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ON SOME PROPERTIES OF DOMAINS OF INTEGRAL OPERATORS

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SUMMARY. A construction of enlarging solid topological spaces of measurable functions is discussed. It is shown that both the domain and the extended domain of an integral operator are invariant under this construction.

1. Introduction. Let X be a measure space, $L^0 = L^0(X)$ be the vector space of measurable finite a.e. scalar-valued functions on X and let $A \subset L^0$ be a topological vector space. Denote $A^\# = \{u \in L^0; \{v \in A; |v(x)| \leqq |u(x)| \text{ a.e.}\} \text{ is bounded in } A\}$. If A is solid then $A \subset A^\#$, otherwise it may happen that $A^\# = \{0\}$. If A is a solid normed space then $A^\#$ is a space defined by a function norm in the sense of [1].

In this paper we study the "functor"$^\#$ as applied to the domain $\mathscr{D}_K$ and the extended domain $\tilde{\mathscr{D}}_K$ of an integral operator K. The conclusion is that both domains are preserved by $^\#$, (Theorem 4.1 and theorem 4.2) in particular if K is defined on A then it is also defined on $A^\#$ and if K extends by continuity to a solid topological vector space A then it also extends by continuity to $A^\#$.

As a preliminary to theorem 4.2 we prove theorem 2.1 which is a new characterization of the space $\tilde{\mathscr{D}}_K$.

Example (4.5) seems to show that Theorem 4.2 is nontrivial; we do not know a proof of (4.5) which would not involve in one way or another the idea of that theorem.

The reference [2] is the background of all the results outlined in Section 2.

2. Notation and prelimaries. We assume that X is σ-finite, by subsets of X we mean measurable subsets, the measure on X we denote by dx and the measure of a set $E \subset X$ we note by $|E|$.

By a metric ρ we shall mean a translation invariant metric and we shall write $\rho(u) = \rho(u, 0)$.

The space L^0 of all measurable, scalar valued, finite a.e. functions on X

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has a natural topology of convergence in measure on all subsets of finite measure. This is a complete vector topology which can be given, e.g., by the metric

$$\rho(u) = \rho_X(u) = \int |u|\,(1 + |u|)^{-1}\,\phi\,dx \tag{2.1}$$

where $\phi \in L^0$, $\phi > 0$, $\int_X \phi dx = 1$.

Above and in what follows we write $u = u(x)$, $|u| = |u(x)|$, and by $u < v$ or $u \leqq v$ we mean the inequalities a.e.

For a subset $A \subset L^0$ and $u \in L^0$ we let

$$A_u = A_{|u|} = \{v \in A;\ |v| \leqq u\}. \tag{2.2}$$

The set A is *solid* if for every $u \in A$ we have $L^0_u \subset A$.

A topological (additive) subgroup of L^0 (in particular a topological vector subspace) is *solid* if its topology can be defined by a base of solid neighborhoods of 0.

A metric ρ on a subspace $A \subset L^0$ is solid if $\rho(v) \leqq \rho(u)$ whenever u, $v \in A$, $|u| \geqq |v|$.

It is known that solid metrizable vector subspaces of L^0 are continuously contained in L^0 (see, e.g., [3]); whether this is true without the hypothesis of metrizability seems to be an open question.

If $A \subset L^0$ and $E \subset X$ then E is an *unfriendly* set for A if $u|_E = 0$ for all $u \in A$, $u|_E$ denoting the restriction of u to E.

Recall (see [2]) that if $A \subset L^0$ is a vector subspace of L^0 (not necessarily solid) then there exists a maximal, unique up to sets of measure 0, unfriendly subset E_A for A, also there exist sequences $X_n \uparrow X \backslash E_A$, $v_n \in A$, such that $|v_n| > 0$ on X_n. If A is solid then for a choice of X_n as above one can take $v_n = \chi_{X_n}$ where χ_E denotes the characteristic function of E.

For a solid metrizable subspace $A \subset L_0$ we denote by A^c the subspace of "norm continuous" functions in A, i.e., $A^c = \{u \in A;\ \chi_{E_n} u \to_A 0$ for every sequence $E_n \subset X$ s.t. $E_n \downarrow \varnothing\}$ where $E_n \downarrow \varnothing$ means that $\{E_n\}$ is decreasing and $|\bigcap E_n| = 0$.

Note that the definition of A^c is meaningful if A is any topological subspace of L^0 containing 0. Of course it is possible that $A^c = \{0\}$.

We recall some facts about integral operators (see [2], [3]).

Let Y be another σ-finite measure space; in the cases not leading to confusion we will write $L^0 = L^0(Y)$.

A kernel is a function $k \in L^0(X \times Y)$ and the corresponding operator (transformation) K is given by

$$\begin{aligned} (Ku)(x) &= \int_Y k(x, y)u(y)dy, \\ \mathscr{D}_K &= \{u \in L^0;\ |K|\,|u|(x) = \int_Y |k(x, y)|\,|u(y)|\,dy < \infty \text{ a.e.}\}. \end{aligned} \tag{2.3}$$

K is a linear operator from $\mathscr{D}_K \subset L^0(Y)$ into $L^0(X)$.

On $\mathscr{D}_K$ there is a natural solid metric

$$\rho_K(u) = \rho_Y(u) + \rho_X(|K|\,|u|) \tag{2.4}$$

where ρ_Y, ρ_X are as in (2.1).

With the metric ρ_K, $\mathscr{D}_K$ is a complete solid vector space and, by the closed graph theorem, if $A \subset_c L^0(Y)$ is a complete metrizable vector subspace of L^0 such that $A \subset \mathscr{D}_K$ then $K|_A\colon A \to L^0(X)$ is continuous.

By the dominated convergence theorem one easily verifies the following.

PROPOSITION 2.1. $\mathscr{D}_K^c = \mathscr{D}_K$.

We assume in what follows that $\mathscr{D}_K$ has no unfriendly sets.

Let $u \in L^0 = L^0(Y)$ and define

$$\tilde\rho_K(u) = \rho_Y(u) + d_K(u),\ d_K(u) = \sup\{\rho_X(Kv);\ v \in (\mathscr{D}_K)_u\} \tag{2.5}$$

where as in (2.2) $(\mathscr{D}_K)_u = \{v \in \mathscr{D}_K;\ |v| \leqq |u|\}$ and ρ_Y, ρ_X are as in (2.1), thus $\rho_Y \leqq 1$, $\rho_X \leqq 1$.

$\tilde\rho_K$ is a complete solid metric on L^0, with this metric L^0 is a metric group (but in general not a metric vector space) which we denote by $\tilde L^0 = \tilde L^0_K$.

The closure $\tilde{\mathscr{D}}_K$ of $\mathscr{D}_K$ in $\tilde L^0$ is a solid metric vector space, this is the *extended domain* of K.

$\tilde{\mathscr{D}}_K$ has the following maximality property.

(2.6) (a) There is a (unique) continuous operator $\tilde K\colon \tilde{\mathscr{D}}_K \to L^0(X)$ such that $\tilde K|_{\mathscr{D}_K} = K$.

(2.7) (b) If A is a solid topological vector subspace of L^0, if $\mathscr{D}_K \cap A$ is dense in A and if there is a continuous extension K_A of K to A then $A \subset \tilde{\mathscr{D}}_K$ and $K_A = \tilde K|_A$.

PROPOSITION 2.2. $\tilde{\mathscr{D}}_K^c = \tilde{\mathscr{D}}_K$.

PROOF. This is an immediate consequence of Prop. 2.1 and of the following general statement. If $A \subset B$ are two solid metric subspaces of L^0 with dense and continuous inclusion and if $A = A^c$ then $B = B^c$. Indeed, if ρ_A, ρ_B are solid metrics defining the topologies on A and B, if $u \in B$ and if $E_n \downarrow \emptyset$, then for every $\varepsilon > 0$ choose $v \in A$ such that $\rho_B(u - v) < \varepsilon/2$ and write

$$\rho_B(\chi_{E_n}u) \leqq \rho_B(\chi_{E_n}v) + \rho_B(\chi_{E_n}(u-v)) \leqq \rho_B(\chi_{E_n}v) + \rho_B(u-v) < \rho_B(\chi_{E_n}v) + \varepsilon/2.$$

Since $\rho_A(\chi_{E_n}v) \to 0$ and the inclusion $A \subset B$ is continuous, it follows that $\rho_B(\chi_{E_n}v) < \varepsilon/2$ for all sufficiently large n.

THEOREM 2.1. $\tilde{\mathscr{D}}_K = (\tilde L^0_K)^c$.

PROOF. By Prop. 2.2 we have the inclusion $\tilde{\mathscr{D}}_K \subset (\tilde L^0)^c$. Suppose that $u \in (\tilde L^0)^c$, we can assume that $u \geqq 0$. Since $\mathscr{D}_K$ has no unfriendly sets there is a sequence $Y_n \uparrow Y$ such that $\chi_{Y_n} \in \mathscr{D}_K$. Let $Y_n' = \{y \in Y_n; u(y) \leqq n\}$, then $Y_n' \uparrow Y$ and $v_n = \chi_{Y_n'} u \in \mathscr{D}_K$. We have

$$\tilde{\rho}_K(u - v_n) = \tilde{\rho}_K(\chi_{Y \setminus Y_n'} u) \to 0$$

since $Y \setminus Y_n' \downarrow \varnothing$ and $u \in (L^0)^c$, and it follows that $u \in \tilde{\mathscr{D}}_K$.

REMARK. One could also consider the space $(\tilde{L}^0)^v = \{u \in L^0; \tilde{\rho}_K(n^{-1}u) \to_{n \to \infty} 0\}$. We don't know whether or not $\tilde{\mathscr{D}}_K = (\tilde{L}^0)^v$.

PROPOSITION 2.3. *If $A \subset \mathscr{D}_K$ is a solid vector space without unfriendly sets then A is dense in $\mathscr{D}_K$ and a fortiori in $\tilde{\mathscr{D}}_K$.*

PROOF. Let $Y_n \uparrow Y$ be such that $\chi_{Y_n} \in A$ and let $u \in \mathscr{D}_K$. Define Y_n' as in the preceding proof with u replaced by $|u|$. Then $\chi_{Y_n'} u \in A$ and $\chi_{Y_n'} u \to_{\mathscr{D}_K} u$ by the dominated convergence theorem.

We have the following necessary condition for a function $u \in L^0$ to belong to $\tilde{\mathscr{D}}_K$(see [2]).

PROPOSITION 2.4. *Let $u \in \tilde{\mathscr{D}}_K$, let $\{E_n\}$ be a partition of Y and let $u_n \in \mathscr{D}_K$ be any sequence such that $|u_n| \leqq \chi_{E_n} |u|$. Then $\sum |Ku_n(x)|^2 < \infty$* a.e.

Except for some special examples we know of no class of kernels K for which the above condition would be also sufficient for u to belong to $\tilde{\mathscr{D}}_K$.

3. Some properties of #. It will be convenient in the next two sections to use the convention that $0/0 = 0$. We recall the definition from §1.

If A is a topological vector subspace of $L^0 = L^0(X)$ then

(3.1) $$A^\# = \{u \in L^0; A_u \text{ is bounded in } A\},$$

where A_u is given by (2.2) and bounded means bounded in the topology of A.

It is easy to find examples of spaces A where $A^\# = \{0\}$, however

(3.2) $$\text{if } A \text{ is solid, then } A \subset A^\#.$$

If $\mathscr{U}$ is a base of neighborhoods of 0 defining the topology in A then a natural topology on $A^\#$ is defined by

(3.3) $$\mathscr{U}^\# = \{U^\#: U \in \mathscr{U}\} \text{ where } U^\# = \{u \in L^0; A_u \subset U\}.$$

It is immediately verified that if A is solid and if $\mathscr{U}$ is a basis of solid neighborhoods of 0 then $\mathscr{U}^\#$ is a basis of solid neighborhoods of 0 defining a vector topology on $A^\#$. If A is a Hausdorff space and has no unfriendly sets, then $A^\#$ is a Hausdorff space; in fact if $u \in \bigcap \{U^\#; U \in \mathscr{U}\}$ and $v \in A_u$ then $v \in \bigcap \{U; U \in \mathscr{U}\} = \{0\}$. Let $X_n \uparrow X$ be such that $\chi_{X_n} \in A$ and let $X_n' = \{x \in X_n; |u(x)| \leqq n\}$. Then $\chi_{X_n'} u \in A_u$ and $\chi_{X_n'} u = 0$. Since $X_n' \uparrow X$ it follows that $u = 0$.

From now on we shall deal only with solid Hausdorff vector subspaces of L^0. If U is a solid neighborhood of 0 in A then $U^\# \cap A = U$ and it follows that the original topology on A and the one induced by $A^\#$ coincide.

If the topology of A is given by a (not necessarily solid) metric ρ then $A^\#$ is a metric space with the solid metric

$$\rho^\#(u) = \sup\{\rho(v);\ v \in A_u\},\ u \in A^\#. \tag{3.4}$$

On A the metrics ρ and $\rho^\#$ are equivalent.

In the case when ρ is a norm, $\rho^\#$ is a (possibly extended valued) function norm in L^0 in the sense of [1].

If A has the weak Fatou property:

$$\begin{aligned}&(\exists u_n \in A_u,\ u_n \to u \text{ a.e.}) \Rightarrow u \in A, \text{ then } A^\# = A\\ &(A^\#)^\# = A^\#, \text{ in particular } \rho^\# = \rho^{\#\#} \text{ if } A \text{ is a metric space.}\end{aligned} \tag{3.5}$$

This follows from the remark that $A_u = \bigcup\{A_v\colon v \in A_u^\#\}$.

PROPOSITION 3.1. *If A, B are topological solid vector subspaces of L^0, if $A \subset B$ with a continuous dense inclusion and if B is metrizable, then $A^\# \subset B^\#$.*

PROOF. Suppose that $u \in A^\#$ but $u \notin B^\#$. Then B_u is unbounded and hence contains an unbounded sequence, say $\{v_n\}$. Since A is dense in B we can find $u_n \in A$ such that $u_n - v_n \to_B 0$. Replacing if necessary u_n by $\min(|u_n|, |v_n|)\,|u_n|^{-1}u_n$ we may assume that $|u_n| \leqq |v_n|$, hence $u_n \in A_u$. It follows that $\{u_n\}$ in bounded in A and $u_n - v_n \to_B 0$ implies that u_n is unbounded in B, which contradicts the continuity of the inclusion $A \subset B$.

The following example (see [1]) shows that without additional hypotheses $A \subset B$ does not imply $A^\# \subset B^\#$.

Let

$$B = \{u \in L^0(\mathbf{R}^1);\ \int_{-\infty}^{\infty} (1 + x^2)^{-2}\,|u|\,(x)dx + \limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{T} |u(x)|dx = \|u\|_B < \infty\}$$

and $A = \{u \in B;\ \limsup\,(1/T)\int_{-T}^{T} |u(x)|dx = 0\}$. B is a normed space and A is a subspace of B with the induced norm. It is easy to see that $u = x^2$ is in $A^\#$ but not in $B^\#$.

It would be of interest to see which properties of A are inherited by $A^\#$. The example of c_0 shows that $A = A^c$ does not imply that $A^\# = (A^\#)^c$ and we don't know a condition on A (and X) which would make this implication valid.

PROPOSITION 3.2. *If A is a complete metric solid vector subspace of L^0 then $A^\#$ is complete.*

PROOF. Suppose that $\{u_n\}$ is a Cauchy sequence in $A^\#$. Then, by the continuity of inclusion $A^\# \subset L^0$, there exists $u \in L^0$ such that $u_n \to_{L^0} u$. We will show that $u \in A^\#$ and that $u_n \to_{A^\#} u$. By choosing if necessary a subsequence we can assume that $\sum |u_{n+1} - u_n| < \infty$ a.e. and that $\sum \rho^\#(u_{n+1} - u_n) < \infty$ where $\rho^\#$ is a metric on $A^\#$ derived from a complete metric ρ on A by (3.4). We have to show that for every $\varepsilon > 0$ there is $\lambda > 0$ such that $\rho(\lambda v) < \varepsilon$ for all $v \in A_u$. To this effect choose n such that $\sum_{\ell=n}^\infty \rho^\#(u_{\ell+1} - u_\ell) < \varepsilon/2$ and $\lambda \in (0, 1]$ such that $\rho^\#(\lambda u_n) < \varepsilon/2$ - this is possible since $u_n \in A^\#$. If $v \in A_u$ then $v = \sum_{\ell=n-1}^\infty v_\ell$ with $v_{n-1} = w_n^{-1} |u_n| v$, $v_\ell = w_n^{-1} |u_{\ell+1} - u_\ell| v$ for $\ell \geqq n$, where $w_n = |u_n| + \sum_{\ell=n}^\infty |u_{\ell+1} - u_\ell| \geqq |u|$. The series $\sum_{\ell=n-1}^\infty v_\ell$ is clearly convergent in L^0, since $|v_\ell| \leqq |u_{\ell+1} - u_\ell|$ it is also convergent in A (at this point the completeness of A is used) and we can write

$$\rho(\lambda v) \leqq \rho(\lambda v_n) + \rho(\lambda \sum_{\ell=1}^\infty v_\ell) \leqq \rho^\#(\lambda u_n)$$

$$+ \rho^\#(\lambda \sum_{\ell=n}^\infty v_\ell) < \varepsilon/2 + \sum_{\ell=n}^\infty \rho^\#(u_{\ell+1} - u_\ell) < \varepsilon,$$

and $u \in A^\#$.

The same argument using the inequality $|u - u_n| \leqq \sum_{\ell=u}^\infty |u_{\ell+1} - u_\ell|$ shows that $\rho^\#(u - u_n) \to 0$.

It is likely that $A^\#$ may be complete without the hypothesis that A is complete.

4. Applications to domains of integral operators.

THEOREM 4.1. *Let K be an integral operator. Then $\mathcal{D}_K = \mathcal{D}_K^\#$.*

PROOF. Suppose that $u \in \mathcal{D}_K^\#$, we may assume that $u \geqq 0$; then (see (2.4)) the set $\{|K|\, |v|;\ v \in \mathcal{D}_K)_u\}$ is bounded in L^0. Let $Y_n \uparrow Y$ be such that $\chi_{Y_n} \in \mathcal{D}_K$ and let $Y_n' = \{y \in Y_n;\ u(y) \leqq n\}$. Then $Y_n' \uparrow Y$ and $u_n = \chi_{Y_n'} u \uparrow u$, in particular $u_n \in (\mathcal{D}_K)_u$. By the known criterion of boundedness in $L^0(X)$ we conclude that for every $E \subset X$ with $|E| < \infty$ and for every $\varepsilon > 0$ there is an $M > 0$ such that for every n $|\{x \in E;\ |K|u_n(x) > M\}| < \varepsilon$. Since the sets $E_n = \{x \in E;\ |K|u_n(x) > M\}$ are increasing $|\bigcup E_n| \leqq \varepsilon$ and $|K|u_m(x) \leqq M$ for all m outside of the set $\bigcup E_n$. It follows from the Fatou's lemma that $|K|\, u < \infty$ a.e. and $u \in \mathcal{D}_K$.

Propositions 2.3 and 3.1 imply now the following corollary.

(4.1) *If $A \subset_c \mathcal{D}_K$ is a solid topological vector space without unfriendly sets then $A^\# \subset \mathcal{D}_K$. If A is a solid complete metric space, then the con-*

tinuous inclusion $A \subset_c \mathscr{D}_K$ *is a consequence of the algebraic inclusion* $A \subset \mathscr{D}_K$.

(4.2) EXAMPLE. Consider a sequence $g_n \in L^0(X)$, $g_n \geqq 0$. Suppose that for every sequence $\xi_n \geqq 0$ with $\lim_{n\to\infty} \xi_n = 0$ we have $\sum_{n=1}^{\infty} \xi_n g_n(x) < \infty$ a.e. Then $\sum_{n=1}^{\infty} g_n(x) < \infty$ a.e.

PROOF. Let $Y = \mathbf{N} = \{1, 2, \ldots\}$ with the natural measure $|\{n\}| = 1$ and define the kernel $k(x, n) = g_n(x)$. The hypothesis means that $c_0 \subset \mathscr{D}_K$ and by (4.1) $\ell^\infty = c_0^\# \subset \mathscr{D}_K$ which is the assertion.

Concerning $\tilde{\mathscr{D}}_K^\#$ we need the following easy observation.

$$\tilde{\rho}_K^\# = \tilde{\rho}_K|_{\mathscr{D}_K^\#}. \tag{4.3}$$

THEOREM 4.2. *Let K be an integral operator. Then* $\tilde{\mathscr{D}}_K^\# = \tilde{\mathscr{D}}_K$.

PROOF. By Theorem 2.1 $\tilde{\mathscr{D}}_K = (\tilde{L}_K^0)^c \supset (\tilde{\mathscr{D}}_K^\#)^c$, and the statement results from the following proposition.

PROPOSITION 4.1. *If K is an integral operator then* $(\tilde{\mathscr{D}}_K^\#)^c = \tilde{\mathscr{D}}_K^\#$.

PROOF. Let $u \in \tilde{\mathscr{D}}_K^\#$, $u \notin (\tilde{\mathscr{D}}_K^\#)^c$. There is then a sequence $E_n \subset Y$, $E_n \downarrow \emptyset$ such that $\tilde{\rho}_K^\#(\chi_{E_n} u) = \tilde{\rho}_K(\chi_{E_n} u) > \alpha$ for all n and some $\alpha > 0$. Since $\rho_Y(\chi_{E_n} u) \to 0$ it follows then from (2.5) that there exists a sequence $v_n \in \mathscr{D}_K$, $|v_n| \leqq \chi_{E_n} u$ such that $\rho_X(Kv_n) > \alpha$ for all sufficiently large n. We use now Prop. 2.1 to conclude that $\rho_X(K\chi_{E_n \setminus E_m} v_n) > \alpha$ for fixed n and all sufficiently large m and to replace $\{E_n\}$, $\{v_n\}$ be sequences with the property that $|v_n| \leqq \chi_{E_n \setminus E_{n+1}} |u|$ and $\rho_X(Kv_n) > \alpha$ for all n, in particular $v_n - s$ have disjoint supports. We show next that for every sequence $\{\xi_n\}$ with $\xi_n \to 0$ we have $\sum_{n=1}^{\infty} \xi_n v_n \in \tilde{\mathscr{D}}_K$. In fact for any m, n we have $|\sum_{\ell=n}^{m} \xi_\ell v_\ell| \leqq \max_{m \leq \ell \leq n} |\xi_\ell| \, |\sum_{\ell=h}^{m} v_\ell|$ and, since $\sum_{\ell=n}^{m} v_\ell \in (\tilde{\mathscr{D}}_K)_u$, for every $\varepsilon > 0$ there is a $\lambda > 0$ such that $\tilde{\rho}_K(\lambda \sum_{\ell=n}^{m} v_\ell) < \varepsilon$ and $\tilde{\rho}_K(\sum_{\ell=m}^{n} \xi_\ell v_\ell) < \varepsilon$ provided $\max_{m \leq \ell \leq n} |\xi_\ell| \leqq \lambda$. It follows that the series $\sum \xi_\ell v_\ell$ is convergent in $\tilde{L}^0$ and since its partial sums are in $\mathscr{D}_K$, the sum is in $\tilde{\mathscr{D}}_K$. We now apply Prop. 2.4 to the function $\sum \xi_\ell v_\ell \in \tilde{\mathscr{D}}_K$ and the sequence $\xi_n v_n \leqq \chi_{E_n \setminus E_{n+1}} \sum \xi_\ell v_\ell$ to conclude that $\sum |\xi_\ell|^2 |Kv_\ell(x)|^2 < \infty$ a.e. and by (4.2) $\sum |Kv_\ell(x)|^2 < \infty$ a.e. This contradicts the property that $\rho_X(Kv_n) > \alpha$ for all n.

Similarly to (4.1) we have the corollary.

(4.4) *If A is as in* (4.1) *and* $A \subset_c \tilde{\mathscr{D}}_K$ *then* $A^\# \subset_c \tilde{\mathscr{D}}_K$.

We also give an example similar to (4.2).

(4.5) *Suppose that* $\{g_n\} \subset L^0$ is such that $\sum \xi_n g_n(x)$ *is convergent* a.e. (*or in* L^0) *for every* $\{\xi_n\} \in c_0$. *Then* $\sum \xi_n g_n$ *is convergent in* L^0 *for every* $\{\xi_n\} \in \ell^\infty$.

To check this statement we notice that with $k(x, n)$ as in (4.2) $\mathscr{D}_K \cap c_0$ contains all sequences with finitely many terms different from 0 and $\mathscr{D}_K \cap c_0$ is dense in c_0. Also, by Banach-Steinhaus principle $T: \{\xi_n\} \in c_0 \to \sum \xi_n g_n(x) \in L^0$ is continuous and on $\mathscr{D}_K \cap c_0$, $T = K$. Since c_0 is solid (2.6)(b) implies that $c_0 \subset_c \tilde{\mathscr{D}}_K$ and by (4.4) $\ell^\infty = c_0^\# \subset_c \tilde{\mathscr{D}}_K$. Since for $\{\xi_n\} \in \ell^\infty$, $\{\xi_1, \xi_2, \ldots, 0, 0 \ldots\} \to_{\mathscr{D}_K} \{\xi_n\}$ (2.6) (a) implies that $\sum \xi_n g_n(x)$ is convergent in measure (on subsets of finite measure).

Added in proof: The author is indebted to Iwo Labuda for the following remark. The operation of enlargement # has been considered for normed spaces by Yu.A. Abramovic, *On maximal normed extension of partially ordered normed spaces*, Vestnik Leningradsk. Univ., **26**, 1 (1970), 7–17, (English translation **3** (1976), 1–12), by Iwo Labuda, *Completeness type properties of locally solid Riesz spaces*, preprint, and by W. Wnuk, *The maximal solid extension of a locally solid Riesz space with Fatou property*.

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