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# OSCILLATION PROPERTIES OF FORCED THIRD ORDER DIFFERENTIAL EQUATIONS

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**Introduction.** A great deal of literature exists on the oscillation and nonoscillation of the equation

$$y''' + q(t)y = 0 \tag{1}$$

where $q(t)$ is a positive continuous function defined on $[0, \infty)$. However, little seems to be known about equations of the type

$$y''' + q(t)y = f(t) \tag{2}$$

where $f(t)$ is continuous and changes sign arbitrarily on $[0, \infty)$. The asymptotic properties of

$$y'' + q(t)y = f(t) \tag{3}$$

have been studied in several works, some which include the investigations of Burton and Grimmer [1], Keener [3] and Hammett [2]. Hammett, in particular, has given conditions under which the nonoscillatory solutions of (3) tend to zero. The main purpose of this work is to carry out a similar study for (2). The techniques used herein are patterned after those in [6] in which Singh concentrated on equations with retarded arguments.

Recall that a solution of (1) or (2) is called *oscillatory* if it has arbitrarily large zeros and nonoscillatory otherwise. A solution $y$ is termed *quickly oscillatory* if there exists an increasing sequence of zeros of $y$, $\{t_i\}_{i=1}^{\infty}$ with the property that $\lim_{i\to\infty}(t_{i+1} - t_i) = 0$. The concept of quickly oscillatory solutions is also considered in other works, see [4] and [7].

**Main result.** It is well-known that if $z$ is a nontrivial solution of $z'' + q(t)z = 0$ having at least two zeros on $[c, d]$, then $(d - c)\int_c^d q(t)dt > 4$. This inequality is sometimes called Lyapunov's inequality. Lovelady in [5] recently obtained analogous results for (1),

THEOREM 1. *If $u$ is a nontrivial solution of* (1) *satisfying $u(a) = u(b) = 0$, and $u(x) \neq 0$ on $(a, b)$, then*

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$$(b-a)^2 \int_a^b q(t)dt > 8, \tag{4}$$

This result is significant because it yields necessary conditions for (1) to have quickly oscillatory solutions.

THEOREM 2. *If equation* (1) *has a quickly oscillatory solution, then*

$$\int_0^\infty q(t)dt = \infty \tag{5}$$

*and* $\limsup_{t\to\infty} q(t) = \infty$.

PROOF. Consider a pair of consecutive zeros $t_{n+1} > t_n$ of a quickly oscillatory solution of (1). Using (4) we obtain

$$\int_0^\infty q(t)dt > \int_{t_n}^{t_{n+1}} q(t)dt > 8/(t_{n+1}-t_n)^2 \to \infty$$

as $n \to \infty$. Hence (5) holds.

Applying the mean-value theorem for integrals we obtain

$$\int_{t_n}^{t_{n+1}} q(t)dt = q(c_n)(t_{n+1} - t_n) > 8/(t_{n+1} - t_n)^2$$

where $t_n < c_n < t_{n+1}$, and it follows that $\limsup_{t\to\infty} q(t) = \infty$ and the proof is complete.

We now investigate the asymptotic behavior of certain solutions of (2).

THEOREM 3. *Suppose* $h > 0$ *is such that* $\liminf_{t\to\infty} \int_t^{t+h} q(t)dt \geqq \varepsilon > 0$ *and* $\int_0^\infty |f(t)|dt < \infty$. *If* $y$ *is a nonoscillatory solution of* (2) *such that* $y(t) \nrightarrow 0$ as $t \to \infty$, *then* $y'(t) \to 0$ as $t \to \infty$.

PROOF. We assume without loss of generality that $y(t) > 0$ on some ray $[t_1, \infty)$. Integrating equation (2) from $t_1$ to $t$ we have

$$y''(t) - y''(t_1) + \int_{t_1}^t q(s)y(s)ds \leqq \int_{t_1}^t |f(t)|\, dt. \tag{6}$$

As $t \to \infty$, the right side of (6) remains bounded. Also, either

$$\int_{t_1}^\infty q(t)y(t)dt = \infty \tag{7}$$

or

$$\int_{t_1}^\infty q(t)y(t)dt < \infty. \tag{8}$$

If (7) holds, then $y''(t) \to -\infty$ as $t \to \infty$, a contradiction, since $y(t) > 0$ for $t > t_1$. Thus (8) holds. Since $\int_{t_1}^\infty q(t)dt = \infty$, it follows that

$$\liminf_{t\to\infty} y(t) = 0. \tag{9}$$

If $y'(t)$ is nonoscillatory, or if $y'(t)$ is oscillatory but does not change sign on $[t_1, \infty)$, then $y(t)$ is monotonic and we have (in view of (9)) that $y(t) \to 0$ as $t \to \infty$, a contradiction to our hypothesis. Therefore we assume $y'(t)$ is oscillatory and changes sign for arbitrarily large values of $t$. Then

$$\liminf_{t\to\infty} |y'(t)| = 0. \tag{10}$$

If $\limsup_{t\to\infty} |y'(t)| \neq 0$, then there is a number $d > 0$ such that

$$\limsup_{t\to\infty} |y'(t)| > d > 0. \tag{11}$$

From (10) and (11) we can obtain increasing sequences $\{T_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ such that

(i) $T_n \to \infty$ as $n \to \infty$, $T_n > t_1$ for $n \geqq 0$,

(ii) $|y'(T_n)| < d/4$ for $n \geqq 0$,

(iii) $d_n \geqq 3/4d$, where $d_n$ is the absolute maxium of $|y'(t)|$ on $[T_{n-1}, T_n]$.

Let $\{z_n\}$ be such that $|y'(z_n)| = d_n$ and $z_n \varepsilon [T_{n-1}, T_n]$. Also let $(a_n, b_n)$ be the largest open interval containing $z_n$ such that $|y'(t)| > d_n/2$ for all $t$ in this interval. Note that $|y'(a_n)| = |y'(b_n)| = d_n/2$ and

$$d_n \geqq |y'(t)| > d_n/2, \text{ where } a_n < t < b_n. \tag{12}$$

Since

$$y'(z_n) = y'(a_n) + \int_{a_n}^{z_n} y''(t)dt$$

we have

$$|y'(z_n)| \leqq |y'(a_n)| + \int_{a_n}^{z_n} |y''(t)|dt, \tag{13}$$

consequently

$$d_n \leqq d_n/2 + \int_{a_n}^{z_n} |y''(t)|dt,$$

or

$$d_n/2 \leqq \int_{a_n}^{z_n} |y''(t)|dt, \tag{14}$$

It is also true that

$$d_n/2 \leqq \int_{z_n}^{b_n} |y''(t)|dt, \tag{15}$$

Adding (14) and (15) we have

$$d_n \leqq \int_{a_n}^{b_n} |y''(t)|dt. \tag{16}$$

Applying the Schwartz inequality

$$d_n^2 \leqq \left[\int_{a_n}^{b_n} |y''(t)|dt\right]^2 \leqq \int_{a_n}^{b_n} dt \int_{a_n}^{b_n} y''^2(t)dt$$
$$= (b_n - a_n) \int_{a_n}^{b_n} y''^2(t)dt.$$

Integrating $\int_{a_n}^{b_n} y''^2(t)dt$ by parts we obtain

$$\begin{aligned} d_n^2/(b_n - a_n) &\leqq y'(b_n)y''(b_n) - y'(a_n)y''(a_n) - \int_{a_n}^{b_n} y'''(t)y'(t)dt \\ &= y'(b_n)y''(b_n) - y'(a_n)y''(a_n) \\ &\quad + \int_{a_n}^{b_n} q(t)y(t)y'(t)dt - \int_{a_n}^{b_n} y'(t)f(t)dt \\ &\leqq K + \int_{a_n}^{b_n} q(t)y(t)\,|y'(t)|dt + \int_{a_n}^{b_n} |y'(t)|\,|f(t)|dt, \end{aligned}$$

where $K = y'(b_n)y''(b_n) - y'(a_n)y''(a_n)$. From our choice of $a_n$ and $b_n$ it follows that $K \leqq 0$.

Thus

$$\begin{aligned} d_n^2/(b_n - a_n) &\leqq \int_{a_n}^{b_n} |y'(t)|q(t)y(t)dt + \int_{a_n}^{b_n} |y'(t)|\,|f(t)|dt \\ &\leqq d_n \int_{a_n}^{b_n} q(t)y(t)dt + d_n \int_{a_n}^{b_n} |f(t)|dt. \end{aligned}$$

After dividing both sides by $d_n$ we get

$$d_n/(b_n - a_n) \leqq \int_{a_n}^{b_n} q(t)y(t)dt + \int_{a_n}^{b_n} |f(t)|dt. \tag{17}$$

In view of (8), the right side of (17) approaches zero as $n \to \infty$. Consequently

$$\lim_{n\to\infty} (b_n - a_n) = \infty. \tag{18}$$

Let $N$ be a positive integer so that $|y'(t)| \geqq 0$ on $[a_N, b_N]$, $t_1 < a_N$ and

$$\int_{a_N}^{b_N} q(t)y(t)dt < 1. \tag{19}$$

If $y'(t) \geqq 0$ on $[a_N, b_N]$, then from our hypothesis and (18) we can choose $N$ to also satisfy

$$\int_{1+a_N}^{b_N} q(t)dt > 8/3d,$$

Applying the mean-value theorem to $y(t)$ over $[a_N, t]$, $a_N \leqq t \leqq b_N$ we have

$$y(t) = y(a_N) + y'(\beta)(t - a_N),\ a_N < \beta < t, \tag{20}$$

so

$$\begin{aligned}\int_{a_N}^{b_N} q(t)y(t)dt &\geqq \int_{a_N}^{b_N} (t - a_N)q(t)y'(\beta)dt \geqq (d_N/2)\int_{a_N}^{b_N}(t - a_N)q(t)dt\\ &\geqq (3/8)d\int_{a_N}^{b_N}(t - a_N)q(t)dt \geqq (3/8)d\int_{1+a_N}^{b_N}(t - a_N)q(t)dt\\ &\geqq (3/8)d\int_{1+a_N}^{b_N} q(t)dt > 1.\end{aligned}$$

But this contradicts (19). Hence, $d = 0$ and $\lim_{t\to\infty} y'(t) = 0$. If $y'(t) < 0$ on $[a_N, b_N]$, then instead of (20) we would use

$$y(t) = y(b_N) + y'(\beta)(t - b_N),\ t < \beta < b_N \tag{20$'$}$$

and the interval $[a_N + 1, b_N]$ above would be replaced by the interval $[a_N, -1 + b_N]$, after which the same conclusion is obtained.

Now for our main results.

THEOREM 4. *Suppose* $\liminf_{t\to\infty} \int_t^{t+h} q(t)dt \geqq \varepsilon > 0$ *for some* $h > 0$ *and* $\int_0^\infty |f(t)|dt < \infty$. *Then every nonoscillatory solution of* (2) *tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be a positive nonoscillatory solution of (2) and suppose $y(t) \neq 0$ on $[a, \infty)$. If $y(t) \not\to 0$ as $t \to \infty$, then by Theorem 3, $y'(t) \to 0$ as $t \to \infty$. And we know from the proof of Theorem 3 that

$$\liminf_{t\to\infty} y(t) = 0. \tag{21}$$

Suppose

$$\limsup_{t\to\infty} y(t) > c > 0. \tag{22}$$

Then in view of (21) and (22) there exists a sequence $\{p_n\}$, $n \geqq 0$ with the following properties:

(i) $p_n \to \infty$ as $n \to \infty$, $p_n \geqq a$ for all $n$,

(ii) $y(p_n) > c$,

(iii) For each $n \geqq 1$, there is number $p_n'$ such that $p_{n-1} < p_n' < p_n$ and $y(p_n') < c/2$.

For $n \geqq 1$, let $\alpha_n$ be the largest number less than $p_n$ such that $y(\alpha_n) = c/2$ and $\beta_n$ be the smallest number greater than $p_n$ such that $y(\beta_n) = c/2$.

Applying the mean-value theorem in the interval $[\alpha_n, p_n]$, there exists a number $t_n$ such that $\alpha_n < t_n < p_n$ and

$$y'(t_n) = (y(p_n) - y(\alpha_n))/(p_n - \alpha_n) > c/2(\beta_n - \alpha_n). \tag{23}$$

From Theorem 3, $y'(t_n) \to 0$ as $n \to \infty$. Therefore it follows from (23) that

$$\lim_{n\to\infty} (\beta_n - \alpha_n) = \infty. \tag{24}$$

Moreover from our choice of $\alpha_n$ and $\beta_n$, $y(t) \geqq c/2$ on $[\alpha_n, \beta_n]$. By our previous Theorem we know that $\int_a^\infty q(t)y(t)dt < \infty$, but

$$\int_a^\infty q(t)y(t)dt > \sum_{n=1}^\infty \int_{\alpha_n}^{\beta_n} q(t)y(t)dt \geqq (c/2)\sum_{n=1}^\infty \int_{\alpha_n}^{\beta_n} q(t)dt \to \infty \text{ as } n \to \infty,$$

a contradiction, so $\lim_{t\to\infty} y(t) = 0$, and our proof is complete.

Finally we examine some oscillatory solutions of (2).

THEOREM 5. *Suppose* $\int_0^\infty q(t)dt < \infty$ *and* $\int |f(t)|dt < \infty$ *and let $y$ be a bounded solution of* (2). *If $y$ is quickly oscillatory and $y'$ bounded, then* $y \to 0$.

PROOF. Suppose $y \nrightarrow 0$ as $t \to \infty$. Then lim sup $|y(t)| > d > 0$ for some constant $d$. Proceeding in a manner similar to Hammett [2] we have a sequence. $t_n$ such that

(i) $t_n \to \infty$ as $n \to \infty$ for each $n \geqq 1$;
(ii) for each $n \geqq 1$, $|y(t_n)| > d$;
(iii) for each $n \geqq 2$, there exist $m_n$ such that $t_{n-1} < m_n < t_n$ and $|y(m_n)| < d/2$.

Let $[p_n, q_n]$ be the smallest closed interval containing $t_n$ such that $|y(p_n)| = |y(q_n)| = d/2$ for $n \geqq 2$. In the interval $(p_n, t_n)$ there exist $r_n$ such that $y'(r_n) = (y(t_n) - y(p_n))/(t_n - p_n)$ which gives

$$\begin{aligned}|y'(r_n)| = |y(t_n) - y(p_n)|/(t_n - p_n) &\geqq \| y(t_n)| - |y(p_n)\|/(t_n - p_n) \\ &> d/2(q_n - p_n),\end{aligned}$$

Since $|y(t)| > 0$ for $t \in (p_n, q_n)$ for each $n \geqq 2$, the pair $(p_n, q_n)$ must lie between two consecutive zeros of $y(t)$. Hence $q_n - p_n \to 0$ as $n \to \infty$, consequently $\limsup_{n\to\infty} |y'(r_n)| = \infty$, a contradiction.

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