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ADAPTIVE TRIANGULAR CUBATURES

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ABSTRACT. Adaptive cubature rules for bivariate problems are given. These rules have wide applicability because they are defined over a general polygonal domain and their good approximation properties make them useful for many integrands. The strategy of the rules is discussed and numerical and graphical examples are given.

1. Introduction. Numerical integration has a long and interesting history. (See [**4**]). Several patterns can be detected in its development. The numerical integration of univariate functions ("quadrature") has received most of the attention with correspondingly less published work on the more difficult and useful numerical integration of functions of more than one variable ("cubature"). A second observation is that numerical integration required considerable interaction by the user, until fairly recently. The idea of "black box" adaptive integration rules is relatively new. Cranley and Patterson [**3**] introduced adaptive quadratures. VanDooren and deRidder [**8**] introduced adaptive cubatures defined over a tensor product domain (a cube). Laurie [**5**] introduced adaptive cubatures over triangles, the first adaptive cubatures defined for intrinsically bivariate domains. General bivariate domains are much better approximated by triangles than by rectangles, so the general triangle seems the best geometric building block for bivariate problems.

We were recently asked to compute some double integrals of singular integrands, integrals which exist only in a rather complicated Cauchy Principal Value sense. The computation of singular integrals is a research topic in its own right and a few quadrature schemes have been devised for singular univariate problems. (See the references in [**4**].) We were interested in potential problems involving surface integrals with singular functions of the form x/R^3 where R can be zero at a point (subsonic flow) or along a curve in the surface (supersonic flow). User interaction in singular problems is pointless because of the large number of calculations needed, so our goal became the development of a reasonably reliable and efficient adaptive cubature. We tried Laurie's scheme for these problems,

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but the unsatisfactory results (See §3) compelled us to develop our own schemes.

This paper contains the components of our strategy in creating appropriate adaptive cubatures. Examples illustrating the algorithm are included.

2. Strategy.

A. Domain of integration. The approximate integration is of the form

$$\iint_D F(u, v)\, du\, dv \cong \sum_{k=1}^{n} A_k F(u_k, v_k). \tag{2.1}$$

The domain D is assumed to be polygonal and hence a union of triangles. We assume that this initial union of triangles is user-supplied.

B. Basic cubature rule. We want a cubature rule, for an arbitrary triangle, of maximal polynomial precision for the number of cubature nodes. Such a rule is the Radon 7–5 rule, a 7 node rule with polynomial precision 5 [7]. The location of these nodes is indicated in Figure 2.1.

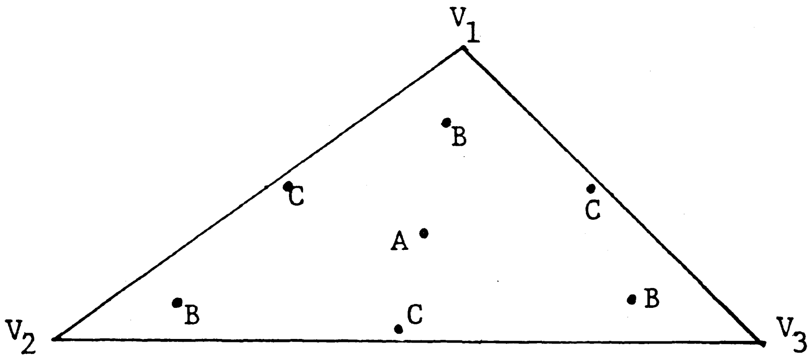


FIGURE 2.1. Location of Radon nodes.

The Radon 7–5 cubature for the triangle $V_1V_2V_3$ is

$$\begin{aligned} &AF\left(\frac{1}{3}V_1 + \frac{1}{3}V_2 + \frac{1}{3}V_3\right) \\ &+B[F(rV_1+rV_2+(1-2r)V_3)+F(rV_1+(1-2r)V_2+rV_3) \\ &+F((1-2r)V_1+rV_2+rV_3)]+C[F(uV_1+uV_2+(1-2u)V_3) \\ &+F(uV_1+(1-2u)V_2+uV_3)+F((1-2u)V_1+uV_2+uV_3)] \end{aligned} \tag{2.2}$$

where T is the area of triangle $V_1V_2V_3$,

$$A = \frac{9}{40}T, \qquad B = \frac{155-\sqrt{15}}{1200}T, \qquad C = \frac{155+\sqrt{15}}{1200}T,$$

$$r = \frac{6-\sqrt{15}}{21}, \qquad u = \frac{6+\sqrt{15}}{21}.$$

We observe that a point such as $rV_1 + rV_2 + (1 - 2r)V_3$ has barycentric coordinates $(r, r, 1 - 2r)$. The fact that the Radon rule is expressed in barycentric coordinates makes formula (2.2) applicable to the general triangle $V_1V_2V_3$. We also used a 13–7 rule due to Cowper [2] and a 4–3 and a 1–1 rule from Stroud [7].

C. Triangle subdivision and error estimation. We begin with the given domain given as a union of triangles $D = \bigcup T_i$. For each triangle T_i we compute the cubature sum $C_i = C_i(F)$. We assign error estimate E_i to

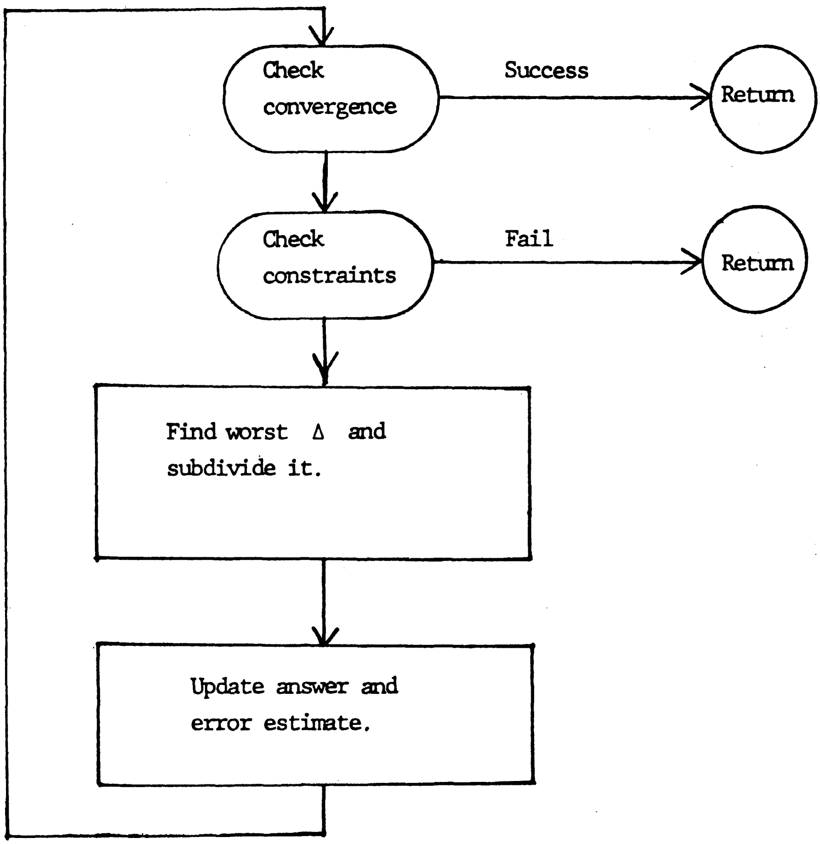


FIGURE 2.2 High Level View of Subdivision Step.

T_i. We are given a user-defined desired accuracy and constraints on computer time and storage. The "refinement" step is as follows:

The triangular subdivision is done as follows: The "worst" triangle T_1 is subdivided by connecting the midpoint of its longest edge to the opposite vertex. The reason for this choice is to generate optimally fat triangles. This ensures good (1) geometric and hence (2) numerical properties of the triangular cubatures. The good geometric properties are that (a) the triangles do shrink to zero and (b) in a controlled way. For example, if an h by h right triangle is repeatedly subdivided by bisecting the longest side, then the longest side decreases by a factor of $1/\sqrt{2}$ per step. (More generally, see [6]. A Sard Kernel analysis [1] applied to this example yields a truncation error of order h^{n+1} where the basic cubature rule has precision n, so the corresponding factor after one step is $(h/\sqrt{2})^{n+1}$. Since $n=5$ and 7 for the 7–5 and 13–7 rules, respectively, the corresponding terms are $h^6/8$ and $h^8/16$. Notice that our method of triangle subdivision is independent of the integrand and the basic cubature rule. Laurie's method of subdivision involves both the integrand and the location of the cubature nodes.

Comparison with Laurie's subdivision scheme: Laurie decides which edge to halve on the basis of second order differences of the integrand along parallels to the edges. This method depends upon the integrand and not upon the geometry. We present pictures in §3 to illustrate how bad the shape of the resulting triangles can be. There is also an analytic argument against this method: Suppose that a quadratic is added to an interesting integrand. This would not affect our schemes of precision two or more. This quadratic does affect the triangle subdivision part of Laurie's scheme. An example is given in §3.

We experimented with making the division of the longest side depend upon the variation of the integrand from an integral mean value, which is in the spirit of Laurie's method of triangle subdivision depending upon the integrand. Our experiment produced worse results than halving the longest side.

Our error estimation is done as follows: The initial errors E_i are set to be "large", i.e., approximately the size of the initial C_i. The iteration step: given an error estimate $E_1 = E_1^{OLD}$ corresponding to worst triangle $T_1 = T_{1,1} \cup T_{1,2}$ calculate the corresponding cubatures $C_{1,1}$ and $C_{1,2}$. Let

$$E_1^{NEW} = |C_1 - C_{1,1} - C_{1,2}| = |C^{OLD} - C^{NEW}| \tag{2.3}$$

where C is the cubature sum over the whole domain D. Assign the error estimates to $T_{1,1}$ and $T_{1,2}$ as follows:

$$E_{1,1} = E_{1,2} = (1/2)\alpha(E_1^{NEW} + E_1^{OLD}) \tag{2.4}$$

where $\alpha = (1/\sqrt{2})^{n+1}$, n being the polynomial precision of the rule, e.g., $\alpha = 1/8$ for the 7–5 rule. Finally, numerical evidence and intuition have led us to the generalization

$$(2.5) \qquad E_{1,1} = E_{1,2} = \alpha(aE_1^{NEW} + bE_1^{OLD})$$

where $a, b \geqq 0$ and $a + b = 1$. (A good choice is $a = 7/8$ and $b = 1/8$.)

Why keep E_1^{OLD} in (2.5)? Sometimes symmetry or a numerical anomaly (e.g., large number minus large number) makes E_1^{NEW} zero to machine accuracy even though the true error is not zero. The retention of past history (E_1^{OLD}) helps avoid this trap.

D. Cullable triangles.

(1) Some triangles have errors so small that they can be disregarded.

(2) A practical constraint on every adaptive calculation is the number of integrand evaluations and machine storage.

To extend the applicability of our algorithm, we consider (1) and (2). This makes the algorithm less adaptive in that the user must specify a criterion for disregarding triangles with "small" cubature error estimates Our implementation: Select a (problem-dependent) "irreducible error tolerance", which is an acceptable error for discarding future consideration of the best triangles. Example: the cubature sum is approximately one, the error tolerance requested is 10^{-6}, and the irreducible error tolerance is 10^{-8}. Then, beginning with the best triangles, "cull out" those triangles whose error estimate sum is less than 10^{-8}.

3. Numerical and graphical examples. We tested this algorithm on a variety of examples. We decided that it might be enlightening to see how the algorithm progressed geometrically and so we used our interactive graphics equipment to display the geometry—the subdividing of the triangles—and simultaneously the numerical results. We recorded these results on 16mm. film. We show some similar pictures in this Section. The information is presented in the following way: Numerical results are given for the problems stated, in sets of five, corresponding to Laurie's rule, 1–1 rule, 4–3 rule, 7–5 rule, and 13–7 rule, respectively. The columns have the following meanings: answer, error made, error estimate, number of integrand evaluations, number of triangles, success ("2") or failure ("4") in achieving the requested absolute error, and an identification of the rule used. The requested absolute errors begin with 1×10^{-1} and decrease by 10^{-1} for each set, ending with 1×10^{-5}. We are limited by storage on our PDP 11/60, so we set the limit of 810 triangles. If storage is unimportant relative to speed, then answers corresponding to the same number of integrand evaluations ("calls") should be compared, e.g., the singular problem with $\varepsilon^2 = 10^{-4}$ and requested absolute error of 10^{-4}

EXAMPLE 1.

$$\int_0^{3\pi}\int_0^{3\pi} \cos(x+y)\, dxdy = -4$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= -3.9928Ø71    ERR=.719E-Ø2 ERR EST=.923E-Ø1 CALLS=   84 TRIANGLES=   7 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -3.97684Ø5    ERR=.232E-Ø1 ERR EST=1.18     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -3.9422472    ERR=.578E-Ø1 ERR EST=.994E-Ø1 CALLS= 1Ø16 TRIANGLES= 128 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= -2.983161Ø    ERR=1.Ø2     ERR EST=.826E-Ø1 CALLS=  168 TRIANGLES=  13 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -4.1839294    ERR=.184     ERR EST=.835E-Ø1 CALLS=  13Ø TRIANGLES=   6 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= -3.9999645    ERR=.355E-Ø4 ERR EST=.876E-Ø2 CALLS=  168 TRIANGLES=  13 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -3.97684Ø5    ERR=.232E-Ø1 ERR EST=1.18     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -3.9988Ø43    ERR=.12ØE-Ø2 ERR EST=.991E-Ø2 CALLS= 3Ø48 TRIANGLES= 382 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= -3.99Ø8867    ERR=.911E-Ø2 ERR EST=.992E-Ø2 CALLS=  77Ø TRIANGLES=  56 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -4.ØØ11573    ERR=.116E-Ø2 ERR EST=.582E-Ø2 CALLS=  364 TRIANGLES=  15 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= -4.ØØØØØ24    ERR=.238E-Ø5 ERR EST=.84ØE-Ø3 CALLS=  49Ø TRIANGLES=  36 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -3.97684Ø5    ERR=.232E-Ø1 ERR EST=1.18     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -3.9998672    ERR=.133E-Ø3 ERR EST=.394E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= -4.ØØØ3572    ERR=.357E-Ø3 ERR EST=.994E-Ø3 CALLS= 1316 TRIANGLES=  95 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -4.ØØ11582    ERR=.116E-Ø2 ERR EST=.657E-Ø3 CALLS=  39Ø TRIANGLES=  16 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= -4.ØØØØØ14    ERR=.143E-Ø5 ERR EST=.964E-Ø4 CALLS= 1344 TRIANGLES=  97 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -3.97684Ø5    ERR=.232E-Ø1 ERR EST=1.18     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -3.9998672    ERR=.133E-Ø3 ERR EST=.394E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= -4.ØØØØØ43    ERR=.429E-Ø5 ERR EST=.993E-Ø4 CALLS= 4Ø46 TRIANGLES= 29Ø RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -4.ØØØØ958    ERR=.958E-Ø4 ERR EST=.962E-Ø4 CALLS= 156Ø TRIANGLES=  61 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= -4.ØØØØØØØ    ERR=.ØØØ     ERR EST=.549E-Ø4 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -3.97684Ø5    ERR=.232E-Ø1 ERR EST=1.18     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -3.9998672    ERR=.133E-Ø3 ERR EST=.394E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= -4.ØØØØØ24    ERR=.238E-Ø5 ERR EST=.998E-Ø5 CALLS= 7266 TRIANGLES= 52Ø RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -3.9999952    ERR=.477E-Ø5 ERR EST=.739E-Ø5 CALLS= 2444 TRIANGLES=  95 RETCD= 2 DEGREE 7, 13 POINT RULE
```

EXAMPLE 2.

$$\int_0^1 \int_0^1 \frac{dxdy}{(x^2+\varepsilon)[(y+\frac{1}{4})^2+\varepsilon]}$$

$$\varepsilon = 1 \times 10^{-4}$$

Laurie claims that this integral equals 499.1249 but we claim that it's 499.1236 which is used in the error column.

```
REQUESTED ABSOLUTE ERROR=0.10000000
ANS=  499.10785     ERR=.157E-01 ERR EST=.996E-01 CALLS= 1596 TRIANGLES= 115 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  493.95807     ERR=5.17     ERR EST=17.3     CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  499.77942     ERR=.656     ERR EST=.491     CALLS= 6472 TRIANGLES= 810 RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  501.02466     ERR=1.90     ERR EST=.999E-01 CALLS= 4284 TRIANGLES= 307 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=  504.12720     ERR=5.00     ERR EST=.989E-01 CALLS= 2886 TRIANGLES= 112 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999998E-02
ANS=  499.12448     ERR=.885E-03 ERR EST=.999E-02 CALLS= 6916 TRIANGLES= 495 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  493.95807     ERR=5.17     ERR EST=17.3     CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  499.77942     ERR=.656     ERR EST=.491     CALLS= 6472 TRIANGLES= 810 RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  499.07855     ERR=.450E-01 ERR EST=.118E-01 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  498.96640     ERR=.157     ERR EST=.100E-01 CALLS= 8632 TRIANGLES= 333 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.10000000E-02
ANS=  499.12488     ERR=.128E-02 ERR EST=.443E-02 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  493.95807     ERR=5.17     ERR EST=17.3     CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  499.77942     ERR=.656     ERR EST=.491     CALLS= 6472 TRIANGLES= 810 RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  499.07855     ERR=.450E-01 ERR EST=.118E-01 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  499.12314     ERR=.458E-03 ERR EST=.996E-03 CALLS=18512 TRIANGLES= 713 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999997E-04
ANS=  499.12488     ERR=.128E-02 ERR EST=.443E-02 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  493.95807     ERR=5.17     ERR EST=17.3     CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  499.77942     ERR=.656     ERR EST=.491     CALLS= 6472 TRIANGLES= 810 RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  499.07855     ERR=.450E-01 ERR EST=.118E-01 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  499.12360     ERR=.000     ERR EST=.603E-03 CALLS=21034 TRIANGLES= 810 RETCD= 4 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999997E-05
ANS=  499.12488     ERR=.128E-02 ERR EST=.443E-02 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  493.95807     ERR=5.17     ERR EST=17.3     CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  499.77942     ERR=.656     ERR EST=.491     CALLS= 6472 TRIANGLES= 810 RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  499.07855     ERR=.450E-01 ERR EST=.118E-01 CALLS=11326 TRIANGLES= 810 RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  499.12360     ERR=.000     ERR EST=.603E-03 CALLS=21034 TRIANGLES= 810 RETCD= 4 DEGREE 7, 13 POINT RULE
```

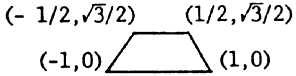
EXAMPLE 3.

$$\int_0^1 \int_0^1 \exp|x+y-1| \; dxdy = 1.436564$$

```
REQUESTED ABSOLUTE ERROR=0.10000000
ANS= 1.4365656   ERR=.167E-05 ERR EST=.289E-03 CALLS=   14 TRIANGLES=   2 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= 1.4265170   ERR=.100E-01 ERR EST=.908E-01 CALLS=   28 TRIANGLES=  15 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= 1.4364703   ERR=.937E-04 ERR EST=.313E-01 CALLS=   56 TRIANGLES=   8 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= 1.4365635   ERR=.477E-06 ERR EST=.684E-01 CALLS=   84 TRIANGLES=   7 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= 1.4365637   ERR=.238E-06 ERR EST=.942E-01 CALLS=  104 TRIANGLES=   5 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999998E-02
ANS= 1.4365656   ERR=.167E-05 ERR EST=.289E-03 CALLS=   14 TRIANGLES=   2 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= 1.4330895   ERR=.347E-02 ERR EST=.996E-02 CALLS=   98 TRIANGLES=  50 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= 1.4365278   ERR=.361E-04 ERR EST=.933E-02 CALLS=  104 TRIANGLES=  14 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= 1.4365636   ERR=.358E-06 ERR EST=.781E-02 CALLS=   98 TRIANGLES=   8 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= 1.4365638   ERR=.119E-06 ERR EST=.195E-02 CALLS=  182 TRIANGLES=   8 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.10000000E-02
ANS= 1.4365656   ERR=.167E-05 ERR EST=.289E-03 CALLS=   14 TRIANGLES=   2 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= 1.4360018   ERR=.562E-03 ERR EST=.996E-03 CALLS=  586 TRIANGLES= 294 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= 1.4365542   ERR=.978E-05 ERR EST=.931E-03 CALLS=  192 TRIANGLES=  25 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= 1.4365637   ERR=.238E-06 ERR EST=.244E-03 CALLS=  210 TRIANGLES=  16 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= 1.4365638   ERR=.119E-06 ERR EST=.992E-03 CALLS=  286 TRIANGLES=  12 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999997E-04
ANS= 1.4365636   ERR=.358E-06 ERR EST=.224E-04 CALLS=   42 TRIANGLES=   4 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= 1.4363487   ERR=.215E-03 ERR EST=.373E-03 CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= 1.4365599   ERR=.405E-05 ERR EST=.989E-04 CALLS=  312 TRIANGLES=  40 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= 1.4365637   ERR=.238E-06 ERR EST=.963E-04 CALLS=  350 TRIANGLES=  26 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= 1.4365638   ERR=.119E-06 ERR EST=.305E-04 CALLS=  390 TRIANGLES=  16 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=0.99999997E-05
ANS= 1.4365636   ERR=.358E-06 ERR EST=.884E-05 CALLS=   70 TRIANGLES=   6 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= 1.4363487   ERR=.215E-03 ERR EST=.373E-03 CALLS= 1618 TRIANGLES= 810 RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= 1.4365628   ERR=.119E-05 ERR EST=.998E-05 CALLS=  504 TRIANGLES=  64 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= 1.4365637   ERR=.238E-06 ERR EST=.764E-05 CALLS=  434 TRIANGLES=  32 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= 1.4365638   ERR=.119E-06 ERR EST=.987E-05 CALLS=  676 TRIANGLES=  27 RETCD= 2 DEGREE 7, 13 POINT RULE
```

EXAMPLE 4.

$$\iint \frac{\ell n r_1}{r_2}\, dxdy \qquad r_1 = \sqrt{(x+1)^2+y^2}$$



$$r_2 = \sqrt{(x-1)^2+y^2}$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= Ø.3475Ø17Ø     ERR=.244E-Ø1 ERR EST=.455E-Ø2 CALLS=   28 TRIANGLES=   4 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.3554Ø572     ERR=.165E-Ø1 ERR EST=.98ØE-Ø1 CALLS=   84 TRIANGLES=  44 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= Ø.34529Ø39     ERR=.266E-Ø1 ERR EST=.662E-Ø1 CALLS=  112 TRIANGLES=  16 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.358ØØ159     ERR=.139E-Ø1 ERR EST=.766E-Ø1 CALLS=  182 TRIANGLES=  15 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.36326984     ERR=.865E-Ø2 ERR EST=.963E-Ø1 CALLS=  286 TRIANGLES=  13 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= Ø.3475Ø17Ø     ERR=.244E-Ø1 ERR EST=.455E-Ø2 CALLS=   28 TRIANGLES=   4 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.37ØØ9773     ERR=.182E-Ø2 ERR EST=.994E-Ø2 CALLS=  38Ø TRIANGLES= 192 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= Ø.3558Ø543     ERR=.161E-Ø1 ERR EST=.9Ø7E-Ø2 CALLS=  24Ø TRIANGLES=  32 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.3642Ø694     ERR=.771E-Ø2 ERR EST=.946E-Ø2 CALLS=  3Ø8 TRIANGLES=  24 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.36327112     ERR=.865E-Ø2 ERR EST=.4Ø4E-Ø2 CALLS=  364 TRIANGLES=  16 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= Ø.37128499     ERR=.635E-Ø3 ERR EST=.876E-Ø3 CALLS=  21Ø TRIANGLES=  17 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.37151229     ERR=.4Ø8E-Ø3 ERR EST=.181E-Ø2 CALLS= 1616 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.37Ø58Ø11     ERR=.134E-Ø2 ERR EST=.992E-Ø3 CALLS=  6Ø8 TRIANGLES=  78 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.365Ø5777     ERR=.686E-Ø2 ERR EST=.714E-Ø3 CALLS=  448 TRIANGLES=  34 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.367Ø9663     ERR=.482E-Ø2 ERR EST=.969E-Ø3 CALLS=  728 TRIANGLES=  3Ø RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= Ø.37198645     ERR=.665E-Ø4 ERR EST=.982E-Ø4 CALLS= 1218 TRIANGLES=  89 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.37151229     ERR=.4Ø8E-Ø3 ERR EST=.181E-Ø2 CALLS= 1616 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.37198785     ERR=.679E-Ø4 ERR EST=.984E-Ø4 CALLS= 1664 TRIANGLES= 21Ø RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.37116683     ERR=.753E-Ø3 ERR EST=.927E-Ø4 CALLS= 1Ø5Ø TRIANGLES=  77 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.367656Ø2     ERR=.426E-Ø2 ERR EST=.977E-Ø4 CALLS=  858 TRIANGLES=  35 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= Ø.37199473     ERR=.747E-Ø4 ERR EST=.997E-Ø5 CALLS= 5516 TRIANGLES= 396 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.37151229     ERR=.4Ø8E-Ø3 ERR EST=.181E-Ø2 CALLS= 1616 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.3719959Ø     ERR=.759E-Ø4 ERR EST=.994E-Ø5 CALLS= 4Ø32 TRIANGLES= 5Ø6 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.37198573     ERR=.657E-Ø4 ERR EST=.988E-Ø5 CALLS= 245Ø TRIANGLES= 177 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.371880Ø5     ERR=.399E-Ø4 ERR EST=.944E-Ø5 CALLS= 221Ø TRIANGLES=  87 RETCD= 2 DEGREE 7, 13 POINT RULE
```

SINGULAR PROBLEM.

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x\,r^3\,dxdy}{(r^2+\varepsilon^2)^3} \qquad r^2 = x^2+y^2$$

$$\varepsilon^2 = 10^{-2}$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.642E-Ø1 CALLS=   7Ø TRIANGLES=   6 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.1Ø6493Ø1E-Ø3 ERR=.1Ø6E-Ø3 ERR EST=.999E-Ø1 CALLS= 126Ø TRIANGLES= 631 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.916E-Ø1 CALLS=  152 TRIANGLES=  2Ø RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.54Ø88473E-Ø1 ERR=.541E-Ø1 ERR EST=.797E-Ø1 CALLS=  126 TRIANGLES=  1Ø RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.623E-Ø1 CALLS=  156 TRIANGLES=   7 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.958E-Ø2 CALLS=  378 TRIANGLES=  28 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.58999285E-Ø4 ERR=.59ØE-Ø4 ERR EST=.769E-Ø1 CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.2Ø5785Ø4E-Ø4 ERR=.2Ø6E-Ø4 ERR EST=.995E-Ø2 CALLS= 1168 TRIANGLES= 147 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS=-Ø.12591481E-Ø4 ERR=.126E-Ø4 ERR EST=.994E-Ø2 CALLS=  952 TRIANGLES=  69 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.6ØØE-Ø2 CALLS=  182 TRIANGLES=   8 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.987E-Ø3 CALLS= 2ØØ2 TRIANGLES= 144 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.58999285E-Ø4 ERR=.59ØE-Ø4 ERR EST=.769E-Ø1 CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.29183429E-Ø5 ERR=.292E-Ø5 ERR EST=.997E-Ø3 CALLS= 3592 TRIANGLES= 45Ø RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.2656Ø295E-Ø3 ERR=.266E-Ø3 ERR EST=.989E-Ø3 CALLS= 182Ø TRIANGLES= 131 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.61157346E-Ø3 ERR=.612E-Ø3 ERR EST=.96ØE-Ø3 CALLS= 1924 TRIANGLES=  75 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= Ø.3259629ØE-Ø8 ERR=.326E-Ø8 ERR EST=.999E-Ø4 CALLS= 8736 TRIANGLES= 625 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.58999285E-Ø4 ERR=.59ØE-Ø4 ERR EST=.769E-Ø1 CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.17229468E-Ø7 ERR=.172E-Ø7 ERR EST=.319E-[illegible] CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.988E-Ø4 CALLS= 357Ø TRIANGLES= 256 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=-Ø.447Ø3484E-Ø7 ERR=.447E-Ø7 ERR EST=.979E-Ø4 CALLS= 3588 TRIANGLES= 139 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= Ø.ØØØØØØØØ     ERR=.ØØØ      ERR EST=.7Ø1E-Ø4 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= Ø.58999285E-Ø4 ERR=.59ØE-Ø4 ERR EST=.769E-Ø1 CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.17229468E-Ø7 ERR=.172E-Ø7 ERR EST=.319E-Ø3 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=-Ø.4284Ø838E-Ø7 ERR=.428E-Ø7 ERR EST=.998E-Ø5 CALLS= 69Ø2 TRIANGLES= 494 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.11175871E-Ø7 ERR=.112E-Ø7 ERR EST=.1ØØE-Ø4 CALLS= 546Ø TRIANGLES= 211 RETCD= 2 DEGREE 7, 13 POINT RULE
```

SINGULAR PROBLEM.

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{x\,r^3}{(r^2+\varepsilon^2)^3} - 100r^2\right) \qquad r^2 = x^2+y^2 \qquad \varepsilon^2 = 10^{-2}$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= -266.7Ø966     ERR=.43ØE-Ø1 ERR EST=.91ØE-Ø1 CALLS=   84 TRIANGLES=   7 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -266.17Ø32     ERR=.496     ERR EST=.979     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.916E-Ø1 CALLS=  152 TRIANGLES=  2Ø RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= -266.61258     ERR=.541E-Ø1 ERR EST=.797E-Ø1 CALLS=  126 TRIANGLES=  1Ø RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -266.66669     ERR=.ØØØ     ERR EST=.623E-Ø1 CALLS=  156 TRIANGLES=   7 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= -266.656Ø1     ERR=.1Ø7E-Ø1 ERR EST=.97ØE-Ø2 CALLS=  3Ø8 TRIANGLES=  23 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -266.17Ø32     ERR=.496     ERR EST=.979     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -266.66663     ERR=.61ØE-Ø4 ERR EST=.993E-Ø2 CALLS= 11Ø8 TRIANGLES= 147 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= -266.66916     ERR=.247E-Ø2 ERR EST=.994E-Ø2 CALLS=  952 TRIANGLES=  69 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -266.66669     ERR=.ØØØ     ERR EST=.6ØØE-Ø2 CALLS=  182 TRIANGLES=   8 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= -266.66666     ERR=.3Ø5E-Ø4 ERR EST=.974E-Ø3 CALLS= 1246 TRIANGLES=  9Ø RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -266.17Ø32     ERR=.496     ERR EST=.979     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.996E-Ø3 CALLS= 3584 TRIANGLES= 449 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= -266.66669     ERR=.ØØØ     ERR EST=.987E-Ø3 CALLS= 1792 TRIANGLES= 129 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -266.666Ø8     ERR=.61ØE-Ø3 ERR EST=.961E-Ø3 CALLS= 1924 TRIANGLES=  75 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= -266.66666     ERR=.3Ø5E-Ø4 ERR EST=.998E-Ø4 CALLS= 6874 TRIANGLES= 492 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -266.17Ø32     ERR=.496     ERR EST=.979     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.313E-Ø3 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.992E-Ø4 CALLS= 357Ø TRIANGLES= 256 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -266.66669     ERR=.ØØØ     ERR EST=.984E-Ø4 CALLS= 3588 TRIANGLES= 139 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= -266.66666     ERR=.3Ø5E-Ø4 ERR EST=.477E-Ø4 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS= -266.17Ø32     ERR=.496     ERR EST=.979     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.313E-Ø3 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= -266.66672     ERR=.3Ø5E-Ø4 ERR EST=.997E-Ø5 CALLS= 6972 TRIANGLES= 499 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= -266.66669     ERR=.ØØØ     ERR EST=.993E-Ø5 CALLS= 5538 TRIANGLES= 214 RETCD= 2 DEGREE 7, 13 POINT RULE
```

SINGULAR PROBLEM.

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x\,r^3}{(r^2+\varepsilon^2)^3}\,dxdy \qquad r^2 = x^2+y^2$$

$$\varepsilon^2 = 10^{-4}$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.96ØE-Ø1 CALLS=  294 TRIANGLES=  22 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.27481932E-Ø3 ERR=.275E-Ø3 ERR EST=.419      CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.15954673E-Ø3 ERR=.16ØE-Ø3 ERR EST=.988E-Ø1 CALLS= 1824 TRIANGLES= 229 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.23545772      ERR=.235      ERR EST=.944E-Ø1 CALLS=  714 TRIANGLES=  52 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=-Ø.58114529E-Ø6 ERR=.581E-Ø6 ERR EST=.878E-Ø1 CALLS=  416 TRIANGLES=  17 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.989E-Ø2 CALLS= 1666 TRIANGLES= 12Ø RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.27481932E-Ø3 ERR=.275E-Ø3 ERR EST=.419      CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.52135438E-Ø5 ERR=.521E-Ø5 ERR EST=.997E-Ø2 CALLS= 4496 TRIANGLES= 563 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS=-Ø.2733618ØE-Ø4 ERR=.273E-Ø4 ERR EST=.997E-Ø2 CALLS= 35ØØ TRIANGLES= 251 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=-Ø.9ØØØ3Ø14E-Ø4 ERR=.9ØØE-Ø4 ERR EST=.963E-Ø2 CALLS= 1924 TRIANGLES=  75 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.997E-Ø3 CALLS= 8526 TRIANGLES= 61Ø RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.27481932E-Ø3 ERR=.275E-Ø3 ERR EST=.419      CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.13727695E-Ø4 ERR=.137E-Ø4 ERR EST=.3Ø5E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.14ØØ7Ø92E-Ø5 ERR=.14ØE-Ø5 ERR EST=.998E-Ø3 CALLS= 5978 TRIANGLES= 428 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.1Ø424Ø78E-Ø1 ERR=.1Ø4E-Ø1 ERR EST=.989E-Ø3 CALLS= 6838 TRIANGLES= 264 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.636E-Ø3 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.27481932E-Ø3 ERR=.275E-Ø3 ERR EST=.419      CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.13727695E-Ø4 ERR=.137E-Ø4 ERR EST=.3Ø5E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.996E-Ø4 CALLS= 9954 TRIANGLES= 712 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.992E-Ø4 CALLS= 9854 TRIANGLES= 38Ø RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ      ERR EST=.636E-Ø3 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.27481932E-Ø3 ERR=.275E-Ø3 ERR EST=.419      CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.13727695E-Ø4 ERR=.137E-Ø4 ERR EST=.3Ø5E-Ø2 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.13923272E-Ø6 ERR=.139E-Ø6 ERR EST=.622E-Ø4 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS= Ø.745Ø58Ø6E-Ø8 ERR=.745E-Ø8 ERR EST=.993E-Ø5 CALLS=1586Ø TRIANGLES= 611 RETCD= 2 DEGREE 7, 13 POINT RULE
```

SINGULAR PROBLEM.

$$\int_{-1}^{1}\int_{-1}^{1} \frac{x\,r^3}{(r^2+\varepsilon^2)^3}\,dxdy \qquad r^2 = x^2+y^2$$

$$\varepsilon^2 = 10^{-6}$$

```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.98ØE-Ø1 CALLS=  742 TRIANGLES=  54 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.35Ø93725E-Ø3 ERR=.351E-Ø3 ERR EST=1.Ø6     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS= Ø.143Ø6977E-Ø3 ERR=.143E-Ø3 ERR EST=.991E-Ø1 CALLS= 3488 TRIANGLES= 437 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS= Ø.45Ø59815E-Ø1 ERR=.451E-Ø1 ERR EST=.99ØE-Ø1 CALLS= 1638 TRIANGLES= 118 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.39584339      ERR=.396     ERR EST=.976E-Ø1 CALLS=  416 TRIANGLES=  17 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.992E-Ø2 CALLS= 3514 TRIANGLES= 252 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.35Ø93725E-Ø3 ERR=.351E-Ø3 ERR EST=1.Ø6     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.289157Ø3E-Ø4 ERR=.289E-Ø4 ERR EST=.2Ø4E-Ø1 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.8285Ø456E-Ø5 ERR=.829E-Ø5 ERR EST=.994E-Ø2 CALLS= 6216 TRIANGLES= 445 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.34272671E-Ø6 ERR=.343E-Ø6 ERR EST=.99ØE-Ø2 CALLS= 793Ø TRIANGLES= 3Ø6 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.191E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.35Ø93725E-Ø3 ERR=.351E-Ø3 ERR EST=1.Ø6     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.289157Ø3E-Ø4 ERR=.289E-Ø4 ERR EST=.2Ø4E-Ø1 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.999E-Ø3 CALLS=11242 TRIANGLES= 8Ø4 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.984E-Ø3 CALLS=1183Ø TRIANGLES= 456 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.191E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.35Ø93725E-Ø3 ERR=.351E-Ø3 ERR EST=1.Ø6     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.289157Ø3E-Ø4 ERR=.289E-Ø4 ERR EST=.2Ø4E-Ø1 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.6519258ØE-Ø7 ERR=.652E-Ø7 ERR EST=.965E-Ø3 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS= Ø.372529Ø3E-Ø8 ERR=.373E-Ø8 ERR EST=.998E-Ø4 CALLS=19396 TRIANGLES= 747 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS= Ø.ØØØØØØØØ      ERR=.ØØØ     ERR EST=.191E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=-Ø.35Ø93725E-Ø3 ERR=.351E-Ø3 ERR EST=1.Ø6     CALLS= 1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=-Ø.289157Ø3E-Ø4 ERR=.289E-Ø4 ERR EST=.2Ø4E-Ø1 CALLS= 6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS= Ø.6519258ØE-Ø7 ERR=.652E-Ø7 ERR EST=.965E-Ø3 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS= Ø.745Ø58Ø6E-Ø8 ERR=.745E-Ø8 ERR EST=.636E-Ø4 CALLS=21Ø34 TRIANGLES= 81Ø RETCD= 4 DEGREE 7, 13 POINT RULE
```

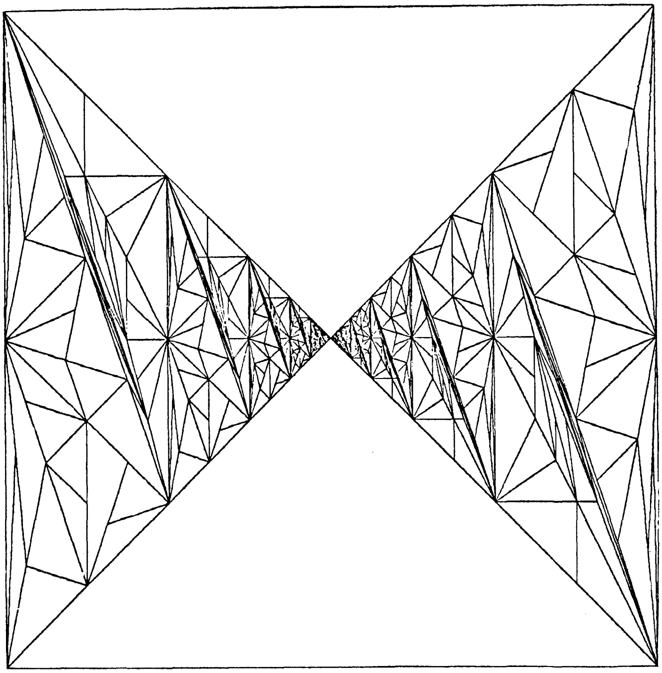
DISCONTINUOUS PROBLEM.

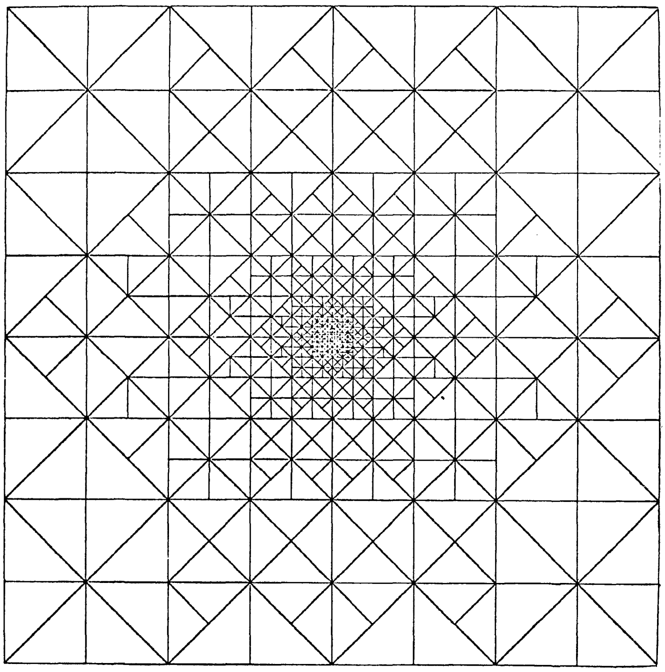
$$\int_{-1}^{1}\int_{-1}^{1} X_C(x,y)\,dx\,dy = \pi$$

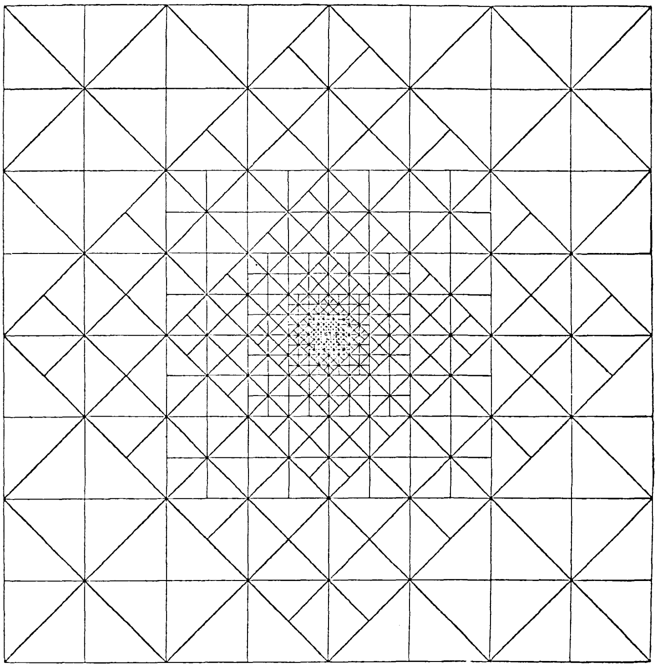
$$C = \{x^2+y^2 \leq 1\}$$

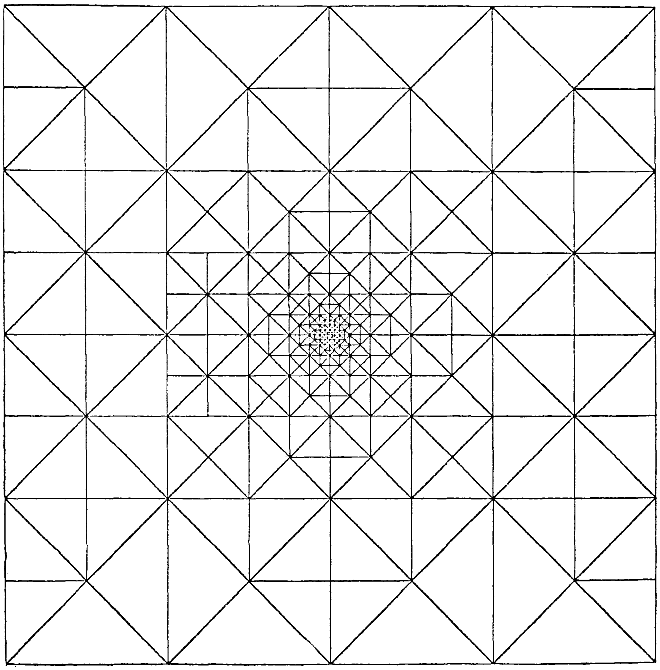
```
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØ
ANS=  2.9693513   ERR=.172      ERR EST=.828E-Ø1 CALLS=    28 TRIANGLES=   3 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  4.ØØØØØØØ   ERR=.858      ERR EST=.956E-Ø1 CALLS=    26 TRIANGLES=  14 RETCD= 2 DEGREE 1,  1 POINT RULE
ANS=  3.2382813   ERR=.967E-Ø1  ERR EST=.98ØE-Ø1 CALLS=   984 TRIANGLES= 124 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS=  2.9731216   ERR=.168      ERR EST=.898E-Ø1 CALLS=    84 TRIANGLES=   7 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=  3.Ø63ØØ66   ERR=.786E-Ø1  ERR EST=.88ØE-Ø1 CALLS=   13Ø TRIANGLES=   6 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999998E-Ø2
ANS=  3.368922Ø   ERR=.227      ERR EST=.999E-Ø2 CALLS=  1Ø36 TRIANGLES=  75 RETCD= 2 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  3.2Ø21484   ERR=.6Ø6E-Ø1  ERR EST=.212E-Ø1 CALLS=  1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  3.177Ø835   ERR=.355E-Ø1  ERR EST=.998E-Ø2 CALLS=  5472 TRIANGLES= 685 RETCD= 2 DEGREE 3,  4 POINT RULE
ANS=  3.1414831   ERR=.11ØE-Ø3  ERR EST=.989E-Ø2 CALLS=  1764 TRIANGLES= 127 RETCD= 2 DEGREE 5,  7 POINT RULE
ANS=  3.168Ø777   ERR=.265E-Ø1  ERR EST=.947E-Ø2 CALLS=  1586 TRIANGLES=  62 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.1ØØØØØØØE-Ø2
ANS=  3.3666737   ERR=.225      ERR EST=.236E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  3.2Ø21484   ERR=.6Ø6E-Ø1  ERR EST=.212E-Ø1 CALLS=  1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  3.1765342   ERR=.349E-Ø1  ERR EST=.65ØE-Ø2 CALLS=  6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  3.149Ø645   ERR=.747E-Ø2  ERR EST=.123E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  3.1429923   ERR=.14ØE-Ø2  ERR EST=.998E-Ø3 CALLS=144Ø4 TRIANGLES= 555 RETCD= 2 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø4
ANS=  3.3666737   ERR=.225      ERR EST=.236E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  3.2Ø21484   ERR=.6Ø6E-Ø1  ERR EST=.212E-Ø1 CALLS=  1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  3.1765342   ERR=.349E-Ø1  ERR EST=.65ØE-Ø2 CALLS=  6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  3.149Ø645   ERR=.747E-Ø2  ERR EST=.123E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  3.1399288   ERR=.166E-Ø2  ERR EST=.497E-Ø3 CALLS=21Ø34 TRIANGLES= 81Ø RETCD= 4 DEGREE 7, 13 POINT RULE
REQUESTED ABSOLUTE ERROR=Ø.99999997E-Ø5
ANS=  3.3666737   ERR=.225      ERR EST=.236E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  7 POINT RULE, LAURIE SPLIT
ANS=  3.2Ø21484   ERR=.6Ø6E-Ø1  ERR EST=.212E-Ø1 CALLS=  1618 TRIANGLES= 81Ø RETCD= 4 DEGREE 1,  1 POINT RULE
ANS=  3.1765342   ERR=.349E-Ø1  ERR EST=.65ØE-Ø2 CALLS=  6472 TRIANGLES= 81Ø RETCD= 4 DEGREE 3,  4 POINT RULE
ANS=  3.149Ø645   ERR=.747E-Ø2  ERR EST=.123E-Ø2 CALLS=11326 TRIANGLES= 81Ø RETCD= 4 DEGREE 5,  7 POINT RULE
ANS=  3.1399288   ERR=.166E-Ø2  ERR EST=.497E-Ø3 CALLS=21Ø34 TRIANGLES= 81Ø RETCD= 4 DEGREE 7, 13 POINT RULE
```

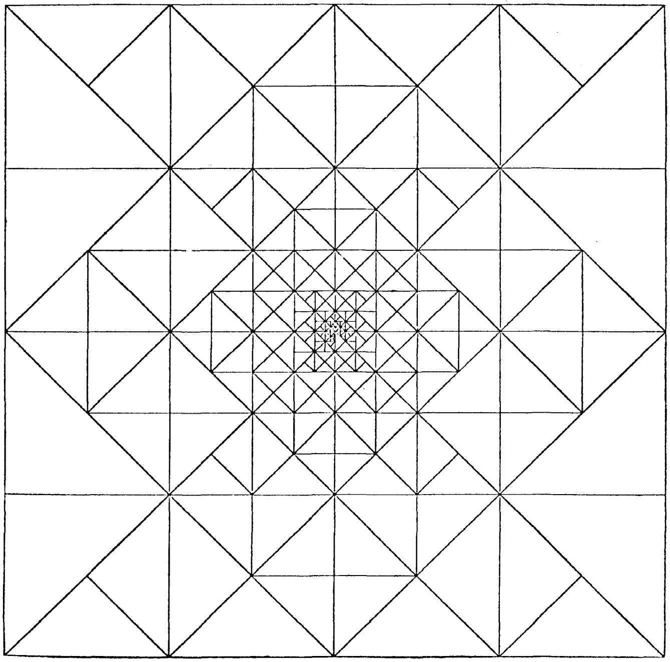
Singular problem with $\varepsilon^2 = 10^{-4}$ Requested absolute error 1×10^{-3}



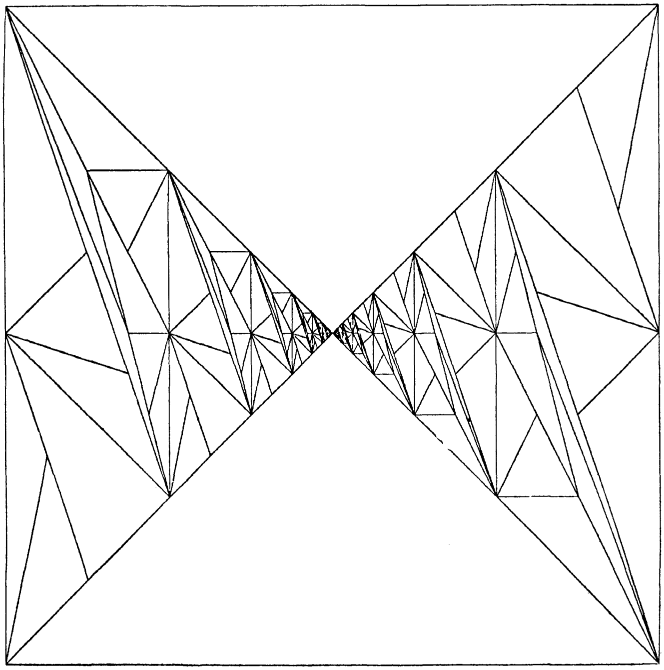


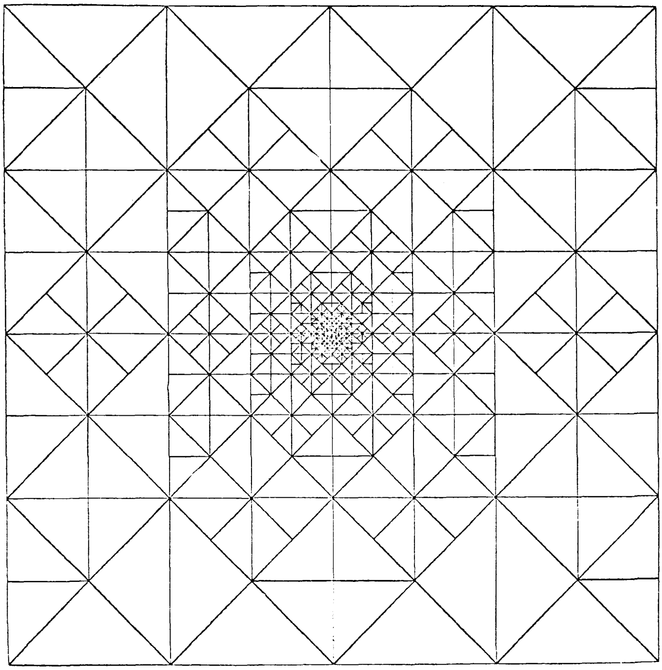


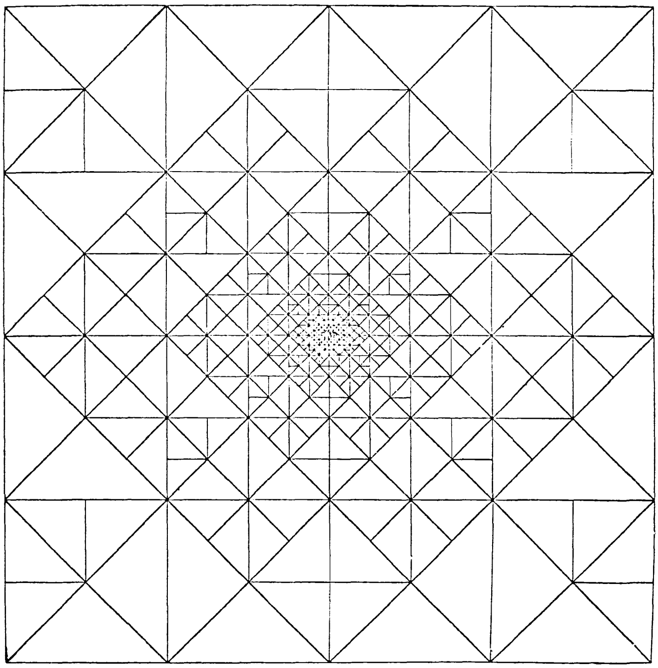


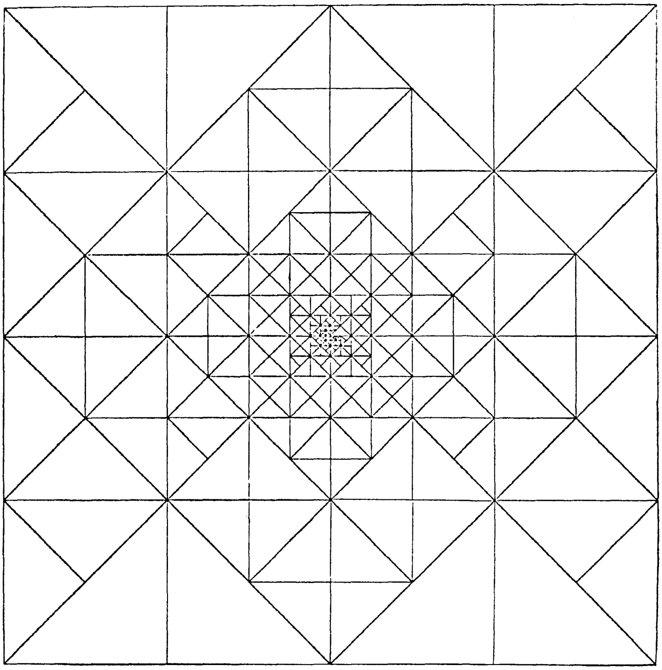


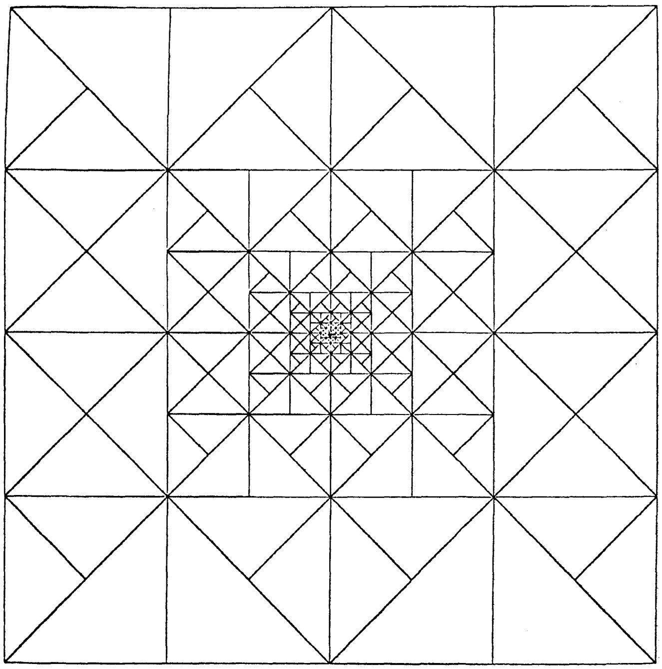
Singular problem with $\varepsilon = 10^{-6}$ Requested absolute error 1×10^{-2}



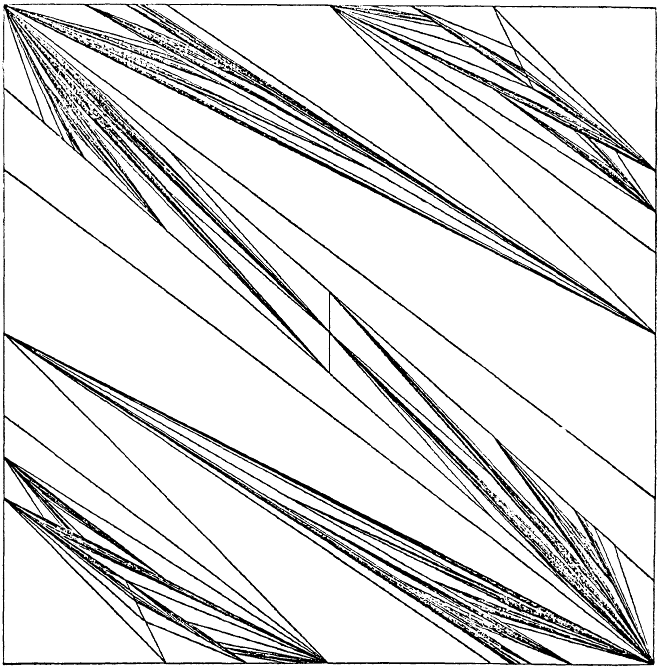


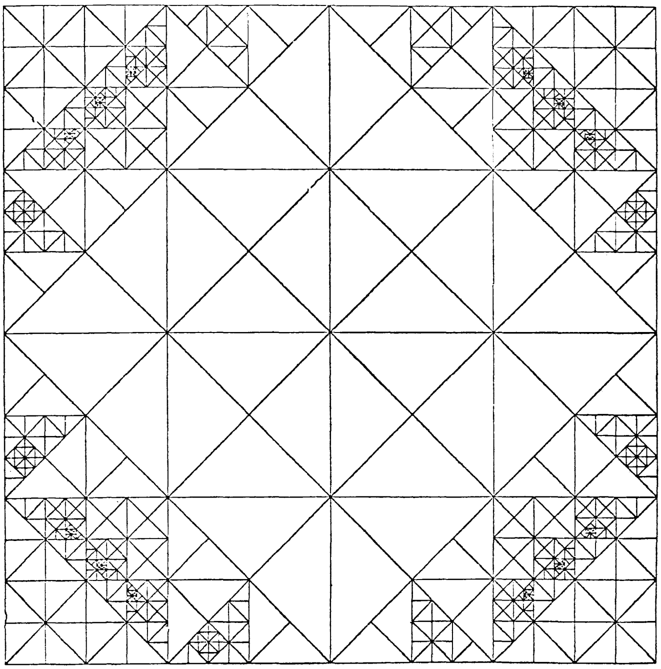


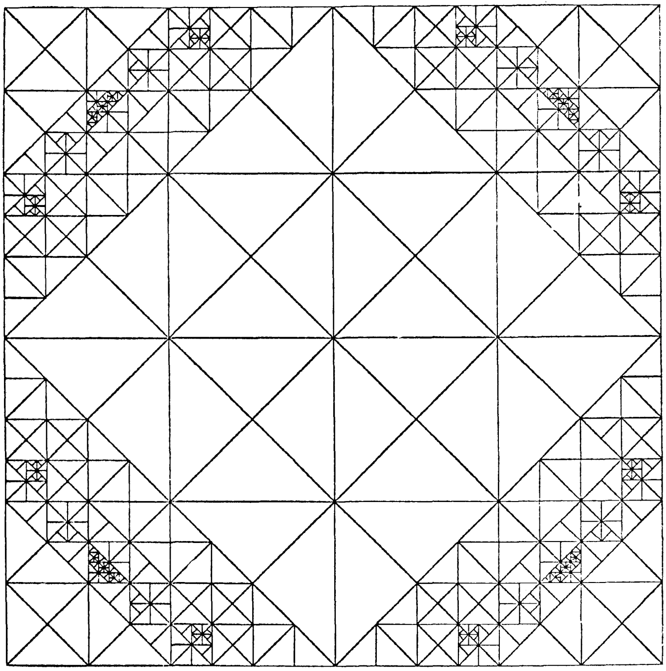


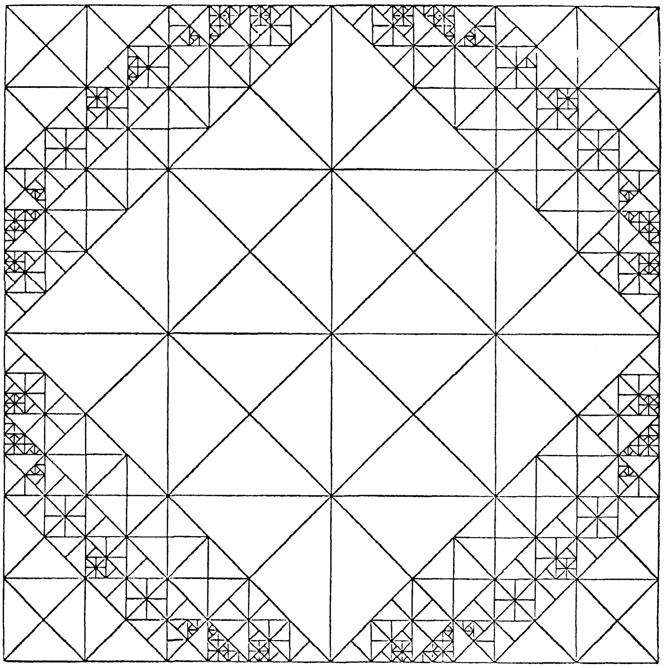


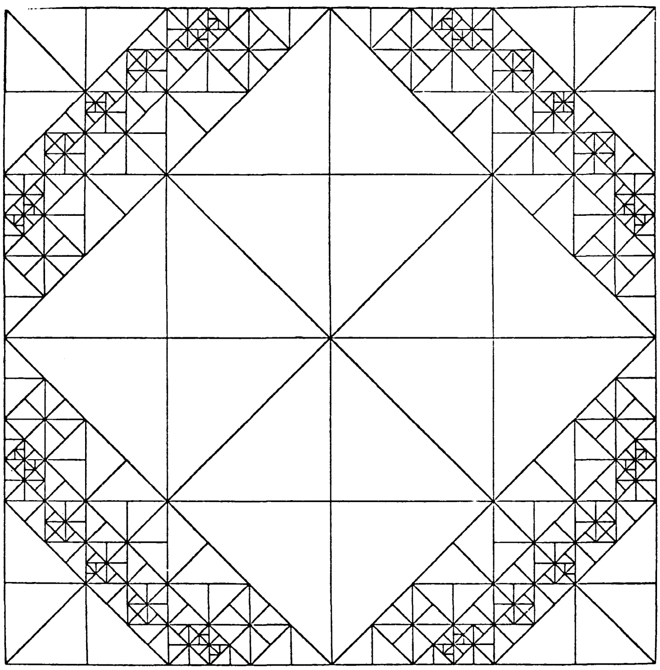
Discontinuous example Requested absolute erroro 1×10^{-3}











for which the 7–5 and 13–7 rules use 9954 and 9854 calls, respectively, and achieve the same accuracy.

After the numerical results, we have three sets of pictures, the first two sets corresponding to the singular problem and the third set corresponding to a discontinuous problem. These pictures correspond to the same sequence: Laurie's rule, the 1–1, 4–3, 7–5, and 13–7 rules, respectively. The two examples are the singular potential problem and a discontinuous problem of the form $\int_{-1}^{1}\int_{-1}^{1} \chi_A(u, v)\, du\, dv$ where χ_A is the characteristic function of the domain A, i.e., $\chi_A = 1$ on A and $\chi_A = 0$ outside A. This discontinuous problem is a model of a surface-surface intersection problem that we were asked to solve.

Examples 1–4 in the following are Laurie's examples. These are followed by the singular problem and then the discontinuous problem.

4. Conclusions. We recommend the 13–7 adaptive cubature, with the 7–5 as second choice. Our error estimation method, although simple, is accurate. Our method of triangle subdivision is effective in that it produces optimally "fat" triangles with the concomitant numerical stability. To distinguish between the various cubatures requires tackling the more difficult examples such as singular and discontinuous integrands.

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