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CONNECTEDNESS PROPERTIES OF SUPPORT POINTS OF CONVEX SETS

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ABSTRACT. It is shown that the set of support points of certain convex subsets of a Banach space is $\mathscr{C}^\infty$.

Let E be a real Banach space and E^* its continuous dual. The natural pairing between these spaces will be denoted by $\langle x, x^* \rangle$ for $x \in E$ and $x^* \in E^*$. If $C \subseteq E$, we will write $M(x^*, C)$ in place of $\sup \{\langle x, x^* \rangle : x \in C\}$. The set of support points of C (written: supp C) is the collection of points $x \in C$ for which there exists $x^* \in E^* \backslash \{0\}$ such that

$$\langle x, x^* \rangle = M(x^*, C).$$

The set C is boundedly (weakly) compact if $C \cap B$ is (weakly) compact for each closed ball in E.

A space Y is said to be k-connected, if it is homotopically trivial over the k-dimensional sphere S^k. If Y is k-connected, for each $k = 0, \ldots, n$, then Y is said to be $\mathscr{C}^n$. An example, the n-dimensional Euclidean sphere S^n is $\mathscr{C}^{n-1}$ but not $\mathscr{C}^n$. A space is said to be $\mathscr{C}^\infty$ if it is $\mathscr{C}^n$ for every n.

If C is a closed convex subset of E, then supp C is known to be a norm dense F_σ subset of the boundary of C (written as bdry C). It is also known [4] that if C contains no hyperplane and is boundedly weakly compact, then supp C is connected.

We show here, that under these same assumptions, supp C is actually arcwise connected. In addition, we show that if C contains no linear variety of finite codimension, then supp C is $\mathscr{C}^\infty$. We also show that if C is boundedly compact, then supp C is contractible.

If $a \in C$, we will use the notation C_a for the union of all open half-spaces not containing C and which are determined by support functionals at a; that is

$$C_a = \bigcup \{(x^* > \langle a, x^* \rangle): x^* \neq 0, \langle a, x^* \rangle = M(x^*, C)\},$$

where $(x^* > \langle a, x^* \rangle) = \{x \in E: \langle x, x^* \rangle > \langle a, x^* \rangle\}$.

We also use the notation X_a for the set $a + \bigcup \{n(C - a): n \in N\}$, and (x, y) for the open line segment $\{\lambda x + (1 - \lambda)y: 0 < \lambda < 1\}$.

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In what follows, C will be a closed convex subset of the Banach space E; without loss of generality we will assume $0 \in C$.

Before we reach our main results, several lemmas are necessary.

LEMMA 1. *For each* $a \in C$ *we have* $\bar{X}_a = E \backslash C_a$.

PROOF. Assume, without loss of generality, that $a = 0$. Suppose $x \in C_0$. Then there exists an $x^* \neq 0$ such that, for each $b \in C$ and $\lambda > 0$,

$$\langle x, x^* \rangle > 0 = M(x^*, C) \geqq \langle \lambda b, x^* \rangle;$$

it follows that $x \notin \bar{X}_0$.

On the other hand, if $x \notin \bar{X}_0$, then, since $\bar{X}_0$ is a closed convex cone, by [1, Corollary 1], there is a $y \in \bar{X}_0$ and $x^* \in E^* \backslash \{0\}$ such that

$$\langle x, x^* \rangle > \langle y, x^* \rangle = M(x^*, \bar{X}_0) = 0.$$

But then x^* supports $\bar{X}_0$ at 0 and so $x \in C_0$.

LEMMA 2. *If* $\operatorname{int} X_0 \neq \emptyset$, *then* $\operatorname{int} C \neq \emptyset$.

PROOF. Let $K^0 \equiv \operatorname{int} X_0$; then K^0 is a Baire space since it is an open subset of the Banach space E.

Note that $C \cap K^0$ is closed relative to K^0 and note also that $K^0 = \bigcup \{n(C \cap K^0): n \geqq 1\}$. Since K^0 is a Baire space, some $n(C \cap K^0)$ has nonempty interior relative to K^0. But then $n(C \cap K^0)$ has nonempty interior relative to E; so the same is true of $C \cap K^0$, and hence of C. (We are grateful to Professor V. L. Klee for pointing out this simple Baire space argument.)

LEMMA 3. *If* $\operatorname{int} C = \emptyset$, *then, for each* $a \in \operatorname{supp} C$, *we have* C_a *dense in* E.

PROOF. (contrapositive). Assume without loss of generality that $0 \in \operatorname{supp} C$ and that $\bar{C}_0 \neq E$. Then there is an $\varepsilon > 0$ and an $a \in E$ such that

$$(*) \qquad B(a; \varepsilon) \subseteq E \backslash \bar{C}_0 \subseteq E \backslash C_0.$$

By Lemma 1, $B(a; \varepsilon) \subseteq \bar{X}_0$ so that

$$B(a; \varepsilon) \backslash X_0 \subseteq \operatorname{bdry} \bar{X}_0.$$

Since $\bar{X}_0$ has interior, $\operatorname{bdry} \bar{X}_0 = \operatorname{supp} \bar{X}_0 \subseteq \bar{C}_0$. From $(*)$ we must have that $B(a; \varepsilon) \subseteq X_0$ and hence, from Lemma 2, that $\operatorname{int} C \neq \emptyset$.

Before proving our final lemma, we recall a definition and some facts from [4]. Suppose $0 \in C \backslash \operatorname{supp} C$ and let

$$F_m = \{x \in C: \exists x^* \in E^*, \|x^*\| \leqq m, \langle x, x^* \rangle = 1 = M(x^*, C)\}.$$

Then each F_m is closed and $\bigcup \{F_m: m \geqq 1\} = \operatorname{supp} C$.

LEMMA 4. *Suppose that* int $C = \emptyset$ *and that* $0 \in C \backslash \text{supp } C$. *Let* S *be a compact subset of* supp C; *then there is an* $x \in E \backslash C$ *such that, for each* $a \in S$, *we have* $C \cap (x, a) \subseteq \text{supp } C$.

PROOF OF LEMMA 4. For each $m \geqq 1$, we have $F_m \cap S$ compact, hence separable. Let A_m be a countable dense subset of $F_m \cap S$ and let $A = \bigcup \{A_m: m \geqq 1\}$. Since each C_a is open and dense and A is countable, we have, by Baire's Theorem, that

$$\bigcap \{C_a: a \in A\} \neq \emptyset.$$

Let x be any element of this intersection and suppose $a \in S$. Then $a \in F_m \cap S$, for some $m \geqq 1$, and there exists a sequence $\{a_n: n \geqq 1\} \subseteq A_m \cap S$ converging to a. Since the normalized support functionals (say $\{a_n^*: n \geqq 1\}$) corresponding to the sequence $\{a_n: n \geqq 1\}$ are uniformly bounded, some subnet converges in the weak* topology to an element a^*.

Now,

$$\begin{aligned} |1 - \langle a, a^*\rangle| &= |\langle a_n, a_n^*\rangle - \langle a, a^*\rangle| \\ &\leqq |\langle a_n - a, a_n^*\rangle| + |\langle a, a_n^* - a^*\rangle| \\ &\leqq \|a_n - a\|m + |\langle a, a_n^* - a^*\rangle| \\ &\to 0, \end{aligned}$$

so $\langle a, a^*\rangle = 1$. If $b \in C$, then for each n, $\langle b, a_n^*\rangle \leqq 1$, hence, $\langle b, a^*\rangle \leqq 1$ and $\langle a, a^*\rangle = M(a^*, C)$. Also since $x \in C_{a_n}$ for each n, we have that $\langle x, a_n^*\rangle > 1$ for each n, so $\langle x, a^*\rangle \geqq 1$.

If $b \in C \cap (x, a)$, then, for some $0 < \lambda < 1$,

$$\begin{aligned} \langle b, a^*\rangle &= \langle \lambda x + (1 - \lambda)a, a^*\rangle \\ &= \lambda \langle x, a^*\rangle + (1 - \lambda) \langle a, a^*\rangle \\ &\geqq 1 = M(a^*, C) \\ &\geqq \langle b, a^*\rangle. \end{aligned}$$

Hence, $b \in \text{supp } C$.

THEOREM 1. *Suppose* E *is an infinite dimensional Banach space and* C *is a boundedly weakly compact convex subset containing no linear variety of finite codimension. Then* supp C *is* $\mathscr{C}^\infty$.

PROOF. If int $C \neq \emptyset$, then supp C = bdry C and, since C contains no linear variety of finite codimension, supp C is an AR(metric) [**3**, Corollary 2].

If C = supp C, then supp C is convex.

Thus we may assume int $C = \emptyset$ and $C \neq \text{supp } C$. Also, using the techniques of [**4**, Theorem 9] we may assume the metric projection, p, of E onto C is single valued and continuous and that $p(E\backslash C) \subseteq \operatorname{supp} C$.

Let $f\colon S^n \to \operatorname{supp} C$ be continuous and let $S = f(S^n)$. By Lemma 4, there exists $x \in E\backslash C$ such that, for each $a \in S$,

$$(**) \qquad (x, a) \cap C \subseteq \operatorname{supp} C.$$

Define the homotopy

$$H\colon [0, 1] \times S^n \to \operatorname{supp} C$$

by $H(t, s) = p(tf(s) + (1 - t)x)$. This map clearly deforms f to a constant map, and the deformation takes place in supp C because of (**).

By using the same ideas, we get the following improvement of a result by Phelps [**4**, Theorem 9], who showed only that supp C is connected.

THEOREM 2. *Suppose E is a Banach space and suppose that C is a closed convex subset of E which is boundedly weakly compact. If C contains no hyperplane, then* supp *C is arcwise connected.*

PROOF. Let $a, b \in \operatorname{supp} C$ and take S in Lemma 4 to be $S = \{a, b\}$. If x is as in Lemma 4, then $p([x, a] \cup [x, b])$ is an arc joining a and b in supp C.

It is clear from the above that the set of support points of a closed convex bounded subset of a reflexive Banach space is highly connected. In all the examples we have considered it is contractible; but the following result is the best we have regarding contractability of supp C.

THEOREM 3. *Let C be a boundedly compact convex subset of an infinite dimensional Banach space E; then* supp *C is contractible.*

PROOF OF THEOREM 3. We assume without loss of generality that C is total and that $0 \in C$; otherwise there is an element of E^* which vanishes on C. Hence, supp $C = C$ is convex.

By considering span C = span $(C - C)$ we can suppose E is separable; this is because $C \cap B[0; 1]$ is compact, hence separable, and span $(C \cap B[0, 1])$ is dense in E, since $C \cap B[0; 1]$ is total.

Since E is separable, we may assume E is locally uniformly convex and, hence, the metric projection $p\colon E \to C$ is single-valued and continuous.

Because C is σ-totally bounded, we have, by [**2**, Theorem 13.3], that $C - C$ is not radial at 0. Thus there is an $0 \neq x_0 \in E$ such that

$$(*) \qquad (C - C) \cap \{tx_0 \colon t \geqq 0\} = \{0\}.$$

If $y \in C$, then $(y, x_0] \subseteq E\backslash C$. Indeed, suppose $x \in (y, x_0] \cap C$; i.e., $x = tx_0 + (1 - t)y$, where $0 < t \leqq 1$. We would then have

$$tx_0 = x - (1 - t)y \in C - C,$$

so, from (*), we would conclude that $t = 0$, which is impossible.

The homotopy

$$H: [0, 1] \times \operatorname{supp} C \to \operatorname{supp} C,$$

defined by $H(t, y) = p((1 - t)y + tx_0)$, deforms the identity on supp C to the constant map $y \to p(x_0)$. Since $(y, x_0] \subseteq E \backslash C$, the deformation takes place in supp C; that is

$$H([0, 1] \times \operatorname{supp} C) \subseteq \operatorname{supp} C,$$

as desired.

REFERENCES

1. E. Bishop and R. R. Phelps, *The support functionals of a convex set*, Proc. Sympos. Pure Math., vol 1, Amer. Math. Soc. (1962), 27–35.

2. J. Kelley and I. Namioka et al., *Linear Topological Spaces*, van Nostrand, 1963.

3. G. Luna, *On the numerical range in reflexive Banach spaces*, Math. Ann. **231** (1977), 33–38.

4. R. R. Phelps, *Topological properties of support points of convex sets*, Israel Journal of Math. **13** (1972), 327–336.

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