ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

SURVEY ON THE TOPOLOGY OF REAL ALGEBRAIC SETS

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Dedicated to the memory of Gus Efroymson

This is a survey of some aspects of the topology of real algebraic sets. It reflects my personal view of the subject. I have been interested in other things lately and have not kept up with various colleague's work so I hope relevant additional information will be brought up in the discussion. The main point I would like to make is that there is much interesting mathematics to be done here. Furthermore, it can be approached at an elementary level. I am particularly gratified with the interest in representing $\mathbf{Z}/2\mathbf{Z}$ homology classes by algebraic subsets.

The first things we will discuss are restrictions on the topology of real algebraic sets. The first restriction is that a real algebraic set is triangulable. The early proofs of this that I know of were incorrect. I do not know of a correct proof before Lojasiewicz, **[14]** but perhaps there were earlier correct proofs. A very nice proof was given by Hironaka **[11]**.

Sullivan found the next restriction in **[17]**. If V is a real algebraic set and $\rho \in V$ then triangulability of V implies that ρ has a neighborhood homeorphic to the cone on some space X. Sullivan's result is that X must have even Euler characteristic. He gets this result by looking at the complexification $V_{\mathbf{C}}$ of V and the involution on $V_{\mathbf{C}}$ induced by complex conjugation.

The next restriction was found by Akbulut and King in **[2]**. If V is a real algebraic set, then a simple construction shows that the one point compactification of V is a real algebraic set also. This shows for example that V is the union of a compact set and set homeomorphic to $X \times [0, 1)$ for some compact polyhedron X with even Euler characteristic.

The final restriction was also found by Akbulut and King in **[4]**. It is complicated to describe but it comes from a synthesis of Hironaka's resolution of singularities of algebraic varieties **[10]**, Sullivan's resolution of singularities of topological spaces **[18]**, and Akbulut and King's notion of A-spaces **[3]**.

One possibility for obtaining more restrictions on the topology of real algebraic sets is to look more carefully at the complexification. A compact real algebraic set has a stratification which is the fixed point set of a certain kind of involution on a compact complex stratified set. It is conceivable that one can obtain information from this beyond Sullivan's even Euler characteristic condition. I don't know of anyone who has looked at this however.

We now look at various sufficient conditions on the topology of real algebraic sets.

In 1936, Seifert **[16]** showed that if $M \subset \mathbf{R}^n$ is a closed compact submanifold with trivial normal bundle, then there is an algebraic set $V \subset \mathbf{R}^n$ and a nonsingular component V_0 of V so that V_0 is isotopic to M via a small isotopy. His basic technique has been used in all the succeeding work on sufficient conditions for algebraicity. The basic technique is to define a space by smooth equations, approximate by polynomials, and use transversality to conclude that the original space is homeomorphic to the space defined by the approximating polynomials (or part of this space). In Seifert's case, he took a smooth function f: $\mathbf{R}^n \to \mathbf{R}^k$ so that 0 is a regular value of f and M is a union of components of $f^{-1}(0)$. Now approximate f by a polynomial ρ and use transversality to conclude that some components of $\rho^{-1}(0)$ are isotopic to M. From this we see the two sources of the extra pieces of $\rho^{-1}(0)$ away from M. One source is that there may be topological obstructions to finding a smooth f so that 0 is a regular value of f and $f^{-1}(0) = M$. The second source is that when approximating by ρ, we may only approximate on compact sets. Consequently, we have no control on $\rho^{-1}(0)$ near infinity.

Seifert did have one important case where there were no extra pieces. If $M \subset \mathbf{R}^n$ has codimension one then he shows that M is isotopic to a real algebraic set via a small isotopy.

The next result was by Nash **[15]**. Nash proved that if $M \subset \mathbf{R}^n$ is any compact smooth manifold, there is a real algebraic set $V \subset \mathbf{R}^n$ so that a sheet of V is isotopic to M by a small isotopy. Unfortunately this sheet might not be a connected component of V, but he showed that it can be if one is willing to cross with some $\mathbf{R}^m$. So there is a real algebraic set $W \subset \mathbf{R}^n \times \mathbf{R}^m$ with a nonsingular component W_0 isotopic to $M \times 0$ via a small isotopy. Nash's method was to find a smooth map f from $\mathbf{R}^n$ to the canonical bundle over a grassmanniam which is transverse to the zero section, so the M is a component of f^{-1} of the zero section. In order to approximate f by a polynomial, he must cross with some $\mathbf{R}^m$ which is the reason that appears. This paper, like much of Nash's work was way ahead of its time.

The next paper is by Wallace **[20]**. He tried to show that if $M \subset \mathbf{R}^n$ is compact and smooth, then M is isotopic to a connected component of a real algebraic subset of $\mathbf{R}^n$. Unfortunately his proof is incorrect. In his final step he applied the following false theorem which is only true if

$n > 2 \dim V$. In the Appendix are some counterexamples. I do not know if other counterexamples appear in the literature.

FALSE THEOREM. (Generic projection). *Suppose* $V \subset \mathbf{R}^n \times \mathbf{R}^m$ *is a compact nonsingular real algebraic set and projection* $\rho\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ *restricts to a smooth imbedding of* V *into* $\mathbf{R}^n$. *Then there is a projection* $q\colon \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ *close to* ρ *so that* $q(V)$ *is a real algebraic set.*

I do not know whether or not Wallace's intended result is true. It is an interesting question.

Despite the falsity of his main theorem, Wallace's paper is important for the ideas it contained. In it he showed that if M bounds, then M is diffeomorphic to a nonsingular real algebraic set (with no extra components). (He also had the hypothesis that M be orientable but suspected that it was unnecessary. Had he known of Seifert's codimension one result he could have easily eliminated this hypothesis.) It is worthwhile going through his argument which we will simplify a little. Suppose M is the boundary of a smooth compact manifold W. Consider the double of W, the manifold obtained by gluing two copies of V together along their boundaries. By Nash's theorem the double of W is diffeomorphic to a nonsingular component V_0 of a real algebraic set V in some $\mathbf{R}^n$. It is not hard to find a codimension one submanifold N of $\mathbf{R}^n$ which intersects V transversely so that $V \cap N$ is diffeomorphic to M. For instance, N could be all points at distance epsilon from one copy of W in V_0. Now by Seifert's result, we may isotop N to a nonsingular algebraic set Z via a small isotopy. But then by transversality, $V \cap Z$ is diffeomorphic to M.

The next step was by Tognoli **[19]**. He proved that any smooth compact manifold is diffeomorphic to a nonsingular real algebraic set. To do this, he noticed that unoriented bordism is generated by nonsingular real algebraic sets. Hence if we have any smooth compact manifold M, there is a nonsingular real algebraic set X so that the disjoint union $M \cup X$ bounds. A straightforward application of Wallace's result would only give you some algebraic set V diffeomorphic to $M \cup X$, but Tognoli was able to do things carefully enough that $V = X \cup M^1$ with M^1 diffeomorphic to M.

He then showed that M^1 must itself be a nonsingular variety and the result follows.

The final step for smooth manifolds is the noncompact case. In **[2]**, Akbulut and King showed that the interior of any compact smooth manifold with boundary is diffeomorphic to a nonsingular real algebraic set and vice versa, so nonsingular real algebraic sets are completely classified.

Tognoli's paper **[19]** introduced some important techniques which we will generalize to the following two theorems.

THEOREM. *Let $f\colon M \to V$ be a smooth map from a smooth closed manifold M to a nonsingular algebraic set V. Then we may approximate f by a rational function $g\colon W \to V$ from a nonsingular algebraic set W diffeomorphic to M if and only if f is bordant to a rational function from a nonsingular algebraic set to V.*

This is the source of the interest in representing $\mathbf{Z}/2\mathbf{Z}$ homology classes by algebraic subsets.

THEOREM. *Let $f\colon W \to V$ be a rational function between two irreducible algebraic sets. Then f has a* mod 2 *degree d which is the* mod 2 *Euler characteristic of a generic inverse image. In other words there is a proper algebraic subset Z of V so that $\chi(f^{-1}(v)) \equiv d(2)$ for all v in $V - Z$.*

Tognoli only showed this for the case $\dim W = \dim V$ but it is not hard to jazz up the proof with controlled vector fields and obtain the above result. If there is any demand for a proof, Akbulut and King will write it up eventually.

The next question is: What about singular algebraic sets? In **[13]** Kuiper showed that some nonsmoothable P.L. manifolds are homeomorphic to components of real algebraic sets. Essentially he considered spaces which were smooth except at isolated points where they looked like the zero set of a v-sufficient jet. He then can use Nash's proof, just making sure that any approximating polynomials have the correct jet at the singular points. In his thesis **[1]**, Akbulut extended this idea to nonisolated singularities and got many more examples. (He could also eliminate the extra components since Tognoli's work was known by then.) These methods could not hope to classify all singularities, but they were a start. Also the cobordism constructions in Akbulut's thesis were useful in later work with King.

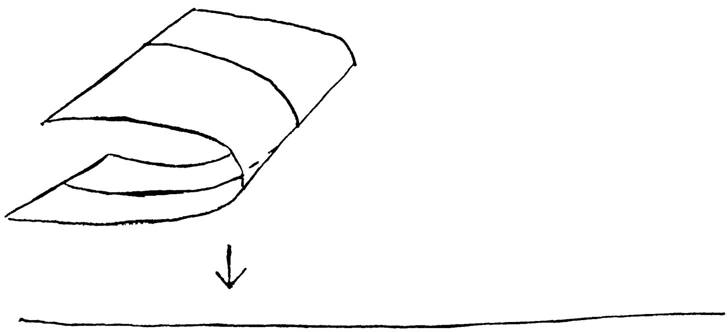
The next step was the isolated singularity case. Hironaka's resolution of singularities implies that if M is the link of an isolated singularity of a real algebraic set, then M bounds a compact manifold W so that there is a finite collection of closed submanifolds $W_i \subset \operatorname{int} W$ so that the quotient space $W/\bigcup V_i$ is homeomorphic to the cone on M. We call such a W a spine manifold. To see that M bounds a spine manifold, note that **[10]** implies the existence of a blow up so that the inverse image of the isolated singularity is a union of submanifolds in general position. Then $\bigcup W_i$ is the inverse image of the isolated singularity and W is the inverse image of a neighborhood homeomorphic to the cone on M. In 1976 Akbulut and King proved the converse; if M is a smooth compact manifold which bounds a spine manifold then the cone on M is homeomorphic to a real algebraic set. In fact they showed that if X is a compact Thom stratified set whose singularities are isolated so that the link of any singularity bounds a spine mainfold, then X is homeomorphic to a real algebraic set. Their method was to show that if N is a smooth closed manifold and $N_i \subset N$ are smooth closed submanifolds in general position, then N is diffeomorphic to a real algebraic set so that each N_i is a subvariety. They also found a construction which allowed them to blow down algebraic subsets to points. Combining these, they obtained the above classification of real algebraic sets with isolated singularities.

Now there was a nagging topological problem i.e., which manifolds bound spine manifolds? Lowell Jones had the crucial idea. If M is a manifold which bounds, then a simple construction shows that the disjoint union of M with a number of copies of the sphere will bound a spine manifold. Akbulut and King then found a way to get rid of the extra spheres by adding one-handles. Consequently a manifold bounds a spine manifold if and only if it bounds. This gives a neater characterization of real algebraic sets with isolated singularities which appeared after much delay in [**2**]. The Tognoli-Benedetti paper with the same result [**9**] is for the most part copied from Akbulut and King's preprint.

The next step after characterizing isolated singularities is to understand nonisolated singularities. Akbulut and King applied the basic philosophy of [**2**] to this problem. This basic philosophy is to take a topological space X which you hope to make algebraic, find a topological resolution of its singularities which has nice properties, make this topological resolution algebraic and then algebraicly blow down the singularities to obtain a real algebraic set homeomorphic to X. To use this method you must decide what a topological resolution of singularities should be.

In 1976, Akbulut and King took a naive notion of topological resolution and got the result that certain spaces called A-spaces are homeomorphic to real algebraic sets (if they are compact or the interior of a compact A-space). The notion of A-space is too weak to possibly characterize real algebraic sets but is of sufficient generality that Akbulut and Taylor [**6**] could prove by homotopy theoretic methods that all P.L. manifolds are A-spaces, and hence the interior of any compact P.L. manifold is homeomorphic to a real algebraic set. A-spaces are stratified sets built up by the processes of crossing with smooth manifolds, coning on compact A-spaces which bound and gluing along boundaries. A-space are resolved by replacing a closed stratum N cross the cone on its link by $N \times W$ where W is an A-space which the link bounds (actually W is a spine A-space). After a finite number of these steps one obtains a smooth manifold together with a collection of submanifolds and various collapsing information. The next step is to make all of that algebraic and then finally blow down to get the result [**3**].

Unfortunately not all real algebraic sets are A-spaces so one must look for a better notion of topological blow up. The simplest example of a real algebraic set which is not an A-space is the Whitney umbrella $x^2 = zy^2$. This fails to be an A-space because the link of its most singular point is a figure eight union, a point which cannot possibly bound a compact A-space. This is true since the isolated point would have to bound an isolated line segment, but then the other end of this line segment would also be in the boundary. If we look at the algebraic resolution of singularities we see folding behavior which is not allowed in an A-space resolution.



By allowing folds in their topological resolutions Akbulut and King showed in 1978 that any compact two dimensional polyhedrom satisfying Sullivan's even local Euler characteristic condition is homeomorphic to a real algebraic set, so two dimensional real algebraic sets are characterized.

Again the proof is by the old process of finding a nice topological resolution, making it algebraic and then algebraicly blowing down. Although this result was announced, the proof was never published. Benedetti and Dedo published an independent proof of this two dimensional result [**8**]. I believe their methods are similar.

The next step is three dimensions. There is some conflict here. Benedetti has announced in [7] that any compact three dimensional polyhedron satisfying Sullivan's even local Euler characteristic condition is homeomorphic to a real algebraic set. On the other hand Akbulut and King have found some necessary conditions on real algebraic sets which are not satisfied by some three dimensional polyhedra satisfying Sullivan's condition [**12**]. Akbulut and King also have a proof that compact three dimensional polyhedra satisfying their more stringent conditions are homeomorphic to algebraic sets. So one way or the other, three dimensional real algebraic sets are classified.

It is worth pointing out that Akbulut and King have shown that real analytic sets of dimension less than or equal to three also have this more stringent resolution, so compact analytic sets of dimension less than or equal to three are homeomorphic to real algebraic sets. This brings up the problem of higher dimensional real analytic sets. Unfortunately the algebraic proof seems to break down in higher dimensions if one tries to carry it over to the analytic case.

Akbulut and King also have a proof that topological spaces of dimension $\leqq 6$ which have a certain type of resolution are homeomorphic to real algebraic sets. This resolution is very similar to the necessary condition on real algebraic sets but at the moment a technical problem makes it not quite the same; it does not yet characterize real algebraic sets of dimension $\leqq 6$.

The only reason for the dimension restriction above is that the following conjecture is only known to be true in low dimensions.

CONJECTURE. Every compact smooth manifold is diffeomorphic to a nonsingular real algebraic set, all of whose $\mathbf{Z}/2\mathbf{Z}$ homology is represented by algebraic subsets.

There are rumors that Benedetti has a counterexample to this conjecture. If these rumors are true it would be extremely interesting (and also unfortunate). The above conjecture makes proofs a lot easier since one can represent all bordism. One has only to look at [3] to see how much more difficult proofs are which do not use this conjecture.

If V is a real algebraic set, let $H_*^a(V)$ be the elements of $H_*(V, \mathbf{Z}/2\mathbf{Z})$ represented by compact algebraic subsets, or equivalently, represented by rational functions $f\colon W \to V$ from compact nonsingular algebraic sets. It is more convenient to take the Poincare dual $H_a^*(V)$. Then Akbulut and King have shown that $H_a^*(V)$ is a subgroup closed under cup products and Steenrod squares. It is not always true that $H_a^*(V) = H^*(V)$; for instance, in the Appendix, are examples of connected nonsingular real algebraic sets so that $H_i^a(V) \neq H_i(V)$ for $i = 2, \ldots, n - 1$.

It is not hard to show that the following is true (both Akbulut and King and Benedetti and Tognoli noticed this independently). Let M be a smooth closed manifold. Then M is diffeomorphic to a nonsingular algebraic variety V so that

1) All classes represented by submanifolds are in $H_a^*(V)$; and

2) If $f\colon V \to W$ is any continuous map to any algebraic set W with $H_a^*(W) = H^*(W)$ then $f^*(H^*(W)) \subset H_a^*(V)$.

The proof is to let $M_i \subset M$ be submanifolds of M in general position representing all classes of type 1) and let $f_i\colon M \to W_i$ $i = 1, \ldots, k$ with $H_a^*(W_i) = H^*(W_i)$ represent all classes coming from 2). Then we make M an algebraic set with each M_i an algebraic subset so that $f_1 \times \cdots \times f_k\colon M \to W_1 \times \cdots \times W_k$ is approximated by a rational function g. We can do this because $H_a^*(W_1 \times \cdots \times W_k) = H^*(W_1 \times \cdots \times W_k)$ implies that all bordism of $W_1 \times \cdots \times W_k$ are algebraic. So smooth maps can be approximated by rational functions. The result now follows from the observation that if $f\colon V \to W$ is a rational function between nonsingular algebraic sets, then $f^*(H_a^*(W)) \subset H_a^*(V)$.

For example the Grassmanian has a simple representation as a nonsingular algebraic set so that all of its $\mathbf{Z}/2\mathbf{Z}$ homology (i.e., the Schubert cells) are algebraic **[2]**. So all the image of K-theory can be made algebraic.

The tempting thing to do is to find algebraic approximations to $K(\mathbf{Z}/2\mathbf{Z}, n)$'s with all algebraic homology since this would prove the conjecture. Except for the obvious $K(\mathbf{Z}/2\mathbf{Z}, 1)$, though, I believe no one has succeeded in doing this.

APPENDIX I

Consider the curve $V = \{y^2 + x^2(x-1)(x-2)\}$ in $\mathbf{R}^2$. V consists of a nonsingular circle and the isolated point $(0, 0)$. We may blow up this point to get a nonsingular curve $W \subset V \times \mathbf{R}^n \subset \mathbf{R}^2 \times \mathbf{R}^n$ so that the blow up map $W \to V$ is induced by projection $\pi\colon \mathbf{R}^2 \times \mathbf{R}^n \to \mathbf{R}^2$. Notice that W is nonsingular and π is a diffeomorphism from W to $V - (0, 0)$.

However, if $\rho\colon \mathbf{R}^2 \times \mathbf{R}^n \to \mathbf{R}^2$ is any linear projection close to the standard one, then $\rho(W)$ is not a real algebraic set. To see this, consider the complexifications. The germ of $V_{\mathbf{C}}$ at $(0, 0)$ consists of two nonsingular sheets $y = \pm ix\sqrt{(1-x)(2-x)}$. Take a small circle C on one of these sheets which encloses $(0, 0)$, for instance $|x| = 1/2$. Notice that this circle has linking number 1 with $\mathbf{R}^2$, i.e., it generates $\pi_1(\mathbf{C}^2 - \mathbf{R}^2)$. Lift C up to $D \subset W_{\mathbf{C}}$, (so $C = \pi(D)$). Then if ρ is any polynomial close to π, $\rho(D)$ will be close to C. In particular $\rho(D)$ will still link $\mathbf{R}^2$, so we will have $\rho(z) \in \mathbf{R}^2$ for some z in the disc which D encloses in W_C. Consequently $\rho(z)$ is in the Zariski closure of $\rho(W)$. But $\rho(Z)$ is near $(0, 0)$, hence far from $\rho(W)$. So $\rho(W)$ is not Zariski closed.

For another example which is more closely connected with **[20]**, let $\mathbf{Z}$ be the real algebraic set $\{(x, y, z) \in \mathbf{R}^3 | z^2 = (4 - x^2 - y^2)^2(1 - x^2 - y^2)\}$. Clearly Z is irreducible and is the disjoint union of a 2-sphere and the circle $\{z = 0,\ x^2 + y^2 = 4\}$. Blow up this circle to get a nonsingular algebraic variety W. Consider the map $\rho\colon \mathbf{R}^3 \to \mathbf{R}^3$ defined by $\rho(x, y, x) = (x - x^3/4, y, z)$. Then $\rho(Z)$ is the union of a sphere and a figure eight which intersects the sphere in four points. Let V be the Zariski closure of $\rho(Z)$. Then by an argument similar to the one in the previous example, if $q\colon W \to \mathbf{R}^3$ is any rational function close to $W \to Z \to^{\rho} \mathbf{R}^3$ then $q(W)$ is not Zariski closed. Furthermore, $q(W)$ is not an isolated sheet of the Zariski closure of $q(W)$. The Zariski closure of $q(W)$ is obtained from $q(W)$ by adding a one dimensional algebraic set which intersects $q(W)$ in at least four points.

APPENDIX II

Let V be an irreducible real algebraic set homeomorphic to two circles, say $x_2^2 = (x_1^2 - 1)(4 - x_1^2)$, $x_i = 0$ $i > 2$. Let M be any closed smooth connected manifold of dimension > 2 so that $H_1(M, \mathbf{Z}/2\mathbf{Z})$ has rank > 1. We may find a nonsingular real algebraic set W diffeomorphic to M so that $V \subset W$, V has trivial normal bundle in W and each component of V represents a different nontrivial element of $H_1(M, \mathbf{Z}/2\mathbf{Z})$. Let $\pi: X \to W$ be the blow up of W along V. Then with $\mathbf{Z}/2\mathbf{Z}$ coefficients, $(*)$ $H_*(X) \approx H_*(W) \oplus H_*(V) \otimes \bar{H}_*(\mathbf{R}P^{n-2})$ where $n = \dim M$. Let α be a nontrivial class in $H_1(V)$ generated by one of the two circles of V. Let γ be the homology class in $H_*(V) \otimes \bar{H}_*(\mathbf{R}P^{n-2})$ representing $\pi^{-1}(V)$. If β_i is the generator of $\bar{H}_i(\mathbf{R}P^{n-2})$, then $(\alpha \otimes \beta_i) \cdot \gamma = \alpha \oplus \beta_{i-1}$ for $i > 1$ and $(\alpha \otimes \beta_1) \cdot \gamma = \alpha$ under the identification $(*)$. (This is easily seen geometrically.) Suppose $\alpha \otimes \beta_i$ is an algebraic homology class. Then it is represented by a nonsingular algebraic subset Z_i of $X \times \mathbf{R}^k$ which we can assume to be transverse to $\pi^{-1}(V) \times \mathbf{R}^k$. But then $Z_i \cap \pi^{-1}(V) \times \mathbf{R}^k$ is an algebraic set representing $\alpha \otimes \beta_i \cdot \gamma = \alpha \otimes \beta_{i-1}$. Continuing in this manner we get Z_1 so that $Y = Z_1 \cap \pi^{-1}(V) \times \mathbf{R}^k$ represents α. Let $\rho: X \times \mathbf{R}^k \to X$ be a projection. Then we know $\pi\rho(Y) \subset V$ and it also represents α.

Hence $\pi\rho: Y \to V$ must have odd degree to the circle representing α and even degree to the other circle which is a contradition. Hence none of the homology classes $\alpha \otimes \beta_i$ can be algebraic. So we have examples of connected real algebraic sets with nonalgebraic homology in all $H_i(X)$, $i = 2, \ldots, n - 1$.

REFERENCES

1. S. Akbulut, *Algebraic equations for a class of P.L. spaces*, Math. Ann. **231** (1977), 19–31.

2. ——— and H. King, *The topology of real algebraic sets with isolated singularities*, Annals of Math. **113** (1981), 425–446.

3. ———, *Real algebraic structures on topological spaces*, Publ. I.H.E.S. **53** (1981), 79–162.

4. ———, *Topological resolutions of real algebraic stes*, preprint.

5. ———, *On submanifolds and homology of real algebraic varieties*, preprint.

6. S. Akbulut and L. Taylor, *A topological resolution theorem*, Publ. I.H.E.S. **53** (1981), 163–195.

7. R. Benedetti, *On certain topological spaces admitting strongly coherent real algebraic structure*, preprint.

8. ——— and M. Dedo, *The topology of two-dimensional real algebraic varieties*, Anali di Matematica pura del Applicata, CXXVII (1981) 141–171.

9. R. Benedetti and A. Tognoli, *Approximation theorems in real algebraic geometry*, preprint.

10. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math. **79** (1964), 109–326.

11. ———, *Triangulations of algebraic sets*, Arcata conference, Proc. of Symposia in Pure Mathematics, A.M.S. **27** (1975), 165–186.

12. H. King, *The topology of real algebraic sets*, Arcata conference, Proc. of Symposia in Pure Mathematics A.M.S. **40** (1983), 42–48.

13. N. Kuiper, *Algebraic equations for nonsmoothable 8-manifolds*, Publ. I.H.E.S. **33** (1968).

14. S. Lojasiewicz, *Triangulation of semi-analytic sets*, Annali Scu. Norm. Sup. Pisa **18** (1964), 449–474.

15. J. Nash, *Real algebraic manifolds*, Ann. of Math. **56** (1952), 405–421.

16. H. Seifert, *Algebraische Approximation von Manningfaltigkeiten*, Math. Zeit. **41** (1936), 1–17.

17. D. Sullivan, *Combinatorial invariants of analytic spaces*, Springer Lecture Notes in Mathematics **192** (1971), 165–168.

18. ———, *Singularities in spaces*, Springer Lecture Notes in Mathematics **209** (1971), 196–206.

19. A. Tognoli, *Su una congettura di Nash*, Annali Sc. Norm. Sup. Pisa **27** (1973), 167–185.

20. A Wallace, *Algebraic approximation of manifolds*, Proc. London Math. Soc. **7** (1957), 196–210.

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