ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 2, Spring 1985

SUMS CONTAINING THE FRACTIONAL PARTS OF NUMBERS

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Dedicated to the Memory of E.G. Straus and R.A. Smith

1. Introduction. Let x be a real number and $[x]$, $\{x\}$ denote respectively the integral part and the fractional part of x. Let k be a positive integer and let a be a real number. The purpose of this paper is to give an asymptotic formula for $\sum_{n\leq x} n^a\{x/n\}^k$.

Smith and Subbarao [5] obtained an asymptotic expression for this sum when $a = 0$, $k = 1$ and $n \equiv b(m)$. More recently MacLeod [4] studied it when a is an integer and k is a positive integer.

To obtain our result, we shall use a result which can be considered as an inversion formula for a class of arithmetic sums. That will be the subject of the following section.

2. Preliminaries. Let f be an arbitrary arithmetic function, arithmetic sums of the form $\sum_{n\leq x} f(n)\,[x/n]$ occur in many situations in the theory of numbers. For example, we have the well-known results

$$\sum_{n\leq x} \sigma(n) = \frac{1}{2}\sum_{n\leq x}\left(\left[\frac{x}{n}\right]^2 + \left[\frac{x}{n}\right]\right)$$

and

$$\sum_{n\leq x} \phi(n) = \frac{1}{2}\sum_{n\leq x} \mu(n)\left[\frac{x}{n}\right]^2 + \frac{1}{2},$$

where σ is the sum of the divisors of n, ϕ is Euler's totient and μ represents the Möbius function, which are used to obtain the average orders of $\sigma(n)$ and $\phi(n)$.

Let k be any non negative integer and let

$$f_k(n) = \sum_{d\mid n} g(d)\left(\frac{n}{d}\right)^k,$$

where g is any arithmetic function. Then we have

Received by the editors on August 15, 1983.

$$(1) \qquad \sum_{n\leq x} f_k(n) = \sum_{n\leq x} g(n)\left(1^k + 2^k + \cdots + \left[\frac{x}{n}\right]^k\right),$$

and using the well-known identity

$$\sum_{n=1}^{m} n^k = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i} B_i m^{k+1-i} + m^k, \qquad k \geqq 1,$$

where B_i are Bernoulli's numbers defined by

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2n+1} = 0,\ B_{2n} = \frac{2(-1)^{n+1}(2n)!\,\zeta(2n)}{(2\pi)^{2n}}, \qquad n = 1, 2, \ldots,$$

and ζ stands for the Riemann zeta function, equation (1) becomes (for $k \geqq 1$)

$$(2) \qquad \sum_{n\leq x} f_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}\left(\binom{k+1}{j} B_j \sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^{k+1-j}\right) + \sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^k.$$

This last transformation is quite trivial. However, let us note that recently Harris and Subbarao **[2]** found an interesting transformation formula for the sums of the type

$$\sum_{\substack{n\leq x\\ (n,m^r)_r=1}} g(n)\left(1^k + 2^k + \cdots + \left[\frac{x}{n}\right]^k\right),$$

where $(n, m^r)_r$ is the greatest r^{th} power common divisor of n and m^r.

Now, we state a result which can be considered as an inversion formula for a class of arithmetic sums.

THEOREM 1. *Let k be any non negative integer and let $f_k(n) = \sum_{d\mid n} g(d) \cdot (n/d)^k$, where g is an arbitrary arithmetic function. Then*

$$(3) \qquad \sum_{n\leq x} f_k(n) = \sum_{n\leq x} g(n)\left(1^k + 2^k + \cdots + \left[\frac{x}{n}\right]^k\right)$$

if and only if

$$(4) \qquad \sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^{k+1} = \sum_{j=0}^{k}\left((-1)^j\binom{k+1}{j+1}\sum_{n\leq x} f_{k-j}(n)\right).$$

PROOF. The proof is trivial for $k = 0$, and, consequently, we will now suppose $k \geqq 1$.

We shall first establish necessity. Since (3) is equivalent to (2), then we can write (3) in the form

$$\sum_{n\leq x} f_k(n) = \frac{1}{k+1}\sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^{k+1} + \frac{1}{2}\sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^k$$
$$+ \frac{1}{k+1}\sum_{j=2}^{k}\binom{k+1}{j} B_j \sum_{n\leq x} g(n)\left[\frac{x}{n}\right]^{k+1-j},$$

and thus we obtain

$$\sum_{j=0}^{k}\left((-1)^j\binom{k+1}{j+1}\sum_{n\le x} f_{k-j}(n)\right) = \sum_{j=0}^{k}\frac{(-1)^j\binom{k+1}{j+1}}{k+1-j}\sum_{n\le x} g(n)\left[\frac{x}{n}\right]^{k+1-j}$$
$$+\frac{1}{2}\sum_{j=0}^{k-1}(-1)^j\binom{k+1}{j+1}\sum_{n\le x} g(n)\left[\frac{x}{n}\right]^{k-j}$$
$$+\sum_{j=0}^{k-2}\left(\frac{(-1)^j\binom{k+1}{j+1}}{k+1-j}\sum_{i=2}^{k-j}\binom{k+1-j}{i}B_i\sum_{n\le x} g(n)\left[\frac{x}{n}\right]^{k+1-j-i}\right)$$
$$=\sum_{n\le x} g(n)\left(\left[\frac{x}{n}\right]^{k+1}+\sum_{j=1}^{k}\left(\frac{(-1)^j\binom{k+1}{j+1}}{k+1-j}-\frac{1}{2}(-1)^j\binom{k+1}{j}\right)\left[\frac{x}{n}\right]^{k+1-j}\right.$$
$$\left.+\sum_{2=0}^{k-2}\frac{(-1)^j\binom{k+1}{j+1}}{k+1-j}\sum_{i=2}^{k-j}\binom{k+1-j}{i}B_i\left[\frac{x}{n}\right]^{k+1-j-i}\right).$$

In this last expression, the coefficient of $[x/n]^k$ is $-1/k\binom{k+1}{2}+1/2\binom{k+1}{1}=0$ whereas the coefficient of $[x/n]_1^{k-m}$, $1\leqq m\leqq k-1$, is

$$\frac{(-1)^{m+1}}{k-m}\binom{k+1}{m+2}-\frac{(-1)^{m-1}}{2}\binom{k+1}{m+1}+\sum_{j=0}^{m-1}\frac{(-1)^j}{k+1-j}\binom{k+1}{j+1}\binom{k+1-j}{m+1-j}B_{m+1-j}.$$

The value of this last sum is also zero. Indeed, since

$$\frac{1}{k+1-j}\binom{k+1}{j+1}\binom{k+1-j}{m+1-j}=\frac{1}{m+1-j}\binom{k+1}{m+1}\binom{m+1}{m-j},$$

it follows that

$$\frac{(-1)^{m-1}}{2}\binom{k+1}{m+1}-\sum_{j=0}^{m-1}\frac{(-1)^j}{k+1-j}\binom{k+1}{j+1}\binom{k+1-j}{m+1-j}B_{m+1-j}$$
$$=(-1)^{m+1}\binom{k+1}{m+1}\left(\frac{1}{2}-\sum_{j=0}^{m-1}\frac{(-1)^{m+1-j}}{m+1-j}\binom{m+1}{m-j}B_{m+1-j}\right),$$

and, from the well-known identity $\binom{m+2}{0}B_0+\binom{m+2}{1}B_1+\cdots+\binom{m+2}{m+1}B_{m+1}=0$, we can deduce that

$$\frac{1}{2}-\sum_{j=0}^{m-1}\frac{(-1)^{m+1-j}}{m+1-j}\binom{m+1}{m-j}B_{m+1-j}=\frac{1}{m+2},$$

and consequently, we have the result of the first part.

The details of the proof of sufficiency are almost identical with those of necessity and thus we shall omit it.

3. Some examples.

THEOREM 2. *Let a be any real number and let k be a positive integer, then*

$$\sum_{n\leq x} n^a\left[\frac{x}{n}\right]^k = \sum_{j=0}^{k-1}\left((-1)^j\binom{k}{j+1}\sum_{n\leq x} n^{k-1-j}\,\sigma_{a-(k-1-j)}(n)\right),$$

where $\sigma_s(n) = \sum_{d|n} d^s$.

PROOF. Let $g(n) = n^a$, then $f_k(n) = n^k\sigma_{a-k}(n) = n^a\sigma_{k-a}(n)$ and thus we obtain

$$\sum_{n\leq x} n^a\sigma_{k-a}(n) = \sum_{n\leq x} n^a\left(1^k + 2^k + \cdots + \left[\frac{x}{n}\right]^k\right)$$

and, using Theorem 1, we get the result.

THEOREM 3. *Let $J_k(n)$ be Jordan's totient defined by*

$$J_k(n) = \sum_{d|n}\mu(d)\left(\frac{n}{d}\right)^k.$$

Then, for all integers $k \geqq 0$, we have

$$\sum_{n\leq x}\mu(n)\left[\frac{x}{n}\right]^{k+1} = \sum_{j=0}^{k}\left((-1)^j\binom{k+1}{j+1}\sum_{n\leq x} J_{k-j}(n)\right).$$

PROOF. Let $g(n) = \mu(n)$. Then

$$\sum_{n\leq x} J_k(n) = \sum_{n\leq x}\mu(n)\left(1^k + 2^k + \cdots + \left[\frac{x}{n}\right]^k\right)$$

and, using Theorem 1, we have the result.

4. The principal result.

THEOREM 4. *Let k be an arbitrary positive integer. Then, for any real number a, we have*

$$\sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k = \begin{cases} \left(\dfrac{1}{a+1-k} - \displaystyle\sum_{i=1}^{k}\frac{k!\,\zeta(a+1+i-k)}{i!(a+1)(a)\cdots(a+1+i-k)}\right)x^{a+1} & \\ \qquad + 0\left(\dfrac{x^{a+1}}{\log x}\right), & \text{if } a > k-1, \\ \left(1-\gamma-\displaystyle\sum_{n=2}^{k}\frac{\zeta(n)-1}{n}\right)x^k + 0\left(\dfrac{x^k}{\log x}\right), & \text{if } a = k-1,\ k\geqq 2, \\ (1-\gamma)x + 0\left(\dfrac{x}{\log x}\right), & \text{if } a = 0,\ k = 1, \\ 0(x^{a+1}), & \text{if } a < k-1, \end{cases}$$

where γ is Euler's constant.

PROOF. Using Theorem 2, we have

$$
\begin{aligned}
\sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k &= \sum_{n\leq x} n^a\left(\frac{n}{n}-\left[\frac{x}{n}\right]\right)^k \\
&= \sum_{j=0}^{k}\left((-1)^j\binom{k}{j}x^{k-j}\sum_{n\leq x} n^{a-k+j}\left[\frac{x}{n}\right]^j\right) \\
&= x^k\sum_{n\leq x} n^{a-k} + \sum_{j=1}^{k}(-1)^j\binom{k}{j}x^{k-j}\sum_{n\leq x} n^{a-k+j}\left[\frac{x}{n}\right]^j \\
&= x^k\sum_{n\leq x} n^{a-k} \\
&\quad + \sum_{j=1}^{k}\left((-1)^j\binom{k}{j}x^{k-j}\sum_{i=0}^{j-1}(-1)^i\binom{j}{1+i}\right. \\
&\quad \left.\cdot\sum_{n\leq x} n^{j-1-i}\,\sigma_{a-k+i+1}(n)\right).
\end{aligned}
\tag{5}
$$

But, using Abel's identity and the asymptotic formula (see [1]) for $\sum_{n\leq x}\sigma_{a-\beta}(n)$, where β is any non negative real number, we obtain

$$
\sum_{n\leq x} n^\beta\sigma_{a-\beta}(n) = \begin{cases} \dfrac{\zeta(a-\beta+1)}{a+1}x^{a+1}+0\left(\dfrac{x^{a+1}}{\log x}\right), & \text{if } a>\beta \\ \dfrac{x^{\beta+1}}{\beta+1}\log x + 0(x^{\beta+1}), & \text{if } a=\beta \\ \dfrac{\zeta(1-a+\beta)}{\beta+1}x^{\beta+1}+0\left(\dfrac{x^{\beta+1}}{\log x}\right), & \text{if } a<\beta. \end{cases}
\tag{6}
$$

First of all, we shall show our result for $a > k-1$, $k \geqq 1$. Using (6), equation (5) becomes

$$
\begin{aligned}
\sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k &= \frac{x^{a+1}}{a-k+1} + 0(x^a) + \sum_{j=1}^{k}(-1)^j\binom{k}{j}x^{k-j}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i+1} \\
&\quad\cdot\left(\frac{\zeta(a-k+i+2)}{a-k+j+1}x^{a-k+j+1} + 0\left(\frac{x^{a-k+j+1}}{\log x}\right)\right) \\
&= x^{a+1}\left(\frac{1}{a-k+1} + \sum_{j=1}^{k}(-1)^j\binom{k}{j}\right. \\
&\quad\left.\cdot\sum_{i=0}^{j-1}(-1)^i\binom{j}{i+1}\frac{\zeta(a-k+i+2)}{a-k+j+1} + 0\left(\frac{1}{\log x}\right)\right) \\
&= x^{a+1}\left(\frac{1}{a-k+1} - \sum_{j=1}^{k}\frac{(-1)^j\binom{k}{j}}{a-k+j+1}\right. \\
&\quad\left.\cdot\sum_{i=1}^{j}(-1)^i\binom{j}{i}\zeta(a-k+i+1) + 0\left(\frac{1}{\log x}\right)\right).
\end{aligned}
$$

Let i_0 be a fixed integer, $1 \leqq i_0 \leqq j \leqq k$, then the double sum of this last equation becomes

$$\sum_{j=i_0}^{k} \frac{(-1)^j\binom{k}{j}}{a-k+j+1}(-1)^{i_0}\binom{j}{i_0}\zeta(a-k+i_0+1)$$
$$= (-1)^{i_0}\zeta(a-k+i_0+1)\binom{k}{i_0}\sum_{j=i_0}^{k}\frac{(-1)^j\binom{k-i_0}{j-i_0}}{a-k+j+1}$$

but

$$\lim_{a\to k-j-1}\frac{(-1)^{i_0}(k-i_0)!\,(a-k+j+1)}{(a+1)(a)\cdots(a-k+i_0+1)} = (-1)^j\binom{k-i_0}{j-i_0}.$$

Hence

$$\sum_{j=i_0}^{k}\frac{(-1)^j\binom{k}{j}}{a-k+j+1}(-1)^{i_0}\binom{j}{i_0}\zeta(a-k+i_0+1)$$
$$= \frac{\binom{k}{i_0}(k-i_0)!\,\zeta(a-k+i_0+1)}{(a+1)(a)\cdots(a-k+i_0+1)},$$

and, thus, for $a > k-1$, we obtain

$$\sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k \tag{7}$$
$$=\left(\frac{1}{a+1-k}-\sum_{i=1}^{k}\frac{k!}{i!}\frac{\zeta(a-k+i+1)}{(a+1)(a)\cdots(a-k+i+1)}\right)x^{a+1}+0\left(\frac{x^{a+1}}{\log x}\right).$$

For the case $a = k-1$, we proceed as follows. From equation (5), we have

$$\sum_{n\leq x} n^{k-1}\left\{\frac{x}{n}\right\}^k = x^k\sum_{n\leq x}\frac{1}{n} - kx^{k-1}\sum_{n\leq x}\left[\frac{x}{n}\right] \tag{8}$$
$$+\sum_{j=2}^{k}(-1)^j\binom{k}{j}x^{k-j}\sum_{n\leq x}n^{j-1}\left[\frac{x}{n}\right]^j.$$

We know the asympotic expressions of the first two sums, so the problem now consists of estimating the sum $\sum_{n\leq x} n^{j-1}[x/n]^j$. But, from example 1, we have, for all j, $2 \leq j \leq k$, the following equation

$$\sum_{n\leq x} n^{j-1}\left[\frac{x}{n}\right]^j = j\sum_{n\leq x} n^{j-1}\sigma_0(n) + \sum_{i=1}^{j-1}(-1)^i\binom{j}{i+1}\sum_{n\leq x}n^{j-1-i}\sigma_i(n).$$

Now, using (6), we obtain, for $2 \leq j \leq k$,

$$\sum_{n\leq x} n^{j-1}\left[\frac{x}{n}\right]^j \tag{9}$$
$$= x^j\left(\log x + 2\gamma - \frac{1}{j} + \frac{1}{j}\sum_{i=1}^{j-1}(-1)^i\binom{j+1}{i}\zeta(1+i)\right)+0\left(\frac{x^j}{\log x}\right).$$

Replacing the asymptotic expression of $\sum_{n\leq x} 1/n$, $\sum_{n\leq x} [x/n]$ and $\sum_{n\leq x} n^{j-1}$ $[x/n]^j$ in (8), we get the result, after having used the identities

$$\sum_{j=1}^{k} \frac{(-1)^{j+1}\binom{k}{j}}{j} = \sum_{j=1}^{k} \frac{1}{j},$$

and

$$\sum_{j=2}^{k} \frac{(-1)^j\binom{k}{j}}{j} \sum_{i=1}^{j-1} (-1)^i \binom{j+1}{i} \zeta(1+i) = -\sum_{j=2}^{k} \frac{\zeta(j)}{j}.$$

Let us remark that we can also obtain an asymptotic formula for $\sum_{n\leq x} n^{k-1}\{x/n\}^k$ in the following way. We replace the estimation of $\zeta(s)$ in a neighborhood of $\sigma = 1$ (see [6]).

$$\zeta(s+2-k) = \frac{1}{s+1-k} + \gamma + 0(s-1+k) \tag{10}$$

in (7), and we take the limit of this new equation when $a \to (k-1)^+$.

For the last case, $(a < k-1)$, we get, after some elementary computations,

$$\sum_{n\leq x} n^a \left\{\frac{n}{n}\right\}^k = 0\left(\frac{x^k}{\log x}\right).$$

But, evidently, we have $\sum_{n\leq x} n^a\{x/n\}^k = 0(x^{a+1})$, so our result does not give much information for this case.

5. An improvement. For any real numbers a with $a \geqq k-1 \geqq 0$, we can prove that

$$\int_1^\infty \frac{\{t\}^k}{t^{a+2}}\,dt = \begin{cases} \dfrac{1}{a+1-k} - \displaystyle\sum_{j=1}^{k} \frac{k!\,\zeta(a+1-k+j)}{j!(a+1)(a)\cdots(a+1-k+j)}, & \text{if } a > k-1 \\ 1-\gamma-\displaystyle\sum_{n=2}^{k}\frac{\zeta(n)-1}{n}, & \text{if } a = k-1,\ k \geqq 2 \\ 1-\gamma, & \text{if } a = k-1,\ k = 1, \end{cases}$$

and then, from Theorem 4, we can state that for all positive integers k and for all real numbers $a \geqq k-1$, we have

$$\sum_{n\leq x} n^a \left\{\frac{x}{n}\right\}^k = \left(\int_1^\infty \frac{\{t\}^k}{t^{a+2}}\,dt\right) x^{a+1} + 0\left(\frac{x^{a+1}}{\log x}\right).$$

We have the following improvement.

THEOREM 5. *Let k be any positive integer. Then for any real number $a > 0$, we have*

$$\sum_{n\leq x} n^a \left\{\frac{x}{n}\right\}^k = C\,x^{a+1} + 0(x^{a+6/13}\log^{7/13} x)$$

where

$$C = \int_1^\infty \frac{\{t\}^k}{t^{a+2}}\,dt.$$

PROOF.

Since

$$\int_x^\infty \frac{\{t\}^k}{t^{a+2}}\,dt = 0\left(\frac{1}{x^{a+1}}\right),$$

then we have

$$\int_1^x t^a\left\{\frac{x}{t}\right\}^k dt = C\,x^{a+1} + 0(1),$$

where C is the constant defined in the statement of this result. Consequently,

$$\begin{aligned}\sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k &= C\,x^{a+1} + 0(1) + \sum_{n\leq x} n^a\left\{\frac{x}{n}\right\}^k - \int_1^\infty t^a\left\{\frac{x}{t}\right\}^k dt\\ &= C\,x^{a+1} + 0(x^a) + \int_0^1 \sum_{x\leq n}\left(n^a\left\{\frac{x}{n}\right\}^k - (n+t)^a\left\{\frac{x}{n+t}\right\}^k\right)dt.\end{aligned}$$

Now, using a result of Kolesnik [3] concerning the sum under the sign of the integral, we have the result. Indeed, Kolesnik has proved, for all t, $0 \leqq t \leqq 1$,

$$\sum_{n\leq x}\left(n^a\left\{\frac{x}{n}\right\}^k - (n+t)^a\left\{\frac{x}{n+t}\right\}^k\right) = 0(x^{a+6/13}\log^{7/13} x),$$

and to obtain this result, he expands $\{y\}^k$ in a Fourier series and uses a theorem of Van der Corput concerning exponential sums.

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