sum_{j=1}^{n-1} b_{k_j^{(m+1)}}\right)\Bigg\} \end{aligned} \tag{7}$$

and note that (7) does indeed define a finite number n_{m+1}. We now define $\pi(n_m + j + 1) = k_j^{(m+1)}$ for $1 \leqq j \leqq n_{m+1} - n_m - 1$.

As defined above, π is a $1-1$ mapping from $\mathbf{Z}^+$ to $\mathbf{Z}^+$; equation (6) guarantees π is onto; thus π is a permutation. Finally, (7) shows that

$$\limsup_{n\to\infty} \sum_1^n a_{\pi(k)} / \sum_1^n b_{\pi(k)} = +\infty.$$

Thus (iii) is violated. This contradiction completes the proof of Theorem 6.

As a consequence of the results presented in this section we state a rearrangement theorem.

THEOREM 7. *Let $\sum a_k$ be a divergent series of positive terms with* $\lim a_k = 0$. *If π is a permutation that maps convergent subseries of $\sum a_k$ onto convergent subseries, and divergent subseries onto divergent subseries, then $\sum a_k$ and $\sum a_{\pi(k)}$ diverge at the same rate.*

3. Rearrangements. In Theorem 7 we gave a sufficient condition that a rearrangement not affect the rate of divergence. Our next result, similar to Riemann's Theorem [5], shows that a divergent series of positive terms can often be rearranged to give a predesignated asymptotic behavior.

THEOREM 8. *Let $\sum a_k$ be a divergent series with* $\lim\, a_k \to 0$. *Let $f(x)$, defined for $x \geqq 0$, be a positive, strictly increasing concave function with*

(i) $\lim_{x\to\infty} f(x) = +\infty$,

(ii) $\lim_{x\to\infty} \{f(x+1) - f(x)\} = 0$,

and

(iii) $\limsup_{n\to\infty} f(n)/A_n \leqq 1$.

Then there is a permutation π such that

$$A_n^\pi \sim f(n) \qquad (as\ n \to \infty) \tag{8}$$

where $A_n^\pi = \sum_1^n a_{\pi(k)}$.

PROOF. Let $b_1 = f(1)$ and $b_n = f(n) - f(n-1) (n = 2, 3, \ldots)$. Since $f(x)$ is concave, we have $b_1 \geqq b_2 \geqq \cdots$. To prove (8) it sufficies to find a permutation π with

$$A_n^\pi \sim B_n = \sum_1^n b_k = f(n). \tag{9}$$

We will obtain (9) by producing π so that

$$\sum_1^n (a_{\pi(k)} - b_k) = A_n^\pi - B_n = o(B_n) \qquad (\text{as } n \to \infty). \tag{10}$$

Now if $\liminf_{n\to\infty} B_n/A_n = 1$, we can let π be the identity mapping. So assume

$$0 \leqq \liminf_{n\to\infty} B_n/A_n = \beta < 1 \tag{11}$$

and

$$\beta \leqq \limsup_{n\to\infty} B_n/A_n = \alpha \leqq 1. \tag{12}$$

By (12) we may produce a sequence $N_1 < N_2 < \cdots < N_m < \cdots$ of positive integers such that

$$B_n/A_n \leqq 1 + 1/m \qquad (n \geqq N_m). \tag{13}$$

By (11), the divergence of $\sum b_k$ and the fact that a_k, $b_k \to 0$, we may find a positive integer N such that

$$B_N/A_N < \frac{1+\beta}{2}, \tag{14}$$

$$\frac{1-\beta}{1+\beta} B_N > \sqrt{B_N} > 1 \tag{15}$$

and so that

$$a_n, b_n < 1/2 \qquad (n > N). \tag{16}$$

We define π by first taking $\pi(k) = k$ for $1 \leqq k \leqq N$. Assume $\pi(k)$ has been defined for $1 \leqq k \leqq n$ $(n \geqq N)$. Then,

I. if $A_n^\pi - B_n \geqq \sqrt{B_n}$, take $\pi(n+1) = \min\{k\colon k \neq \pi(j), 1 \leqq j \leqq n$ and $a_k \leqq b_{n+1}\}$, and

II. if $A_n^\pi - B_n < \sqrt{B_n}$, take $\pi(n+1) = \min\{k\colon k \neq \pi(j), 1 \leqq j \leqq n\}$.

We first observe that if I is used to define $\pi(n+1)$, then $A_{n+1}^\pi - B_{n+1} \leqq A_n^\pi - B_n$. Thus, since $\sqrt{B_n}$ increases to ∞, we see that II will be used infinitely often in defining π. This guarantees that π is a permutation of $\mathbf{Z}^+$.

From (14) and (15) we see

$$A_N^\pi - B_N > \frac{1-\beta}{1+\beta} B_N > \sqrt{B_N}. \tag{17}$$

Thus I is used to define $\pi(N+1)$. Let M_1 be the first integer for which we use II to define $\pi(M_1)$, and take $n > N_m > M_1$ for some positive integer m (see (13)). We now consider three cases.

Case 1. If $0 \leqq A_n^\pi - B_n < \sqrt{B_n}$, there is nothing to consider: we have the "little oh" relationship desired in (10).

Case 2. Suppose $A_n^\pi - B_n \geqq \sqrt{B_n}$. We let

$$n' = \max\{k \leqq n\colon \text{II was used to define } \pi(k)\}. \tag{18}$$

Since $n > M_1$, we know $n' \geqq M_1$, and it then follows from II and (18) that

$$A_{n'-1}^\pi - B_{n'-1} < \sqrt{B_{n'-1}} \tag{19}$$

and $A^\pi_{n'} - B_{n'} \geqq \sqrt{B_{n'}}$. By (16) and (19) we then see

$$A^\pi_{n'} - B_{n'} = (A^\pi_{n'-1} - B_{n'-1}) + (a_{\pi(n')} - b_{n'}) \leqq \sqrt{B_{n'}} + \frac{1}{2}.$$

However, by the definition of n', we will use I in defining $\pi(k)$, $k = n' + 1, n' + 2, \ldots, n$. Thus (even in the event $n = n'$) it follows by I that

$$A^\pi_n - B_n \leqq A^\pi_{n'} - B_{n'} \leqq \sqrt{B_{n'}} + \frac{1}{2} \leqq \sqrt{B_n} + \frac{1}{2}.$$

Since we are assuming $A^\pi_n - B_n \geqq \sqrt{B_n}$, we again have the "$o$" relationship of (10).

Case 3. If $A^\pi_n - B_n < 0$, we show that $A^\pi_n = A_n$. We let

(20) $$n'' = \max\{k \leqq n\colon \text{I was used to define } \pi(k)\}.$$

By (17) we know $n'' > N$ and from I we have

$$A^\pi_{n''-1} - B_{n''-1} \geqq \sqrt{B_{n''-1}} > 1.$$

But then by (16)

(21) $$A^\pi_{n''} - B_{n''} \geqq \sqrt{B_{n''-1}} - \frac{1}{2} > 0.$$

Now by the definition of n'' in (20) we must use II to define $\pi(k)$ for $n'' < k \leqq n$. But by (21) and the fact that $A^\pi_n - B_n < 0$, it follows that for some k', with $n'' < k' \leqq n$ we must have $a_{\pi(k')} < b_{k'}$. Observe that the definitions I and II imply that $\pi(k') \leqq k'$ whenever II is used to obtain k'. We claim that $\pi(k') = k'$. Note that if $\pi(k') < k'$, then in some previous application of I to define some $\pi(k'')$ $(k'' \leqq n'')$ we would have taken

(22) $$\pi(k'') = \min\{k\colon k \neq \pi(j),\ 1 \leqq j \leqq k'' - 1 \text{ and } a_k \leqq b_{k''}\},$$

with $\pi(k'') > k' > \pi(k')$. But b_k is a decreasing sequence, so $b_{k''} \geqq b_{k'} > a_{\pi(k')}$ and the choice of a $\pi(k'') > k' > \pi(k')$ violates (22). This contradiction shows $\pi(k') = k'$ and it easily follows that $\pi(k) = k$ for $k' \leqq k \leqq n$. Thus we obtain $\pi(n) = n$ when II is used to define $\pi(n)$ in Case 3, and it follows that $A^\pi_n = A^n$ as claimed.

We have shown that in Case 3, we actually have $A^\pi_n - B_n = A_n - B_n < 0$ and hence $B_n/A_n > 1$. However by (13) and our choice $n > N_m$ it follows that $1 > A_n/B_n = A^\pi_n/B_n \geqq m/(m+1)$ giving $|A^\pi_n - B_n| \leqq (1/m)B_n$. So for $n > N_m$ (combining cases 1, 2 and 3) we have

(23) $$|A^\pi_n - B_n| \leqq \max\left\{\frac{1}{m} B_n,\ \sqrt{B_n} + \frac{1}{2}\right\}.$$

Since m may be chosen arbitrarily large and $B_n \to \infty$, (23) is the $o(B_n)$ relation in (10).

4. Examples. Our first example shows that the converse of Theorem 2 does not hold even if $\sum a_k$ and $\sum b_k$ are series of decreasing terms. To this end let $a_1 = 1$ and take $a_2 = a_3 = \cdots = a_{28} = 3^{-3}$. Having defined a_k for $1 \leqq k \leqq N_n = \sum_1^n (2j-1)^{2j-1}$, we take $a_k = (2n+1)^{-(2n+1)}$ for $N_n + 1 \leqq k \leqq N_{n+1}$. Thus the terms of $\sum a_k$ are defined in blocks and the sum of the terms in each block is one. The series $\sum b_k$ is defined in an analogous way. Let $b_1 = b_2 = b_3 = b_4 = 2^{-2}$. Having defined b_k for $1 \leqq k \leqq M_n = \sum_1^n (2j)^{2j}$, we take $b_k = (2n+2)^{-(2n+2)}$ for $M_n + 1 \leqq k \leqq M_{n+1}$.

For the series as defined above, it is easily checked that $A_n \sim B_n$, so the two series do diverge at the same rate. We now consider the subseries of the two series determined by the set $N^1_{A,B} = \{k\colon a_k \geqq b_k\}$ of positive integers. We then find that

$$\sum_{k \in N^1_{A,B}} a_k = 1 + \sum_{n=1}^{\infty} c_n (2n+1)^{-(2n+1)} \tag{24}$$

and

$$\sum_{k \in N^1_{A,B}} b_k = \frac{1}{4} + \sum_{n=1}^{\infty} c_n (2n+2)^{-(2n+2)} \tag{25}$$

where

$$c_n = 1 + \sum_{j=1}^{n} [(2j+1)^{2j+1} - (2j)^{2j}]. \tag{26}$$

Let $\bar{A}_n$ and $\bar{B}_n$ denote partial sums of

$$\sum_{k \in N^1_{A,B}} a_k \text{ and } \sum_{k \in N^1_{A,n}} b_k$$

respectively. Then from (24), (25), (26) we have

$$\limsup_{n\to\infty} \bar{A}_n / \bar{B}_n \geqq \limsup_{N\to\infty} \frac{1 + \sum_{n=1}^{N} c_n (2n+1)^{-(2n+1)}}{\frac{1}{4} + \sum_{n=1}^{N} c_n (2n+2)^{-(2n+2)}}$$

$$\geqq \limsup_{N\to\infty} \frac{1 + \sum_{n=1}^{N} [(2n+1)^{2n+1} - (2n)^{2n}](2n+1)^{-(2n+1)}}{\frac{1}{4} + \sum_{n=1}^{N} (2n+1)^{2n+1}(2n+2)^{-(2n+2)}}$$

$$\geqq \limsup_{N\to\infty} \frac{1 + N - \sum_1^N \frac{1}{2n+1}}{\frac{1}{4} + \sum_1^N \frac{1}{2n+1}} = +\infty.$$

Thus, though the subseries

$$\sum_{k \in N^1_{A,B}} a_k \text{ and } \sum_{k \in N^1_{A,B}} b_k$$

both diverge, they do not diverge at the same rate. But then the hypotheses of Corollary 3 (and hence of Theorem 2) cannot be satisfied. Thus there is a sequence $\{n_k\}$ of positive integers for which $\sum a_{n_k}$ diverges and $\sum b_{n_k}$ converges.

Our next example shows that Theorem 8 may not hold if the concavity hypothesis is lifted from $f(x)$. Let $a_k = 1/\sqrt{k}(k = 1, 2, \ldots)$, it is easily shown that $A_n \sim 2\sqrt{n}$ and in fact that $|A_n - 2\sqrt{n}| \leqq 3$ for $n \geqq 3$. For our example it suffices to construct a series $\sum b_k$ with $\lim b_k = 0$ and $\limsup_{n\to\infty} B_n/A_n \leqq 1$ but such that no rearrangement of $\sum a_k$ is asymtotic to $\sum b_k$.

We define $\{b_k\}$ inductively. Let $0 < \varepsilon < 1/100$ be given. There is a large, positive integer N_1 so that

$$(39) \qquad |2\sqrt{n}/A_n - 1| < \varepsilon \qquad (n \geqq N_1).$$

Let $b_k = a_k$ for $1 \leqq k \leqq N_1$. Now suppose positive integers $N_1 < N_2 < \cdots < N_m$ have been defined and b_k has been defined for $k \leqq N_m$ with $B_{N_m} = A_{N_m}$. Let

$$(40) \qquad N'_m = \min\{n > N_m : A_n \geqq 2A_{N_m}\}$$

and define

$$(41) \qquad b_k = \frac{1}{N'_m - N_m} \qquad (N_m < k \leqq N'_m).$$

Let $N''_m = [(N'_m - N_m)/50]$ and define $N_{m+1} = N'_m + N''_m$. For $N'_m < k \leqq N_{m+1}$ we take

$$(42) \qquad b_k = (A_{N_{m+1}} - A_{N_m} - 1)/N''_m,$$

which gives $B_{N_{m+1}} = A_{N_{m+1}}$. The process described above defines a series $\sum b_k$ with

$$\frac{1}{2} = \liminf_{n\to\infty} B_n/A_n < \limsup_{n\to\infty} B_n/A_n = 1.$$

From (40), (41), (42) and the fact that $a_k = 1/\sqrt{k}$ it easily follows that $\lim b_k = 0$.

Now suppose π is a permutation for which $A^\pi_n \sim B_n$. Then there is $N > 0$ so that

$$(43) \qquad |A^\pi_n/B_n - 1| < \varepsilon/3 \qquad (n > N).$$

In particular (43) must hold for $n = N_m, N_m', N_{m+1}$ for all sufficiently large m; we show this is impossible.

If (43) holds with $n = N_m$, it follows that

$$\begin{aligned}
|A^{\pi}_{N_m} - 2\sqrt{N_m}| &\leqq |A^{\pi}_{N_m} - B_{N_m}| + |B_{N_m} - 2\sqrt{N_m}| \\
&< \frac{\varepsilon}{3} B_{N_m} + |B_{N_m} - 2\sqrt{N_m}| \\
&= \frac{\varepsilon}{3} A_{N_m} + |A_{N_m} - 2\sqrt{N_m}| \qquad (44)\\
&\leqq \frac{\varepsilon}{3}\{|A_{N_m} - 2\sqrt{N_m}| + 2\sqrt{N_m}\} + 3 \\
&< \varepsilon\sqrt{N_m}
\end{aligned}$$

for sufficiently large m. It follows from (44) that

$$(2 - \varepsilon)\sqrt{N_m} < A^{\pi}_{N_m} < (2 + \varepsilon)\sqrt{N_m}. \qquad (45)$$

Now let $S_m = \{k: 1 \leqq k \leqq N_m \text{ and } k \neq \pi(j),\ 1 \leqq j \leqq N_m\}$. Then by (45), the sum of those of the first N_m elements of $\sum a_k$ that are not among the first N_m elements of $\sum a_{\pi(k)}$ is

$$\sum_{k \in S_m} a_k \leqq \varepsilon\sqrt{N_m} + \int_{N_m}^{2N_m} \frac{dx}{\sqrt{x}} < .85\sqrt{N_m} \qquad (46)$$

The definition of N_m' shows that $N_m' \approx 4N_m$(or, sufficient for our purposes $3N_m < N_m' < 5N_m$). Since (43) holds for $n = N_m'$ and $B_{N_m'} = A_{N_m} + 1$, we have by reasoning analogous to that for (44)

$$\begin{aligned}
|A^{\pi}_{N_m'} - (2\sqrt{N_m} + 1)| &\leqq |A^{\pi}_{N_m'} - B_{N_m'}| + |B_{N_m'} - (2\sqrt{N_m} + 1)| \\
&< \frac{\varepsilon}{3} B_{N_m'} + 3 \\
&< \varepsilon\sqrt{N_m}
\end{aligned}$$

for large m. Thus for large m,

$$(2 - \varepsilon)\sqrt{N_m} \leqq A^{\pi}_{N_m'} \leqq (2 + \varepsilon)\sqrt{N_m}. \qquad (47)$$

Now by our definitions of N_{m+1} and N_m'', we have

$$N_{m+1} = N_m' + N_m'' = N_m' + [(N_m' - N_m)/50] \leqq N_m' + 2N_m/25.$$

But referring to (45) with m replaced by $m + 1$ shows

$$A^{\pi}_{N_{m+1}} > (2 - \varepsilon)\sqrt{N_{m+1}} > (2 - \varepsilon)\sqrt{N_m'} > (2 - \varepsilon)\sqrt{3N_m}. \qquad (48)$$

Thus by (47) and (48)

$$(49) \qquad A^{\pi}_{N_{m+1}} - A^{\pi}_{N'_m} = \sum_{k=N'_m+1}^{N_{m+1}} a_{\pi(k)}$$
$$> [(2\sqrt{3}-2) - \varepsilon(\sqrt{3}+1)]\sqrt{N_m} > 1.1\sqrt{N_m}.$$

But this "growth" illustrated by (49) is for the terms $a_{\pi(k)}$ $(N'_m < k \leqq N_{m+1} < N'_m + 2N_m/25)$ of $\sum a_{\pi(k)}$. However, referring to (46) we see that

$$\sum_{k=N'_m+1}^{N_{m+1}} a_{\pi(k)} \leqq \sum_{k\in S_m} a_k + 2\int_{N_m}^{27/25\, N_m} \frac{dx}{\sqrt{x}} < 1.1\sqrt{N_m}$$

in contradiction of (49). Hence there is no permutation π with $A^{\pi}_n \sim B_n$.

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