ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 1, Winter 1983

LOCALIZATION WITH RESPECT TO A CLASS OF SPACES

GERALD A. ANDERSON

1. Introduction. Let $\mathscr{F}$ be a set of compact Hausdorff spaces. Given a Hausdorff space X, we construct the universal example of a map $X \to Y$ so that every map $\Delta \to Y$, $\Delta \in \mathscr{F}$, is null-homotopic. For suitably chosen $\mathscr{F}$, this localization is shown to be the Quillen plus construction, if X is a CW-complex with $[\pi_1(X), \pi_1(X)]$ perfect, and the Sullivan localization, if X is a nilpotent CW-complex with the Malcev-Lazard completion of $\pi_1(X)$ equal to 0.

2. $\mathscr{F}$-localization. Throughout this paper, we will assume all spaces pointed, maps basepoint preserving and homotopies relative to the basepoint. Cones, suspensions and mapping cylinders and cones will all be reduced. Let $\mathscr{F}$ be any set of spaces. We say that a space X is *$\mathscr{F}$-local* if every map f: $\Delta \to X$, $\Delta \in \mathscr{F}$, is null-homotopic. An *$\mathscr{F}$-localization* of a space X is a map $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ so that

(i) $X_{\mathscr{F}}$ is $\mathscr{F}$-local and

(ii) if Y is $\mathscr{F}$-local and g: $X \to Y$, then there is a map h: $X_{\mathscr{F}} \to Y$ so that $h \circ L_{\mathscr{F}} \cong g$.

Given X, define $F_{\mathscr{F}}(X)$ (or just $F(X)$ if $\mathscr{F}$ is understood) to be the space obtained from X by adjoining the mapping cones of all maps $\Delta \to X$, $\Delta \in \mathscr{F}$. Note that $X \subset F_{\mathscr{F}}(X)$. Define $F^0_{\mathscr{F}}(X) = X, F^n_{\mathscr{F}}(X) = F_{\mathscr{F}}(F^{n-1}_{\mathscr{F}}(X))$ and let $X_{\mathscr{F}} = \varinjlim F^n_{\mathscr{F}}(X)$; define $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ to be the obvious inclusion.

THEOREM 2.1. *If X is Hausdorff and each space in $\mathscr{F}$ is compact Hausdorff, then $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ is an $\mathscr{F}$-localization.*

PROOF. Let $\Delta \in \mathscr{F}, f$: $\Delta \to X_{\mathscr{F}}$. Since Δ is compact, there is an integer n so that $f(\Delta) \subset F^n_{\mathscr{F}}(X)$. Therefore, f is null-homotopic in $F^{n+1}_{\mathscr{F}}(X)$ and so in $X_{\mathscr{F}}$.

Suppose g: $X \to Y$ where Y is $\mathscr{F}$-local. Clearly, g extends to a map $F_{\mathscr{F}}(X) \to F_{\mathscr{F}}(Y)$, and so to a map $g_{\mathscr{F}}: X_{\mathscr{F}} \to Y_{\mathscr{F}}$. But $L_{\mathscr{F}}(Y)$: $Y \to Y_{\mathscr{F}}$ is a homotopy equivalence, and letting ϕ be a homotopy inverse, $h = \phi \circ g_{\mathscr{F}}$ satisfies the relation $h \circ L_{\mathscr{F}}(X) \cong g$.

AMS (MOR) subject classification: 57D99.
Received by the editors on July 7, 1978.

REMARKS. (1) If $\mathscr{F}$ is a set of compact Hausdorff spaces and $\mathscr{F} \subset \mathscr{F}'$, then $X_{\mathscr{F}'}$, is $\mathscr{F}$-local by the proof of the theorem.

(2) If X is a CW-complex and $\mathscr{F}$ is a set of CW-complexes, then X has the homotopy type of a CW-complex. To see this, define $\hat{F}_{\mathscr{F}}(X)$ to be the CW-complex obtained from X by adjoining the cones of all cellular maps $\Delta \to X$, $\Delta \in \mathscr{F}$. Then $\hat{F}_{\mathscr{F}}(X) \subset F_{\mathscr{F}}(X)$ is a homotopy equivalence, and the construction follows as above.

(3) This construction has been used by Spanier [7] (with $X = M(\pi, n)$, $\mathscr{F} = \{S^{n+1}, S^{n+2}, \dots\}$) to obtain spaces of type $K(\pi, n)$, and by Anderson [1] (with $\mathscr{F} = \{\Sigma^k M(\mathbf{Z}/p, 1): k = 0, 1, \dots, p \in K)\}$ to obtain a localization of the homotopy groups of a space, away from a set of primes K.

3. Λ-acyclic localization. Let Λ be a subring of $\mathbf{Q}$ and define $\mathscr{F}(\Lambda)$ to be a set of representatives of the homotopy types of finite CW-complexes Δ with $\tilde{H}_*(\Delta; \Lambda) = 0$. In this section, we characterize the $\mathscr{F}(\Lambda)$-localization of a CW-complex. We let $X_\Lambda = X_{\mathscr{F}(\Lambda)}$ and $L_\Lambda = L_{\mathscr{F}(\Lambda)}$.

LEMMA 3.1. *L_Λ is the composition $g \circ f$ of a homotopy equivalence f and a Λ-acyclic resolution g.*

PROOF. Define, as in §2, $G^1(X)$ to be X with the mapping cylinders of all cellular maps $\Delta \to X$ adjoined; obviously $f_1: X \subset G^1(X)$ is a homotopy equivalence and the natural collapse $g_1: G^1(X) \to \hat{F}^1(X)$ is a Λ-acyclic resolution. Define $G^n(X) = G^1(G^{n-1}(X))$ and $f_n: X \to G^n(X)$, $g_n: G^n(X) \to \hat{F}^n(X)$ by

$$X \xrightarrow{f_{n-1}} G^{n-1}(X) \xrightarrow{f_1} G^n(X),$$

$$G^n(X) = G^1(G^{n-1}(X)) \xrightarrow{g_1} \hat{F}^1(G^{n-1}(X)) \xrightarrow{g_{n-1}} \hat{F}^1(\hat{F}^{n-1}) = \hat{F}^n(X);$$

f_n, g_n are compatible with the inclusions $G^n(X) \subset G^{n+1}(X)$, $\hat{F}^n(X) \subset \hat{F}^{n+1}(X)$. Passing to the direct limit, we get the result.

We say that a group P is *Λ-perfect* if $(P/[P, P]) \otimes \Lambda = 0$. Given a group G, define G_Λ to be the smallest normal subgroup of G containing all finitely presented Λ-perfect subgroups.

THEOREM 3.1. *Let X be a CW-complex. Then $L_\Lambda: X \to X_\Lambda$ is the unique map (up to homotopy) satisfying.*

(1) $(L_\Lambda)_*: H_*(X; L_\Lambda^*\xi \otimes \Lambda) \cong H_*(X_\Lambda; \xi \otimes \Lambda)$ *for every coefficient bundle ξ over X_Λ,*

(2) $\pi_n(X_\Lambda)$ *is a Λ-module for $n \geqq 2$, and*

(3) $\pi_1(X_\Lambda) \cong \pi_1(X)/(\pi_1(X))_\Lambda$.

PROOF. For (1), it suffices to show that $H_*(X; (\xi|X) \otimes \Lambda) \cong H_*(\hat{F}(X); \xi \otimes \Lambda)$ for every coefficient bundle ξ over $\hat{F}(X)$. (See the remarks following Theorem 1.1.) Define $\hat{X} = X \bigcup M_f$, taken over all cellular maps $f: \Delta \to X$,

$\Delta \in \mathscr{F}(\Lambda)$. Then $X \cong \hat{X}$ and $\hat{F}(X) = \hat{X} \cup \bigcup c(\partial_+ M_f)$ where $\partial_+ M_f$ denotes the top of the mapping cylinder M_f. The result now follows from the appropriate Mayer-Vietoris sequence since $\bigvee c(\partial_+ M_f)$ is contractible, $\hat{X} \cap \bigvee c(\partial_+ M_f) = \partial_+ M_f$ is Λ-acyclic, and $\xi | \bigvee \partial_+ M_f$ is trivial since it extends over the contractible space $\bigvee c(\partial_+ M_f)$.

Let $K = \{p: p \text{ a prime}, \Lambda \otimes \mathbf{Z}/p = 0\}$ and let M_p be the Moore space $M(\mathbf{Z}/p, 1)$, $p \in K$. Then $\Sigma^k M_p \in \mathscr{F}(\Lambda)$, $k = 0, 1, \ldots$, and so $[\Sigma^k M_p, X_\Lambda] = 0$. By [6], there is an exact sequence

$$0 \to \pi_{k+1}(X_\Lambda) \otimes \mathbf{Z}/p \to [\Sigma^k M_p, X_\Lambda] \to \operatorname{Tor}(\pi_k(X_\Lambda); \mathbf{Z}/p) \to 0$$

and (2) follows from [1], Theorem 1.5.

We now show that there is an exact sequence

$$0 \to \pi_1(X)_\Lambda \to \pi_1(X) \xrightarrow{(L_\Lambda)_\#} \pi_1(X_\Lambda) \to 0.$$

(i) $L_\#$ is surjective: Immediate from the Van Kampen theorem.

(ii) $\ker(L_\#) \subset \pi_1(X)_\Lambda$: Let $[f] \in \ker(L_\Lambda)_\#$. Then there exists a map $F: D^2 \to X_\Lambda$ so that $F|S^1 \cong L_\Lambda \circ f$. By Lemma 3.1, there is a space X' containing X as a strong deformation retract and a Λ-acyclic resolution $p: X' \to X_\Lambda$ so that $p|X = L_\Lambda$. Let Δ be the pull-back of the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{p'} & D^2 \\ \downarrow & & \downarrow F \\ X' & \xrightarrow{p} & X_\Lambda \end{array}$$

Since p is a Λ-acyclic resolution, p' is also, and so Δ is Λ-acyclic; Δ is compact since p' is proper.

Clearly, f factors through Δ, i.e., there exist maps $G: \Delta \to X$, $g: S^1 \to \Delta$ so that $G \circ g \cong f$. Therefore, $[f] = G_\#[g]$ lies in the subgroup $G_\#(\pi_1(\Delta))$. Since $H_1(\Delta; \Lambda) = 0$, $\pi_1(\Delta)$ is Λ-perfect, and it follows that $G_\#(\pi_1(\Delta))$ is also. Thus $[f] \in \pi_1(X)_\Lambda$.

(iii) $\pi_1(X)_\Lambda \subset \ker(L_\Lambda)_\#$: Let P be a finitely presented Λ-perfect subgroup of $\pi_1(X)$ and $[f] \in P$. Let $p: Y \to X$ be the covering space corresponding to the subgroup P and let $\tilde{f}: S^1 \to Y$ be a lift of f. Since $\pi_1(Y) \cong P$ is finitely presented, there is a finite subcomplex Y_0 of Y so that $\pi_1(Y_0) \cong P$ and $\tilde{f}$ is deformable into Y_0. It follows from the proof of Proposition 2.2 of [5] that $\tilde{f}$, and therefore f, factors through a finite Λ-acyclic CW-complex, since $\tilde{H}_n(Y_0; \Lambda) = 0$ for $n \leq 1$. Thus $[f] \in \ker(L_\Lambda)_\#$. Since $\ker(L_\Lambda)_\#$ is normal, it contains $\pi_1(X)_\Lambda$.

Uniqueness of L_Λ follows from obstruction theory.

REMARK. Let c be an infinite cardinal and define $\mathscr{F}_c(\Lambda)$ to be a complete set of representatives of the homotopy types of Λ-acyclic CW-complexes with $< c$ cells. By the proof of the theorem $L_{\mathscr{F}_c(\Lambda)}\colon X \to X_{\mathscr{F}_c(\Lambda)}$ satisfies conditions (1) and (2) (since $X_{\mathscr{F}_c(\Lambda)}$ is $\mathscr{F}(\Lambda)$-local) and

$$\pi_1(X_{\mathscr{F}_c(\Lambda)}) \cong \pi_1(X)/(\pi_1(X))_{\Lambda, c},$$

where $(\quad)_{\Lambda, c}$ denotes the normal closure of all Λ-perfect subgroups with a presentation of cardinality $< c$.

EXAMPLES. (1) If $[\pi_1(X), \pi_1(X)]$ is perfect and has a presentation of cardinality $< c$. then $L_{\mathscr{F}_c(\mathbf{Z})}\colon X \to X_{\mathscr{F}_c(\mathbf{Z})}$ is the plus construction of Quillen (see, for example, [4]).

(2) If X is a nilpotent space and the Malcev-Lazard completion ([3]) $\pi_1(X) \otimes \Lambda$ is 0, then $L_\Lambda\colon X \to X_\Lambda$ is the localization of X, away from the set of primes invertible in Λ, of Sullivan [8].

We may generalize the class $\mathscr{F}_c(\Lambda)$ as follows. Let h_* be a generalized (reduced) homology theory defined on the category of CW-complexes and let $\mathscr{F}_c(h_*)$ be a complete set of homotopy types of CW-complexes Δ with $< c$ cells, such that $h_*(\Delta) = 0$. Recall from [2] that a CW-complex X is said to be h_*-local if for any h_*-equivalence $f\colon A \to B, f^\#\colon [B, X] \to [A, X]$ is bijective, and that any CW-complex X has an h_*-localization $X\hat{_{h_*}}$, provided h_* satisfies the limit axiom.

Clearly, any h_*-local space is $\mathscr{F}_c(h_*)$-local, and so there is a map $X_{\mathscr{F}_c(h_*)} \to X\hat{_{h_*}}$ commuting, up to homotopy, with the natural maps from X. By Theorem 3.1 and [2], this map is not in general a homotopy equivalence (for any cardinal c).

PROPOSITION 3.2. *Let c_0 be the cardinality of $h_*(S^0)$. If $c > c_0$ and X is $\mathscr{F}_c(h_*)$-local, then ΩX is h_*-local.*

PROOF. Let $f\colon A \to B$ be an h_*-equivalence where A, B are CW-complexes with $< c$ cells. Then both C_f and ΣC_f are in $\mathscr{F}_c(h_*)$. Since the sequence $C_f \to \Sigma A \to \Sigma B \to \Sigma C_f$ is coexact, $(\Sigma f)^\#\colon [\Sigma B, X] \xrightarrow{\cong} [\Sigma A, X]$ and the result follows as in [2], Lemma 3.3.

REFERENCES

1. D.W. Anderson, *Localizing CW-complexes*, III. J. Math. **16** (1972), 519–525.

2. A.K. Bousfield, *Homological localization of spaces, groups and π-modules*, in *Localization in Group Theory and Homotopy Theory*, Lecture Notes in Mathematics **418**, Springer-Verlag, Berlin, 1974.

3. ———, and D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, Berlin, 1972.

4. S.M. Gersten, *Higher K-theories of rings*, in *Algebraic K-Theory* I, Lecture Notes in Mathematics **341**, Springer-Verlag, Berlin, 1973.

5. C.R.F. Maunder, *General position theorems for homology manifolds*, J. London Math. Soc. **4** (1972), 760–768.

6. F.P. Peterson, *Generalized cohomotopy groups*, Amer. J. Math **78** (1956), 259–281.

7. E.H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

8. D. Sullivan, Genetics of homotopy theory and the Adams conjecture, Ann. of Math. **100** (1974), 1–79.

QUEST RESEARCH CORPORATION, NAVAL OPERATIONS DIVISION, 6858 OLD DOMINION DR. MCLEAN, VA 22101