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DEGREE THEORY AND THE UNIQUENESS OF SOLUTIONS TO NODAL PROBLEMS

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ABSTRACT. The work consists of a uniqueness theorem for a class of nonlinear nodal problems. Uniqueness is shown subject to an oscillation assumption on certain solutions of the associated variational equation.

The proof is degree-theoretic and reveals the consequences of attempting a Kellogg-type uniqueness argument without the attendant continuity and spectral assumptions. First, the problem is formulated as a pair of operator equations acting in a suitable Banach space. An abstract degree computation is then applied leading to a global degree formula. Finally, a local degree is obtained which when coupled to the global degree yields an identity holding only in the event of uniqueness.

Applications to the Emden-Fowler equation are included.

1. Introduction. This investigation treats the uniqueness of solutions to a class of second-order scalar differential equations of the form

$$y''(t) + f(t, y(t)) = 0, 0 < t < 1, \tag{1.1}$$

with boundary and nodal condition

$$\begin{gathered} y(0+) = y(1-) = 0, y \text{ has exactly } n \text{ simple zeros } 0 < a_1 < a_2 < \cdots < a_n \\ < 1 \text{ with } y(t) > 0 \text{ for } 0 < t < a_1 \text{ (the } a_j \text{ are not prescribed).} \end{gathered} \tag{1.2}$$

The principal motivation comes from nonlinearities f: $[0, 1] \times \mathbf{R} \to \mathbf{R}$ of the form $f(t, x) = u(t) \times |x|^\nu$.

In 1955, Kolodner [8] demonstrated uniqueness for a problem like (1.1)–(1.2) in the case of a certain sublinear f arising in the deflection theory of a heavy rotating string. The Emden-Fowler equations of astrophysics, and certain other superlinear equations, were shown by Coffman [3] to have a unique solution satisfying condition (1.2). Of central importance in the work of Coffman is the variational equation

$$z'' + f_x(t, y)(t)z = 0, \tag{1.3}$$

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where, via Prüfer transformation methods, uniqueness in (1.1) – (1.2) was shown to hold subject to special oscillation conditions on certain solutions of (1.3). Explicit conditions on f establishing these oscillation conditions were obtained by Coffman in the case that $n = 0$. The cases where $n > 0$ have proven less tractable.

This work examines the question of uniqueness in the light of Leray-Schauder degree theory. Briefly, it is shown that oscillation conditions, like those in [3], permit the computation of the local degree which when coupled with the global degree formula of Bates and Gustafson [1], yields a counting argument like the one given by Kellogg [7].

The uniqueness argument depends on the following elements of degree theory: the excision, addition, and homotopy theorems, the Schauder homotopy theorem, and the Leray-Schauder index theorem. Secondary tools used include many of the standard results from the theory of second-order scalar differential equations, as well as elements of the Riesz theory of compact operators.

An outline of the proof is as follows. First, the boundary value problem (1.1) – (1.2) is formulated as a fixed point problem for a nonlinear operator acting in a special Banach space (§3). Using [1], the degree of the operator is obtained relative to a bounded open set containing the fixed points (§4). The number of fixed points is shown to be finite (§5) subject to an assumption on (1.3).

Next, we give the Frechét differential of the operator at a given fixed point (§6), and the Schauder homotopy theorem is applied to obtain the local degree as the index of the derivative (§7). However, a further (non-standard) homotopy is required before applying the index theorem, and it is at this point that the special oscillation conditions are needed (§8).

Finally, the index is found to equal the global degree (§9), and the uniqueness result then follows from the same device as employed in [7].

The work concludes with an application to the classical Emden-Fowler equations (§10).

2. Preliminaries. The following notation and definitions are to be used throughout. $C([a, b])$, the space of all real continuous functions x on $[a, b]$ with norm $\|x\|_{a,b} = \max\{|x(t)|: a \leqq t \leqq b\}$. $C^2((a, b))$, the space of all real and twice continuously differentiable functions on (a, b). L^p, $1 < p < \infty$, the space of all real measurable functions x on $[0, 1]$ such that $\|x\|_p = (\int_0^1 |x(t)|^p\, dt)^{1/p} < \infty$, L^∞, the space of all real measurable functions x on $[0, 1]$ such that $\|x\|_\infty = \text{ess sup}\{|x(t)|: 0 \leqq t \leqq 1\} < \infty$. $G(t, s; a, b)$, the Green's function for the boundary value problem

$$z''(t) = g(t),\ a \leqq t \leqq b,$$

$$z(a) = z(b) = 0,$$

given by $G(t, s; a, b) = (t - a)(b - s)/(b - a)$ $(t \leqq s)$, $= (s - a)(b - t)/(b - a)$ $(s \leqq t)$.

Let g: $[0, 1] \times \mathbf{R} \to \mathbf{R}$ and suppose that g maps L^p into L^q as a substitution operator, where $1 < p, q < \infty$ satisfy $p^{-1} + q^{-1} = 1$. Then g satisfies the boundedness condition provided that

$$(2.1) \qquad \sup \{\|g(\cdot, x(\cdot))\|_q \colon \|x\|_p \leqq r\} < \infty,$$

whenever $0 < r < \infty$, and the continuity condition at x provided $x \in L^p$ and

$$(2.2) \qquad \sup \{\|g(\cdot, y(\cdot)) - g(\cdot, x(\cdot))\|_q \colon y \in L^p, \|y - x\|_p \leqq r\} \to 0$$

The function f satisfies the differentiability condition provided f_x exists and satisfies

(2.3) f_x: $[0, 1] \times \mathbf{R} \to [0, \infty)$ continuously and $f_x(s, x) = 0$ only if $x = 0$.

Let α be a fixed constant and define g by $g(t, x) = f(t, \alpha\,|x|)$. If f satisfies (2.3), then an easy calculation shows that $g_x\,(t, x) = \operatorname{sgn}(x)\, f_x(t, \alpha\,|x|)$ for $x \neq 0$, $= 0$ for $x = 0$. If f(resp. f_x) satisfies (2.1) and (2.2), then g(resp. g_x) does also.

Suppose that f and f_x are continuous on $[0, 1] \times \mathbf{R}$. If n is a fixed nonnegative integer, then the nodal problem determined by f and n is that of solving (1.1) subject to (1.2).

If y satisfies (1.1), then the linear equation

$$(2.4) \qquad z'' + f_x(t, y(t)) = 0, \; 0 < t < 1,$$

is called the variational equation corresponding to y.

If $z \not\equiv 0$ satisfies (2.4) and y has zeros at

$$0 = a_0 < a_1 < a_2 < a_3 < \cdots a_n < a_{n+1} = 1,$$

then z may satisfy one or more of the following oscillation conditions:

(2.5) If $0 \leqq j \leqq n$ and $z(a_j) = 0$, then $z(a_{j+1})\, z'(a_j) < 0$.

(2.6) If $z(0) = 0$, then $(-1)^j z(a_j) z'(0) < 0$, $1 \leqq j \leqq n + 1$,

(2.7) If $z(0) = 0$, then $z(1) \neq 0$.

REMARK 2.2. In certain results to follow it will be assumed that a particular substitution map takes L^p into L^q. This is not a weak assumption, since the map then satisfies (2.1) and (2.2) (KRASNOSELSKII [9]).

3. Operator formulation. In this section, we obtain a pair of nonlinear operator equations which are equivalent, in a sense to be made precise, to the nodal problem (1.1)–(1.2).

First, make the assumptions (to hold henceforth) that (i) f: $[0, 1] \times$

$\mathbf{R} \to \mathbf{R}$ is continuous, (ii) $f(t, x)\, x \geqq 0$ for $0 \leqq t \leqq 1$ and $x \in R$, and (iii) f satisfies conditions (2.1), (2.2), and (2.3), Next, define the set $T = \{(a_1, a_2, \ldots, a_n) \in R^n\colon 0 < a_1 < a_2 < \cdots < a_n < 1\}$ and for a given n-tuple in T, let $J_i = (a_i, a_{i+1})$, $0 \leqq i \leqq n$, where $a_0 = 0$ and $a_{n+1} = 1$. Then for $(a, x) \in T \times L^p$, the nodal operator B is given by

$$B(a, x)(t) = \sum_{i=0}^{n} (-1)^i \int_{a_i}^{a_{i+1}} G(t, s; a_i, a_{i+1}) f(s, (-1)^i |x(s)|)\, ds\, \chi_{J_i}(t), \tag{3.1}$$

and the matching operator $F\colon T \times L^p \to R^n$ by

$$(-1)^i F_i(a, x) = \tag{3.2}$$

$$\int_{a_i}^{a_{i+1}} \frac{a_{i+1} - s}{a_{i+1} - a_i} f(s, (-1)^i |x(s)|)\, ds + \int_{a_{i-1}}^{a_1} \frac{s - a_{i-1}}{a_i - a_{i-1}} f(s, (-1)^{i-1} |x(s)|)\, ds,$$

for $1 \leqq i \leqq n$.

If $x \in L^p$, then each function $f(\cdot, \pm |x(\cdot)|) \in L^1$ and the integrals in (3.1) – (3.2) are finite. Extend these operators over $cl(T) \times L^p$ by taking the integrals to be zero whenever the corresponding J_i is empty and let it be noted that for any $(a, x) \in cl(T) \times L^p$ the function given by (3.1) is nonnegative.

Lemma 3.1. *For $(a, x) \in cl(T) \times L^p$ the function defined by* (3.1) *may be identified in L^p with an element of* $C([0, 1])$. *Hence $B\colon cl(T) \times L^p \to L^\infty$.*

Proof. This follows easily from the properties of the Green's functions and the assumption that $f(\cdot, x(\cdot)) \in L^q$ whenever $x \in L^p$.

Lemma 3.2. *If $(a, x) \in T \times L^p \backslash \{0\}$ satisfies*

$$F(a, x) = 0,\ B(a, x)(t) = x(t),\ \textit{a.e. in } [0, 1], \tag{3.3}$$

then the function $\sum_{i=0}^{n} (-1)^i x(\cdot) \chi_{J_i}(\cdot)$ may be identified in L^p with a solution of the nodal problem (1.1)–(1.2).

Conversely, given that y satisfies (1.1), *setting $a = (a_1, a_2, \ldots, a_n)$ for a_i the i-th interior zero of y, and $x(\cdot) = \sum_{i=0}^{n} (-1)^i y(\cdot) J_i(\ )$, yields a solution of* (3.3).

Proof. Assume $(a, x) \in T \times L^p \backslash \{0\}$ satisfies (3.3). Applying Lemma 3.1 to $x = B(a, x)$, we may identify x in L^p with a member of $C([0, 1])$. It then follows that $x(a_i) = 0$ for $0 < i < n + 1$, and $x''(s) = -(-1)^i f(s, (-1)^i |x(s)|)$, $s \in J^i$. Therefore, if $y(\cdot) = \sum_{i=0}^{n} (-1)^i x(\cdot) \chi_{J_i}(\cdot)$, then $y''(s) = (-1)^i x''(s) = -f(s, (-1)^i x(s)) = -f(s, y(s))$, for $s \in J_i$. The function y will then satisfy (1.1) provided $y'(a_i +) = y'(a_i -)$, $1 \leqq i \leqq n$. But this is just what the equation $F(a, x) = 0$ means

$$\begin{aligned} y'(a_i +) &= (-1)^i x'(a_i +) \\ &= \int_{a_i}^{a_{i+1}} G_t(a_i, s; a_i, a_{i+1}) f(s, (-1)^i |x(s)|) ds \\ &= \int_{a_{i-1}}^{a_i} G_t(a_i, s; a_{i-1}, a_i) f(s, (-1)^{i-1} |x(s)|) \, ds \\ &= (-1)^{i-1} x'(a_i -) \\ &= y'(a_i -). \end{aligned}$$

The converse should be apparent in view of the above computations.

4. Global degree formula. Given below is a degree computation for abstract operator equations in Banach spaces. The computation is then used to obtain a global degree formula needed in the proof of uniqueness.

Let E be a real Banach space, K a cone in E, and T an open bounded subset of $\mathbf{R}^n$. For $0 < r < R$, define $M = \{x \in E : r \leqq \|x\| \leqq R\}$. and suppose that the following maps are given-B: $cl(T) \times E \to K$, F: $cl(T) \times E \to \mathbf{R}^n$, h and k: $cl(T) \to K$. Define A: $cl(T) \times E \to \mathbf{R}^n \times K$ by $(a, x) = (a + F(a, x), B(a, x))$, and assume the following conditions hold:

(i) The complete continuity of B and F.
(ii) The continuity of h and k.
(iii) The nonsingularity condition: If $x = B(a, x) \neq 0$, or $x \geqq \tau k(a)$, or $x \geqq \tau h(a)$ for some $\tau > 0$, $(a, x) \in T \times K$, then $F(a, x) \neq 0$.
(iv) The smallness condition on k: For $a \in T$, $0 < \|k(a)\| < r$.
(v) The compression condition on h: For $a \in T$, $\|h(a)\| \geqq R$.
Then we have

LEMMA 4.1. Assume (i) – (v) together with
(4.1) If $(a, x) \in T \times M$, $x = B(a, x) + \lambda h(a)$, $F(a, x) = 0$, for some $0 \leqq \lambda \leqq 1$, then $\|x\| \neq R$;
(4.2) If $(a, x) \in T \times M$, $x = B(a, x) + (1 - \lambda)k(a)$, $F(a, x) = 0$, for some $0 \leqq \lambda \leqq 1$, then $\|x\| \neq r$.
Then the following degree identity holds

$$d(I - A, T \times B_R \backslash cl(B_r), 0) = -d(-F(\cdot, k(\cdot)), T, 0),$$

where B_ε denotes the ε-ball about 0 in E.

PROOF. Apply the proof of Theorem 4.1 in [1].

Consider now the real Banach space $E = L^p$, the cone K of all nonnegative functions in E, and the operator F given by (3.2). It can be shown [6] that F is completely continuous on $cl(T) \times E$, so that $d(-F(\cdot, x(\cdot)), T, 0)$ is defined for any continuous x: $cl(T) \to E$ such that $F(a, x(a)) \neq 0$ for $a \in \partial T$. The value of this degree, subject to certain conditions on x, is given next.

LEMMA 4.2. *Suppose x: $cl(T) \to E$ is continuous such that the range of x may be identified in L^p with a subset of* $C([0, 1])$ *and* $x(a)(t) = 0$ *precisely at the a_i's. Then the Brouwer degree*

$$d(-F(\cdot, x(\cdot)), T, 0) = 1.$$

PROOF. Apply the proof of Lemma 5.4 in [1].

While our operator formulation of the nodal problem follows that given in [1], it does differ in one important way. The principal underlying function space is L^p ([0, 1]) instead of $C([0, 1])$. The change is needed for the purpose of calculating a local degree at the fixed points of A. This requires two successive homotopies. First, a standard Schauder homotopy, and secondly, a special or nonstandard homotopy. It is the requirement that A have a Frechét derivative in the first homotopy that drives us (reluctantly) into the L^p setting. Fortunately, this change does not impair the approach followed in [1] to establish the applicability of Lemma 4.1 (all necessary modifications appear in [6]).

The most difficult task in applying the Lemma is the establishment of a priori bounds in (4.1) and (4.2). For this we need the additional assumption

$$\begin{aligned} &f(t, x)/x \geqq u(t)\,|x|^\nu,\ 0 \leqq t \leqq 1 \text{ and } |x| \geqq x_0 \text{ for some} \\ (4.3)\qquad &\nu, x_0 > 0, u \in C([0, 1]), u \geqq 0, \text{ and } u \not\equiv 0 \text{ on any interval} \\ &\text{of length } (3(n+1))^{-1}. \end{aligned}$$

We have then from Lemmas 4.1 and 4.2.

THEOREM 4.3. *For the operator A defined above and subject to the accumulated assumptions on f, there exists* $0 < r < R$ *such that*

$$d(I - A, T \times B_R \backslash cl(B_r), 0) = -1. \tag{4.4}$$

REMARK 4.4. It can be shown using (2.7) and (4.3) that r and R may be chosen so that all solutions of (1.1) – (1.2) are contained by $B_R \backslash cl(B_r)$. Hence, (4.4) is termed a global degree formula. Of course, the immediate degree-theoretic significance of (4.4) is that there are solutions in $B_R \backslash cl(B_r)$.

5. Finiteness of the solution set. We next establish, subject to condition (2.7), that the cardinality of the solution set of (3.3) is finite.

LEMMA 5.1. *Assume that* (3.3) *has among its solutions a sequence $v^m = (a^m, x_m)$ with $v^m \to v$ in $R^n \times L^p$. Then v is a solution of* (3.3).

PROOF. Since B and F are continuous we need to check only that $a \in T$, and this follows from the nonsingularity condition.

LEMMA 5.2. *If $v^m = (a^m, x_m)$ is a sequence of solutions of* (3.3), *then a subsequence exists which converges in L^∞ to a solution of* (3.3).

PROOF. This is a consequence of the compactness argument given for Theorem 9.5 of [1]. By the preceding lemma the limit is a solution of (3.3).

THEOREM 5.3. *If* (3.3) *has more than a finite number of solutions, then there exists a function* y, *satisfying* (1.1) – (1.2), *at which condition* (2.7) *does not hold.*

PROOF. By Lemma 5.2 we may find solutions $v^m = (a^m, x_m)$, $v = (a, x)$ with $v^m \to v$ in $\mathbf{R}^n \times L^p$. Equivalently, for $\bar{a}_i = a^m \cdot c_i$, $0 \leqq i \leqq n$, the functions

$$y_m(t) = \sum_{i=0}^{n}(-1)^i x_m(t)\,\chi_{J_i}(t),\ y(t) = \sum_{i=0}^{n}(-1)^i x(t)\,\chi_{J_i}(t)$$

each satisfy (1.1) – (1.2) and $\|y_m - y\|_{0,1} \to 0$.

Subtracting $y'' + f(s, y) = 0$ from $y_m'' + f(s, y_m) = 0$, adding terms and using Taylor's Theorem gives

$$(y_m - y)'' + f_x(s, y)(y_m - y) = \int_0^1 [f_x(s, y) - f_x(s, y + \lambda(y_m - y))]d\lambda(y_m - y).$$

Dividing by $\|y_m - y\|_{0,1}$ and setting $z_m = (y_m - y)/\|y_m - y\|_{0,1}$ yields

$$z_m'' + f_x(s, y)z_m = \int_0^1 [f_x(s, y) - f_x(s, y + \lambda(y_m - y))]d\lambda\, z_m.$$

The functions z_m are equicontinuous and for each m, $\|z_m\| = 1$. Applying the Ascoli-Arzéla Theorem yields a function z which is the uniform limit of a subsequence of $\{z_m\}$. The function z is nontrivial, vanishes at 0 and 1, and satisfies $z'' + f_x(s, y(s))z = 0$. Hence, (2.7) does not hold at y.

REMARK 5.4. From Theorem 5.3 it follows that condition (2.7) is sufficient for local uniqueness. In particular, the zero function is isolated.

6. The Frechét differentials of B and F. For a given pair $(a, x) \in T \times C([0, 1])$, we now state the Frechét differentials of B and F. Their computation appears in [6] and, although cumbersome, is straight forward. We do need one last condition on f, however, which is that f_x take L^p into L^q as a substitution map.

Set $g^{(i)}(s, x) = f(s, (-1)^i |x|)$. Then the differentiability condition on f shows that

$$g_x^{(i)}(s, x) = \begin{cases} (-1)^i f_x(s, (-1)^i x), & x > 0, \\ (-1)^{i+1} f_x(s, (-1)^{i+1} x), & x < 0. \end{cases}$$

In addition, it will help to have the auxiliary functions

$$D_i(t, s) = \frac{(t - a_{i+1})(s - a_{i+1})}{(a_{i+1} - a_i)^2}, \ (t, s) \in J_1^2,$$

$$D^i(t, s) = \frac{(t - a_{i-1})(s - a_{i-1})}{(a_i - a_{i-1})^2}, \ (t, s) \in J_{i-1}^2,$$

$$\tilde{F}(s, x) = \sum_{i=0}^{n} (-1)^i g^{(i)}(s, x)\, \chi_{J_i}(s),$$

$$\tilde{F}_x(s, x) = \sum_{i=0}^{n} (-1)^i g_x^{(i)}(s, x)\, \chi_{J_i}(s),$$

$$G_a(t, s) = \sum_{i=1}^{n} D_i(t, s)\, \chi_{J_{i-1} \times J_{i-1}}(t, s),$$

$$G^a(t, s) = \sum_{i=1}^{n} D^i(t, s)\, \chi_{J_i \times J_i}(t, s).$$

Then B has a Frechét differential given by

$$\dot{B}(a, x)(h, k) = B_a(h) + B_x(k), \ (h, k) \in \mathbf{R}^n \times L^p, \tag{6.1}$$

for the operators B_a, B_x defined by

$$\begin{aligned} B_a(h)(t) = \int_0^1 [G_a(t, s)\, \tilde{F}(s, x(s)) \sum_{i=1}^{n} h_i\, \chi_{J_i}(s) \\ - G^a(t, s)\, \tilde{F}(s, x(s)) \sum_{i=1}^{n} h_i\, \chi_{J_i}(s)]\, ds, \end{aligned} \tag{6.2}$$

$$B_x(k)(t) = \int_0^1 G(t, s)\, \tilde{F}_x(s, x(s)) k(s) ds. \tag{6.3}$$

For the differential of F, set $\Delta = \min\{a_{i+1} - a_i : 0 \leqq i \leqq n\}$, and let $U = \{\alpha = (a_1, \ldots, a_n) : a_i - \Delta/4 < \alpha_i < a_i + \Delta/4, 1 \leqq i \leqq n\}$, with $\alpha_0 = \alpha_{n+1} = 0$. Define $\phi_i : U \to \mathbf{R}$ by $\phi_i(\alpha) =$

$$(-1)^i \left(\int_{\alpha_{i-1}}^{\alpha_i} \frac{s - \alpha_{i-1}}{\alpha_i - \alpha_{i-1}} g^{(i-1)}(s, x(s)) ds + \int_{\alpha_i}^{\alpha_{i+1}} \frac{\alpha_{i+1} - s}{\alpha_{i+1} - \alpha_i} g^{(i)}(s, x(s)) ds \right),$$

for $1 < i < n$. Denote by $[\phi_i]_j(\alpha)$ the partial derivative of ϕ_i with respect to α_j and take $[\phi_i]_0 = [\phi_n]_{n+1} = 0$.

Define the operators F_a and F_x by

$$F_a(h) \cdot e_i = \sum_{j=i-1}^{i+1} [\phi_i]_j(a) h_j, \ 1 < i < n, \tag{6.4}$$

$$\begin{aligned} F_x(k) \cdot e_i = (-1)^i \Big(\int_{a_{i-1}}^{a_i} \frac{s - a_{i-1}}{a_i - a_{i-1}} g_x^{(i-1)}(s, x(s))\, k(s) ds \\ + \int_{a_i}^{a_{i+1}} \frac{a_{i+1} - s}{a_{i+1} - a_i} g_x^{(i)}(s, x(s))\, k(s) ds \Big), \end{aligned} \tag{6.5}$$

for $1 \leqq i \leqq n$, then F has the Frechét differential

(6.6) $\dot{F}(a, x)/(h, k) = F_a(h) + F_x(k), (h, k) \in R^n \times L^p.$

7. The Schauder homotopy. Let (a, x) be a fixed solution to (3.3), $\dot{B} = \dot{B}(a, x)$, and $\dot{F} = \dot{F}(a, x)$. Given $(h, k) \in \mathbf{R}^n \times L^p$, put $v = (h, k)$ and define $I_2 v = k$.

LEMMA 7.1. *If $k = \dot{B}v$ and $u = \dot{F}v$, then k may be identified in L^p with a function in $C^2(J_i)$, $0 \leqq i \leqq n$, satisfying*

(i) $k(0+) = k(1-) = 0$,
(ii) $k(a_i+) + k(a_i-) = 0$,
(iii) $k'(a_i+) + k'(a_i-) = u_i$,
(iv) $k'' + f_x(t, (-1)^i x(t))k = 0, a_i < t < a_{i+1}$.

PROOF. The equation $k = \dot{B}v$ means that we may take

$$\begin{aligned} k(t) &= (B_x(k) + B_a(h))(t) \\ &= \sum_{i=0}^{n} \int_{a_i}^{a_{i+1}} G(t, s; a_i\, a_{i+1})(-1)^i g_x^{(i)}(s, x(s))k(s)ds\, \chi_{J_i}(t) \\ &\quad + \sum_{i=0}^{n-1} (-1)^i h_{i+1} \int_{a_i}^{a_{i+1}} \frac{(t-a_i)(s-a_i)}{(a_{i+1}-a_i)^2} g^{(i)}(s, x(s))ds\, \chi_{J_i}(t) \\ &\quad - \sum_{i=1}^{n} (-1)^i h_i \int_{a_i}^{a_{i+1}} \frac{(t-a_{i+1})(s-a_{i+1})}{(a_{i+1}-a_i)^2} g^{(i)}(s, x(s))ds\, \chi_{J_i}(t) \end{aligned}$$

Hence, $k(0+) = k(1-) = 0$.
If $1 \leqq i \leqq n$, then

$$k(a_i+) = (-1)^i h_i \int_{a_i}^{a_{i+1}} \frac{s-a_{i+1}}{a_i-a_i} g^{(i)}(s, x(s))ds,$$

$$k(a_i-) = (-1)^{i-1} h_i \int_{a_{i-1}}^{a_i} \frac{a-s_{i-1}}{a_i-a_{i-1}} g^{(i-1)}(s, x(s))ds,$$

and $k(a_i+) + k(a_i-) = -h_i\dot{F}_i(a, x) = 0$.
If $1 \leqq i \leqq n$ and we take $h_0, h_{n+1} = 0$, then

$$(-1)^i k'(a_i+) = \int_{a_i}^{a_{i+1}} \frac{a_{i+1}-s}{a_{i+1}-a_i} g_x^{(i)}(a, x(s))k(s)ds$$

$$+ h_i \int_{a_i}^{a_{i+1}} \frac{a_{i+1}-s}{a_{i+1}-a_i} g^{(i)}(s, x(s))ds + h_{i+1} \int_{a_i}^{a_{i+1}} \frac{s-a_i}{a_{i+1}-a_i} g^{(i)}(s, x(s))ds,$$

$$(-1)^i k'(a_i-) = \int_{a_{i-1}}^{a_i} \frac{s-a_{i-1}}{a_i-a_{i-1}} g_x^{(i)}(s, x(s))k(s)ds$$

$$+ h_i \int_{a_{i-1}}^{a_i} \frac{a_{i-1}-s}{(a_i-a_{i-1})^2} g^{(i-1)}(s, x(s))ds + h_{i-1} \int_{a_{i-1}}^{a_i} \frac{s-a_i}{(a_i-a_{i-1})^2} g^{(i-1)}(s, x(s))ds.$$

Careful comparison with $\dot{F}_i v$ now shows

$$k'(a_i+) + k'(a_i-) = u_i.$$

Finally, differentiating the relation $k = \dot{B}v$ twice with respect to t, $a_i < t < a_{i+1}$, gives (iv).

THEOREM 7.2. *Suppose that (a, x) is a solution of (3.3) at which condition (2.7) holds. If B_ε is an open ball about 0 in $\mathbf{R}^n \times L^p$ and $cl(B_\varepsilon) + (a, x) \subseteq T \times B_R \backslash cl(B_r)$ contains no other solution, then*

$$d(I - A, B_\varepsilon + (a, x), 0) = d(\dot{F}, I_2 - \dot{B}), B_\varepsilon, 0). \tag{7.1}$$

This is a statement of the Schauder Homotopy Theorem for the operator A at the fixed point (a, x). The homotopy in this instance is given by

$$H((h, k), \lambda) = \begin{cases} (h+\lambda^{-1}(F(a+\lambda h, x+\lambda k)+a), \lambda^{-1}(B(a+\lambda h, x+\lambda k)-x)), 0<\lambda<1, \\ (h+F(a, x)(h, x), B(a, x)(h, k)), \lambda = 0 \end{cases}$$

where $(h, k) \in cl(B_\varepsilon)$. The complete continuity of H follows from the compactness and differentiability results given in [6]. The condition that H not have a fixed point on ∂B_ε is satisfied by the assumption that (a, x) is isolated for $0 < \lambda < 1$, and by Lemma 7.1 for $\lambda = 0$. For a proof of the homotopy theorem refer to Krasnosel'skii [9].

Assume condition (2.7) holds at each solution of (3.3). Then Theorem 5.3 implies that the solution set is finite. Enumerate it as $(a^1, x_1), \ldots, (a^m, x_m)$. For each j, choose an $\varepsilon_j > 0$ such that the sets $B_{\varepsilon_j} + (a^j, x_j)$ are pairwise disjoint and satisfy the conditions of Theorem 7.2. Then from the excision and addition theorems for Leray-Schauder degree and the preceding theorem we have

THEOREM 7.3. *Under the above assumptions*

$$d(I-A, T\times B_R\backslash cl(B_r), 0) = \sum_{j=1}^{m} d((-\dot{F}(a^j, x_j), I_2-\dot{B}(a^j, x_j)), B_{\varepsilon_j}, 0).$$

8. Degree formula-local. We now compute $d(I - A, B_\varepsilon + (a, x), 0)$, where (a, x) and B_ε satisfy the assumptions of Theorem 7.2.

Let v, I_2, $\dot{F}$, and $\dot{B}$ be defined as above. Additionally, set $I_1 v = h$. Recall that the variational equation corresponding to

$$y(t) = \sum_{i=0}^{n} (-1)^i x(t)\, \chi_{J_i}(t), \ t \in I,$$

is given by

$$z'' + f_x(t, y(t))z = 0, \ t \in I. \tag{8.1}$$

For $v \in \mathbf{R}^n \times L^p$, $0 < \lambda < 1$, and $1 < j < n$ define $H\colon (\mathbf{R}^n \times L^p) \times [0, 1] \times \{1, 2, \ldots, n\} \to \mathbf{R}^n \times L^p$ by

$$(I_1 H(v, \lambda, j)) \cdot e_i = \begin{cases} h_i - \dot{F}_i v, \ 1 < i < j, \\ \lambda(h_i - F_i v), \ i = j \\ \qquad 0, j < i < n, \end{cases} \tag{8.2}$$

$$I_2 H(v, \lambda, j) = \dot{B}v. \tag{8.3}$$

Then H is a completely continuous operator in v, λ such that $H(v, 1, n) = (h - \dot{F}v, \dot{B}v)$, and $H(v, 0, 1) = (0, \dot{B}v)$.

THEOREM 8.1. *Assume* (8.1) *satisfies the oscillation conditions* (2.5) *and* (2.6), *then*

$$\begin{aligned} d((-\dot{F}, I_2 - \dot{B}), \Omega, 0) &= (-1)^n d((\dot{F}, I_2 - \dot{B}), B_\varepsilon, 0) \\ &= (-1)^n d((I_1, I_2 - \dot{B}), B_\varepsilon, 0). \end{aligned} \tag{8.4}$$

PROOF. The first equality follows from the multiplication formula for degree; for the second, it is enough to check that $h(v, \lambda, j) = v$ only if $v = 0$.

Suppose that $H(v, \lambda, j) = v$ for some v, λ, and j. This means $\dot{B}v = k$ and

$$h_i = \dot{F}_i v = h_i,\ 1 < i < j,$$

$$(h_j - \dot{F}_j v) = h_j,$$

$$0 = h_i,\ j < i < n.$$

So that $k(a_i -) = (-1)^{i-1} h_i \int_{a_{i-1}}^{a_i} (s - a_{i-1})/(a_i - a_{i-1})\, g^{(i-1)}(s, x(s))ds = 0$, $j < i < n$.

If $\lambda = 1$ and $j = n$, then $v = 0$ by condition (2.7).

If $\lambda = 0$, then $h_j = 0$ and $k(a_j -) = 0$. From Lemma 7.1 we have $k(a_i +) + k(a_i -) = 0$, $1 < i < n$. $k'(a_i +) + k'(a_i -) = 0$, $1 < i < j$. It follows from (2.5) that $k \equiv 0$ on $[a_j, 1]$. In addition, we may continue $z(t) = k(t) \sum_{i=0}^{n} (-1)^i x_{J_i}(t)$ over $[0, 1]$ to obtain the solution to (8.1) satisfying $z(0) = 0$, $z'(0) = k'(0+)$. But condition (2.6) requires that $z(a_j) \neq 0$ unless $z \equiv 0$. Thus, $k \equiv 0$ and therefore $v = 0$.

If $\lambda = 1$ and $j < n$, then proceed as above for the case $\lambda = 0$, replacing j by $j + 1$ throughout.

If $0 < \lambda < 1$, then Lemma 7.1 gives

$$k(a_i +) + k(a_i -) = 0,\ 1 \leqq i \leqq n,$$

$$k'(a_i +) + k'(a_i -) = 0,\ 1 \leqq i < j,$$

$$k'(a_j +) + k'(a_j -) = -\lambda h_j/(1 - \lambda).$$

As in the cases $\lambda = 0, 1$, $k \equiv 0$ on $[a_{j+1}, 1]$. Let z_0, z_1 be the solutions to (8.1) satisfying $z_0(a_j) = k(a_j -)$, $z_0'(a_j) = k'(a_j -)$ and $z_1(a_j) = k(a_j +)$, $z_1'(a_j) = k'(a_j +)$. The Wronskian, W, of z_0 and z_1 is independent of t and is given by

$$\begin{aligned} W &= W(t) \\ &= z_0(t)z_1'(t) - z_0'(t)z_1(t) \\ &= z_0(a_j)z_1'(a_j) - z_0'(a_j)z_1(a_j) \\ &= k(a_j -)k'(a_j +) - k'(a_j -)k(a_j +) \\ &= k(a_j -)[k'(a_j +) + k'(a_j -)] \\ &= -k(a_j -)\lambda h_j/(1 - \lambda). \end{aligned}$$

Since $k(a_j -) = (-1)^{j-1} h_j \int_{a_{j-1}}^{a_j} (s - a_{j-1})/(a_j - a_{j-1}) \, g^{(j-1)}(s, x(x))ds$ and $(-1)^{j-1} g^{(j-1)}(s, x(s)) = (-1)^{j-1} f(s,(-1)^{j-1}|x(x)|) \geqq 0$, it follows that the common value of W is negative.

Therefore $z_0(a_{j+1})z_1'(a_{j+1}) = W(a_{j+1}) < 0$ since $z_1(a_{j+1}) = k(a_{j+1}-) = k(a_{j+1} +) + k(a_{j+1} -) = 0$. Hence, $-\operatorname{sgn} z_0(a_{j+1}) = \operatorname{sgn} z_1'(a_{j+1})$.

Condition (2.6) requires that $-\operatorname{sgn} z_0(a_{j+1}) = (-1)^j \operatorname{sgn} z_0'(0)$ and $\operatorname{sgn} z_0(a_j) = (-1)^j \operatorname{sgn} z_0'(0)$, while (2.5) requires that $\operatorname{sgn} z_1(a_j) = \operatorname{sgn} z_1'(a_{j+1}) = -\operatorname{sgn} z_0(a_{j+1})$. This contradicts $z_0(a_j) + z_1(a_j) = 0$ unless $z \equiv 0$.

Thus, $\nu = 0$ and the identity (8.4) is established following an application of the homotopy invariance theorem.

Now let λ be an eigenvalue of the operator $\tilde{A} = (I_1, I_2 - \dot{B})$ and define β by

$$\beta \equiv \sum_{\lambda<1} \beta(\lambda), \tag{8.5}$$

where $\beta(\lambda) = \dim(\bigcup_{n\geqq 0} \operatorname{Ker}(\lambda I - \tilde{A})^n)$.

Because B is completely continuous the differential $\dot{B}$ is compact. The Riesz theory for compact linear operators then implies that the sum in (8.5) is finite.

We now give the main result of this section.

THEOREM 8.2. *Assume that identities* (7.1) *and* (8.4) *hold for* (a, x) *and* B_ε. *Then* $d(I - A, B_\varepsilon + (a, x), 0) = (-1)^\beta$.

PROOF. This follows by applying the Index Theorem of Leray-Schauder to the operator $\tilde{A}$ at 0 relative to B_ε. For a complete statement and proof of the index theorem refer to Krasnosel'skii [9].

9. The parity of β. It will be shown that β and $n + 1$ are of equal parity, that is $(-1)^\beta = (-1)^{n+1}$.

Let $J = [a, b]$ be a finite interval, $f \in C(J)$, and define $L: L^p(J) \to L^p(J)$ by

$$L\ell(t) = \int_a^b G(t, s; a, b)f(s)\ell(s)ds, \ t \in J. \tag{9.1}$$

LEMMA 9.1. *If* $k \in \operatorname{Ker}(I - L) \cap \operatorname{Im}(I - L)$, *then* $k \equiv 0$ *on* J.

PROOF. Because G is continuous on $J \times J$ the operator L is $C(J)$ valued, that is $L\ell$ may be identified in L^p with a continuous function on J. If $k = \ell - L\ell$, then $\ell = k + L\ell = LK + L\ell$ and the functions k and ℓ may be so identified.

For continuous k and ℓ the relations $k = Lk$ and $\ell = k + L\ell$ may be twice differentiated to obtain

(i) $k, \ell \in C(J) \cap C^2(\text{int } J)$,
(ii) $k'' + f(s)k = 0$,
(iii) $\ell'' + f(s)\ell = k''$.

In addition, from the properties of the Green's function, $k(a) = k(b) = 0$ and $L\ell(a) = L\ell(b) = 0$, so that $\ell(a) = \ell(b) = 0$, by virtue of $\ell \equiv k + L\ell$.

Multiplying (ii) by ℓ, (iii) by k, and subtracting gives $-(\ell k' - \ell' k) = kk''$. Then integrating from a to b we obtain

$$\begin{aligned} 0 &= -(\ell k' - \ell' k)(b) + (\ell' k - \ell' k)(a) \\ &= (kk')(b) - (kk')(a) - \int_a^b (k')^2 ds \\ &= -\int_a^b (k')^2 ds. \end{aligned}$$

Hence, $k \equiv 0$ and the proof is complete.

LEMMA 9.2. *Suppose that $f(s)$ in* (9.1) *is nonnegative and $\not\equiv 0$ on J. Let $k = z$ solve the initial value problem*

$$\text{(9.2)} \qquad \begin{cases} z'' + f(s)z = 0, \ s \in J, \\ z(a) = 0, \ z'(a) = 1. \end{cases}$$

Then the zeros of k in $(a, b]$ and the eigenvalues $\lambda \geqq 1$ of the operator L are equal in number.

PROOF. Let $0 \leqq \mu \leqq 1$ and denote by $\phi(s, \mu)$ the solution to

$$\text{(9.3)} \qquad \begin{cases} \phi' = \mu f(s) \sin^2\phi + \cos^2\phi, \ s \in J, \\ \phi(a) = 0. \end{cases}$$

Assume $0 \leqq \mu < \nu \leqq 1$ and $\phi(b, \mu) = \phi(b, \nu)$. From standard comparison theory [2] and the condition $f \geqq 0$ it follows that $\phi(s, \mu) = \phi(s, \nu)$ for $s \in J$. A simple calculation then shows that $\sin \phi(s, \mu) = 0$, whenever $f(s) \neq 0$.

The hypothesis on f allows us to choose $c < d$ such that $f > 0$ on (c, d). Then the preceding remarks imply that $\phi(s, \mu) \equiv \phi(a, \mu) = 0$. This contradicts (9.3).

Therefore, we may conclude that $\mu \to \phi(b, \mu)$ is strictly increasing for $0 \leqq \mu \leqq 1$.

By the properties of the Prüfer Transformation, the number of zeros of k in $(a, b]$ is equal to $[\phi(b, 1)/\pi]$ ($[u]$ = greatest integer $\leqq u$).

If $\lambda \geqq 1$ is an eigenvalue of L, then the problem $y'' + \lambda^{-1}f(s)y = 0$, $y(a) = y(b) = 0$, has a solution $\not\equiv 0$, so $\phi(b, \lambda^{-1}) = j\pi$ for some integer j. But $\mu \to \phi(b, \mu)$ is increasing on $[0, 1]$ so λ^{-1} can take on exactly $[\phi(b, 1)/\pi] - [\phi(b, 0)/\pi]$ values. However, $\phi(b, 0) = \tan^{-1}(b - a) \in (0, \pi/2)$. Therefore card(eigenvalues $\lambda \geqq 1$ of L) $= [\phi(b, 1)/\pi] - [\phi(b, 0)/\pi] = [\phi(b, 1)/\pi]$ = card(zeros of k in $(a, b]$). This proves the lemma.

Let $\lambda > 1$ be an eigenvalue of $\tilde{A}$ and put $\mu = \lambda^{-1}$. If $(h, \ell) \in \operatorname{Ker}((I - \mu\tilde{A})^2)$, then $h = 0$ and $\ell \in \operatorname{Ker}((I_2 - \mu B_x)^2)$. Assume $\ell \notin \operatorname{Ker}(I_2 - \mu B_x)$, then $k \equiv \ell - B_x \ell \in \operatorname{Ker}(I_2 - \mu B_x) \cap \operatorname{Im}(I_2 - \mu B_x)$ and $k \not\equiv 0$ on some J_i. Lemma 9.1 applies with $J = cl(J_i)$ and $f(s) = \mu f_x(s, (-1)^i x(s))$ to obtain $k \equiv 0$ on J, a contradiction. Hence, the ascent of $\mathrm{I} - \mu\tilde{A}$ is one and the multiplicity of λ is given by

$$\begin{aligned} \beta(\lambda) &= \dim[\ker(I - \mu\tilde{A})] \\ &= \dim[\ker(I_2 - \mu B_x)]. \end{aligned} \tag{9.4}$$

Enumerate the eigenvalues $\lambda > 1$ of $\tilde{A}$ as $\lambda_1, \lambda_2, \ldots \lambda_p$ Let. $\mu_r = \lambda_r^{-1}$ and define the sets

$$F_q = \{r\colon \exists\, k \in \operatorname{Ker}(I_2 - \mu B_x) \text{ with } k \not\equiv 0 \text{ on } J_q\},$$
$$E_r = \{q\colon \exists\, k \in \operatorname{Ker}(I_2 - \mu B_x) \text{ with } k \not\equiv 0 \text{ on } J_q\}.$$

LEMMA 9.3. $\bigcup_{q=0}^{n} \{q\} \times F_q = \bigcup_{r=1}^{q} E_r \times \{r\}$.

LEMMA 9.4. $\beta(\lambda_r) = \operatorname{card}(E_r \times \{r\})$.

PROOF. For each $q \in E_r$ take $k \in \operatorname{Ker}(I_2 - \mu B_x)$ with $k \not\equiv 0$ on J_q and set $k_q(t) = k(t)\,\chi_{J_q}(t)$, $t \in I$. Then the functions k_q are linearly independent and span $\operatorname{Ker}(I_2 - \mu B_x)$. Hence, E_r is in one to one correspondence with a basis for $\operatorname{Ker}(I - \mu\tilde{A})$ which, in view of (9.4), completes the proof.

We may now prove the parity theorem.

THEOREM 9.5. If (8.1) satisfies condition (2.5) and $f_x(s, y(s)) \geqq 0$, $s \in I$, and $\not\equiv 0$ on each subinterval J_i, then each eigenvalue $\lambda > 1$ of the operator $\tilde{A}$ has multiplicity, $\beta(\lambda)$, equal to $\dim[\operatorname{Ker}(I_2 - \lambda^{-1} B_x)]$ and the integer β defined by (8.5) satisfies $(-1)^{\beta} = (-1)^{n+1}$.

PROOF. The first statement is taken from (9.4).

Let $0 \leqq q \leqq n$ and apply Lemma 9.2 to the equation $z'' + f_x(s, y(s))z = 0$, $s \in cl(J_q)$. Since (8.1) satisfies (2.5) the function k of Lemma 9.2 vanishes an odd number of times in $(a_q, a_{q+1}]$, say $2n_q + 1$. Hence, we may compute as follows $\beta = \sum_{r=1}^{p} \beta(\lambda_r) = \sum_{r=1}^{p} \operatorname{card}(E_r \times \{r\}) =$

$\sum_{q=0}^{n}\text{card}(\{q\} \times F_q) = \sum_{q=0}^{n}(2n_q + 1) = (\text{even number}) + (n + 1)$. So that finally, $(-1)^\beta = (-1)^{n+1}$.

We are now ready to state the uniqueness result, but first recall the hypotheses on f. These are: items (i), (ii), (iii) of §3, condition (4.3), and the assumption that f_x take L^p into L^q as a substitution map.

THEOREM 9.6. *If the variational equation* (1.3) *corresponding to a given solution of* (1.1)–(1.2) *satisfies the oscillation conditions* (2.5) *and* (2.6), *then the nodal problem has exactly one solution.*

PROOF. Coupling together the preceding theorems, we get

$$\begin{aligned} 1 &= -d(I - A, TxB_R \backslash cl(B_r), 0) \\ &= -\sum_{j=1}^{m} d(I - \dot{A}(a^j, x_j), B_{\varepsilon_j}, 0) \\ &= -\sum_{j=1}^{m} (-1)^n d(I_1, I_2 - \dot{B}(a^j, x_j), B_{\varepsilon_j}, 0) \\ &= -\sum_{j=1}^{m} (-1) \\ &= m. \end{aligned}$$

10. Application to the Emden-Fowler equation. We conclude with an application to equations of the form

$$y'' + u(t)y|y|^\nu = 0, \quad a < t < b, \tag{10.1}$$

where $0 < a < b$, $u: [a, b] \to (0, \infty)$ is continuously differentiable, and ν is a fixed positive number.

For $f(t, x) = u(t)x|x|^\nu$, the differentiability condition (2.4) holds and the variational equation corresponding to a solution y is given by

$$z'' + (1 + \nu)u(t)|y(t)|^\nu z = 0, \quad a < t < b. \tag{10.2}$$

If we take $p = 2 + \nu$ and $q = (2 + \nu)/(1 + \nu)$, then f and f_x act as substitution operators from L^p into L^q.

Finally, the lower bound condition (4.3) is satisfied since $u > 0$.

It follows from Theorem 4.3 that the nodal problem for (10.1) has at least one solution. Uniqueness will follow from Theorem 9.1 provided that the oscillation conditions (2.5) and (2.6), hold at an arbitrary solution (the extension from [0, 1] to arbitrary intervals is easy).

Expanding on a technique due to Kolodner [**8**], Coffman [**3**] is able to prove the following.

Assume that $u' < 0$ and the function h defined by $h(t) = t + 2u(t)/u'(t)$ is nonpositive and monotone-nonincreasing. Then the conditions (2.5) and (2.6) hold at any solution to the nodal problem.

If $u(t) = t^r$, then the conditions of Coffman reduce to $2 \leqq r < 0$. However, this special case may be handled for arbitrary r by employing a different attack. This will now be demonstrated.

Assume that y is a solution of the nodal problem for (10.1) and that z satisfies the corresponding variational equation with the initial data $z(a) = 0$, $z'(a) = 1$. Change variables [5]

$$(10.3) \qquad y = t^m v, \; x = \ln(t), \; m = -(r + 2)/\nu,$$

to transform (10.1) into the autonomous equation

$$(10.4) \qquad v + (2m - 1)\dot{v} + (|v|^\nu - m(1 - m))v = 0,$$

where dot denotes differentiation with respect to s. The dsme change of variables transforms (10.2) into the variational equation for (10.4) given by

$$(10.5) \qquad \ddot{w} + (2m - 1)\dot{w} + ((1 + \nu)|v(x)|^\nu - m(1 - m))w = 0.$$

It is important to note that because (10.4) is autonomous, $\dot{v}$ satisfies (10.5).

LEMMA 10.1. *If between consecutive zeros of the function v defined by* (10.3) *$\dot{v}$ vanishes exactly once, then on (a, b) the zeros of z and y separate each other, i.e., conditions* (2.5) *and* (2.6) *hold at y.*

PROOF. Define z_0 by $z_0(t) = t^r\dot{v}(\ln(t))$, $a \leqq t \leqq b$. Then z_0 satisfies (10.2) and because the transformation (10.3) preserves the relative order of the zeros of v and $\dot{v}$, it follows that on $[a, b]$ the zeros of z_0 are separated by those of y.

Let ϕ_1 (resp. ϕ_2) be the solution to the Prüfer equation for (10.1) (resp. (10.2)) given by

$$\phi' = \cos^2\phi + ct^r|y(t)|^\nu \sin^2\phi, \; a \leqq t \leqq b,$$
$$\phi(a) = 0,$$

where $c = 1$ (resp. $c = 1 + \nu$). Since $t^r|y(t)|^\nu < (1 + \nu)t^r|y(t)|^\nu$, except at the zeros of y, it follows that $\phi_1(t) < \phi_2(t)$ for $a < t \leqq b$.

Let c_i, d_i, and e_i be the i-th zeros in $(a, b]$ of z_0, z, and y, respectively. Set $d_0 = e_0 = a$ and let $0 \leqq i \leqq n$ be fixed. The opening remarks concerning z_0 and y imply $e_i < c_{i+1} < e_{i+1}$. The inequality $\phi_2(d_{i+1}) > \phi_1(d_{i+1})$ implies that $d_{i+1} < e_{i+1}$. And from the Sturm Separation Theorem it follows that $d_i < c_{i+1} < d_{i+1}$. Finally, since z_0 does not vanish on $(d_{n+1}, b]$, neither does z.

LEMMA 10.2. *Assume that F: $[0, \infty) \to \mathbf{R}$ is continuous, has at most one zero, and is eventually positive. If the equation*

$$(10.9) \qquad v + k\dot{v} + F(|v|)v = 0,$$

k a constant, has unique solutions to initial value problems, then between consecutive zeros of a solution v of (10.8), *$\dot{v}$ vanishes exactly once.*

PROOF. If f has no positive zero, then the conclusion may be reached by applying the usual Prüfer methods. The correct Prüfer equation is given by $\phi' = e^{-ks}\cos^2\phi + e^{ks}F(|v(s)|)\sin^2\phi$.

Thus, we may assume that F vanishes at $v_0 > 0$. Let c and d be consecutive zeros of v, and with no loss of generality assume that $v < 0$ on (c, d). If $\dot{v}(s) = 0$, then $v(s) \neq -v_0$ since this would imply that $v \equiv -v_0$. Hence, either $-v_0 < v(s) < 0$ or $v(s) < -v_0$. In the first case, $\ddot{v}(s) < 0$ and s is a local maximum. In the second, $\ddot{v}(s) > 0$ and s is a local minimum.

Thus, if $\dot{v}$ vanished more than once on (c, d), then on the same interval v would cross the line $-v_0$ at least four times. Suppose it did and let the last and third to the last crossings be at d_1 and c_1, respectively. If $k \geqq 0$, then multiplying both sides of (19.8) by $\dot{v}$ and integrating from c_1 to d_1 gives $0 \geqq \int_{c_1}^{d_1} \ddot{v}\,\dot{v}\,ds$. Therefore, $\dot{v}(c_1) \geqq \dot{v}(d_1) > 0$.

Set $u(s) = v(s + c_1 - d_1)$. Then y satisfies (10.9), $u(d_1) = v(c_1) = -v_0$, and $\dot{u}(d_1) = \dot{v}(c_1) \geqq \dot{v}(d_1)$. Now consider $(v, \dot{v})$ and $(u, \dot{u})$ as solutions of the first order autonomous system $\dot{x} = y$, $\dot{y} = -ky - F(|x|)x$. Since on $[d_1, d]$ v vanishes and $\dot{v}$ does not, the trajectory of $(v, \dot{v})$ crosses the y-axis without first crossing the x-axis. Because of the assumption of uniqueness for initial value problems and the data at d_1, the trajectory of $(u, \dot{u})$ must do the same. But this is impossible since $\dot{u}$ vanishes in advance of u.

If $k < 0$, set $w(s) = v(-s)$ and note that w satisfies the equation $\ddot{w} - k\dot{w} + F(|w|)w = 0$. Now repeat the argument replacing v by w throughout.

These lemmas demonstrate that if y is a solution of the nodal problem for (10.1) with $u(t) = t^r$, then the oscillation conditions (2.5) and (2.6) hold at y. Hence, the problem has exactly one solution.

REMARK 10.5. In the case that $u(t) = e^{rt}$, $-\infty < r < \infty$, (10.1) is transformed by $Y(t) = e^{rt/\nu}\,y(t)$ into $y'' + (-r/\nu)Y' + Y|Y|^{\nu} = 0$. Lemmas 10.2 and 10.3 then show that (2.5) and (2.6) hold at any solution of the nodal problem.

REMARK 10.6. If $u(t) = t^r$ for $r < 0$, then f is singular at $t = 0$. For this special case, existence and uniqueness for nodal problems with left endpoint equal to zero may be treated by first examining the positive case ($n = 0$) and then employing the appropriate phase-plane methods and asymptotic formulas. One finds that solutions (i) exist and are unique for $-2 < r < 0$, (ii) exist and are unique (if we require that $y'(+0) < \infty$) for $-2 - \nu/2 < r \leqq -2$, and (iii) do not exist for $r \leqq -2 - \nu/2$.

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